Nature of the instability in a system of two weakly coupled waves

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1. A vast number of physical problems reduce to an interaction between two waves. Such interactions are widely used in amplifiers, if the instability is connective, and in oscillators, if it is absolute. The nature of the instability, however, is usually not a simple question. The purpose of the present study is twofold. The first goal is to study the nature of the instability in a situation in which it is not clear whether we are dealing with an amplifier or an oscillator. The second is to propose a method for studying the nature of the instability in multiparameter problems; this method radically shortens the formal mathematical aspect of the analysis.

Let us assume that two branches of the spectrum of a dissipationless system, $\omega=\omega_1(\beta)$ and $\omega=\omega_2(\beta)$ intersect and are coupled with each other; the coupling region in the phase plane occupies a small interval of frequencies and wave numbers. The dispersion relation of the system in this case is [1,2]

$$[\omega-\omega_1(\beta)][\omega-\omega_2(\beta)]=\pm\varepsilon^2. \qquad (1)$$

Here ε is a small parameter, equal in order of magnitude to the frequency dimension of the coupling region. We will treat the case in which the system is unstable, which corresponds to the minus sign on the right side of (1).

Since the coupling region is small, we approximate the functions $\omega=\omega_1(\beta)$ and $\omega=\omega_2(\beta)$ in this region by simply the first few terms of Taylor series. It is frequently sufficient to use only the linear terms:

$$[\omega-\omega_0-\omega_1'(\beta_0)(\beta-\beta_0)][\omega-\omega_0-\omega_2'(\beta_0)(\beta-\beta_0)]=\pm\varepsilon^2. \qquad (2)$$

If the group velocities of the waves $\omega_1'(\beta)$ and $\omega_2'(\beta)$ have different signs, the instability is absolute; if the signs are the same, the instability is convective. This result is attributable to Sturrock [2] and is a universal criterion for distinguishing the nature of the instability of two weakly coupled waves. If, however, the group velocity of one of the waves vanishes in the coupling region, this analysis of the nature of the instability is no longer valid.[1)] *A* situation of this sort arises in a problem characterized by some parameter *B* if, upon a change in *B*, the point at which the dispersion characteristics intersect jumps from the ascending part to the descending part of one of the characteristics, and the instability accordingly goes over from| convective to absolute (Fig.1). We will analