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ONE FORM OF EXCITATION EQUATIONS OF A PERIODIC WAVEGUIDE

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Consider a problem on excitation of electromagnetic field in a periodic waveguide with ideally conducting walls by a given current density $\mathbf{j}(\mathbf{r},t)$. Let us depart from the Maxwell equations

$$\begin{aligned} &\operatorname{rot}\mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{j},\ \operatorname{rot}\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \\ &\operatorname{div}\mathbf{B} = 0, \qquad \operatorname{div}\mathbf{D} = 0, \end{aligned} \tag{1}$$

material equations $\mathbf{D} = \varepsilon(x,y,z)\mathbf{E},\ \mathbf{B} = \mu(x,y,z)\mathbf{H}$, where ε and μ are scalar *d*-periodic functions of x, and usual boundary conditions (also *d*-periodic) on the waveguide walls.

Next, we will need a special form of discrete Fourier transform of a function $\Phi(x)$:

$$\Phi_\beta(x) = \sum_{n=-\infty}^{\infty} \Phi(x+nd)e^{i\beta nd}, \tag{2}$$

where β is some real-value parameter. The transform $\Phi_\beta(x)$ definitely exists if the function $\Phi(x)$ approach zero sufficiently fast at $x \to \pm\infty$. Evidently,

$$\Phi_\beta(x+d) = \Phi_\beta(x)e^{-i\beta d},\ \Phi_\beta(x) = \Phi_{\beta+2\pi/d}(x). \tag{3}$$

Integrating (2) over an interval of β from 0 to $2\pi/d$ we obtain a formula for the inverse transform:

$$\Phi(x) = \frac{d}{2\pi}\int_0^{2\pi/d} \Phi_\beta(x)d\beta. \tag{4}$$

Now, the transform (2) applied to the equations (1) yields

$$\operatorname{rot}\mathbf{H}_\beta = \frac{\partial \mathbf{D}_\beta}{\partial t} + \mathbf{j}_\beta,\ \operatorname{rot}\mathbf{E}_\beta = -\frac{\partial \mathbf{B}_\beta}{\partial t}, \tag{5}$$

$$\operatorname{div}\mathbf{B}_\beta = 0, \qquad \operatorname{div}\mathbf{D}_\beta = 0; \tag{6}$$

note that the material equations and the boundary conditions remain unchanged because of the periodicity of the waveguide.

To solve equations (5) and (6) let us construct a set of solenoidal eigenfunctions $\mathbf{E}_{s,\beta}(\mathbf{r})$ (analogously for the other field vectors), which satisfy the conditions

$$\mathbf{E}_{s,\beta}(x+d,y,z) = \mathbf{E}_{s,\beta}(x,y,z)e^{-i\beta d} \tag{7}$$

inside the waveguide, the boundary conditions on the walls, and the equations

$$\operatorname{rot}\mathbf{E}_{s,\beta} + \Omega_s(\beta)\mathbf{B}_\beta = 0,\ \operatorname{rot}\mathbf{H}_{s,\beta} + \Omega_s(\beta)\mathbf{D}_{s,\beta} = 0. \tag{8}$$

At a given real β the eigenproblem (7)-(8) appears to be self-conjugate and possesses a spectrum of eigenvalues $\Omega_s(\beta)$, which all are real. It may be shown [1] (in the same way as it is done in the theory of excitation of resonators [2]) that without degeneracy under appropriate normalization of the eigenfunctions

$$\int_{V_0} (\mathbf{D}_{s,\beta}\mathbf{E}^*_{p,\beta} + \mathbf{H}_{s,\beta}\mathbf{B}^*_{p,\beta})dV = \delta_{s,p}, \tag{9}$$

where V_0 is an arbitrary volume in the waveguide bounded by two transversal cross-sections separating a spatial period of the waveguide. In a case of degeneracy one can redefine the eigenfunctions in such way that the relations (9) remain valid (cf. [2]).

Let us search a solution of Eqs. (5), (6) in the form

$$\mathbf{E}_\beta = \sum_s C_{s,\beta}(t)\mathbf{E}_{s,\beta}(\mathbf{r}) - \mathrm{grad}\,\Phi_\beta,\ \mathbf{H}_\beta = -i\sum_s C_{s,\beta}(t)\mathbf{H}_{s,\beta}(\mathbf{r}), \tag{10}$$

imposing a condition on Φ_β

$$\mathrm{div}(\varepsilon\,\mathrm{grad}\,\Phi_\beta) = -\rho_\beta. \tag{11}$$

Now, multiply the first equation (5) by $i\mathbf{H}_p^*$, and the second by $i\mathbf{E}_p^*$, sum them and integrate the result over the volume V_0. The term containing Φ_β disappears, as ρ_β and $\mathbf{j}_\beta$ obey the continuity equation. Accounting the orthogonality relation (9), we obtain the following equations for the coefficients $C_{s,\beta}$:

$$\frac{\partial C_{s,\beta}}{\partial t} - i\Omega_s(\beta)C_{s,\beta} = -\int_{V_0} j_\beta \mathbf{E}^*_{s,\beta}dV = -\int_V j\mathbf{E}^*_{s,\beta}dV, \tag{12}$$

where V designates the entire volume of the waveguide. Let us decompose both parts of the relation (12) to Fourier series over the argument β:

$$\frac{\partial C_{s,n}}{\partial t} - i\sum_{m=-\infty}^{\infty}\Omega_{s,m}C_{s,n-m} = -\int_V j\mathbf{E}^*_{s,n}dV, \tag{13}$$

where

$$C_{s,n} = \frac{1}{2\pi}\int_0^{2\pi} C_{s,\beta}e^{-i\beta nd}d(\beta d);\ \Omega_{s,n} = \frac{1}{2\pi}\int_0^{2\pi}\Omega_s(\beta)e^{-i\beta nd}d(\beta d); \tag{14}$$

$$\mathbf{E}_{s,n}(x,y,z) = \frac{1}{2\pi}\int_0^{2\pi}\mathbf{E}_{s,\beta}e^{-i\beta nd}d(\beta d) = \mathbf{E}_{s,0}(x-nd,y,z). \tag{15}$$

Undertaken the transform (4) we finally get from Eqs. (10)

$$\begin{aligned}\mathbf{E} &= \sum_s\left[\sum_{m=-\infty}^{\infty}C_{s,n}(t)\mathbf{E}_{s,0}(x-nd,y,z)\right] - \mathrm{grad}\,\Phi,\\ \mathbf{H} &= -\sum_s\left[\sum_{m=-\infty}^{\infty}iC_{s,n}(t)\mathbf{H}_{s,0}(x-nd,y,z)\right],\end{aligned} \tag{16}$$

where the coefficients $C_{s,\,n}$ has to be determined from equations (13) and Φ from the Poisson equation

$$\mathrm{div}(\varepsilon\,\mathrm{grad}\,\Phi) = -\rho \tag{11}$$

obtained by transformation (4) of the relation (11). Quasi-static field of space charge is extracted in (15) explicitly, like in the theory of excitation of resonators [2].

To reveal physical sense of the quantities $C_{s,n}$, $\Omega_{s,n}$, $\mathbf{E}_{s,n}$ consider a coupled resonator waveguide. If we decrease to zero a size of coupling holes, the surfaces where the conditions (7) must be valid disappear, and the eigenfrequences $\Omega_s(\beta)$ and the functions $\mathbf{E}_{s,\beta}(\mathbf{r})$ become independent on β. Then, the Fourier transforms of these quantities in respect to the argument β contain only constant terms $\left(\mathbf{E}_{s,0}(\mathbf{r}),\ \Omega_{s,0}\right)$, and (13) becomes the set of uncoupled equations for separated resonators. The coefficient $C_{s,n}$ characterizes then the complex amplitudes of the *s*-th mode of the *n*-th resonator.

A weak coupling between the resonators will be characterized evidently by the terms $C_{s,n\pm 1}$ (each resonator is coupled with the nearest neighbors). In this approximation, as follows, one can neglect by the values of $\Omega_{s,m}$ with all m except $m = 0, \pm 1$. With $\Omega_1 = \Omega_{-1}$ the system (13) gives rise to a dispersion relation $\omega = \Omega_0 + 2\Omega_1 \cos\beta d$ typical to a chain of weakly coupled oscillators [3]. Under increase of coupling, other terms will become relevant, with $m = \pm 2, \pm 3$ etc., which correspond to coupling with the resonators of number $n \pm 2$, $n \pm 3$ etc.

Next, accounting the sense of the coefficients $C_{s,n}$, from the relation (15) one can conclude that the function $\mathbf{E}_{z,n}(\mathbf{r})$ determines the field at the point $\mathbf{r}$ in the case $C_{s,n} = 1$, $C_{p,m} = 0$ ($s \neq p$ or $m \neq n$). The same (conjugate) function enters the integral expression in the right-hand part of (13), i.e. it characterizes the effect of the source placed at the point $\mathbf{r}$ onto the coefficient $C_{s,n}$.

Thus, it is clear that the equations (13) and (15) are of simple form just in the cases when the so-called discrete approach is appropriate for description of the excitation of the transmission line [3]. So, they may be treated as electrodynamic foundation and generalization of this approach (including non-stationary processes). Consideration based on the equations (13) and (15) has some advantages over the usually used equivalent scheme approach, which are generally intrinsic to a consistent electrodynamic description [2]. Indeed, if we are given the dispersion relation and configuration of the electromagnetic field in the eigenmode at different β (for their computations many methods considered in literature may be used, see e.g. [4]), the excitation equations are determined completely and definitely in general form. Note that even accounting a restricted number of coupling coefficients $\Omega_{s,m}$ in (13) we get an approximate description of the waveguide properties in a wide frequency band in contrast to the method suggested in Ref. [5].

For a ring geometry of the periodic structure, characteristic, say, to magnetrons, one has to set $C_{s,\, n+N} = C_{s,\, n}$, where N is the number of the resonator cells. In the case of perfectly matched load at the ends of the waveguide one can think of the system as continued to infinity in both directions, and, solving the initial problem not deal with the end boundary conditions at all. In other cases a rigorous definition of boundary conditions is more problematic and requires concretization of the situation. It may be recommended to apply a phenomenological approach accounting the above physical interpretation of the equations (13). For instance, suppose we account in the Fourier decomposition of $\Omega_s(\beta)$ only the terms of numbers -1, 0, $+1$, and the waveguide is bounded to the left by the cell n=0. Then, excluding in the respective relation (13) the term $C_{s,-1}$ and adding the term $\gamma C_{s,0}$ modeling the active load, we obtain the expression

$$\frac{\partial C_{s,0}}{\partial t} - i\Omega_{s,0} C_{s,0} - i\Omega_{s,1} C_{s,1} + \gamma C_{s,0} = -\int_V j\mathbf{E}^*_{s,0} dV\,, \qquad (17)$$

which play the role of the boundary condition. The rest equations remain unchanged. In a similar way one can consider a case when larger number of the coefficients $\Omega_{s,m}$ is accounted.

REFERENCES

1. V.A. Solntsev, JTP, 1968, 38, 1, 100. (In Russian.)
2. L.A. Vainstein, Electromagnetic waves, Sov. Radio, 1957; G.V. Kisun'ko, Electrodynamics of hollow systems, VKAS, 1949; L.A. Vainstein and V.A. Solntsev, Lectures on ultra-high-frequency electronics, Sov. Radio, 1973. (In Russian.)
3. L.V. Bulgakova, D.I. Trubetskov, V.L. Fisher, and V.N. Shevchik, Lectures on electronics of UHF devices of O-type, Saratov Univ., 1974. (In Russian.)
4. R.A. Silin and V.P. Sazonov, Slow-wave systems, Sov. Radio, 1966. (In Russian.)
5. S.P. Kuznetsov and D.I. Trubetskov, Two lections on non-stationary theory of interaction of electronic beams with electromagnetic waves, Lectures on UHF electronics (3-d Winter School-Seminar of Engineers), book 5, Saratov Univ., 1974. (In Russian.)
6. W.I. Kleen. Electronics of Microwave Tubes, Academic Press, 1958.

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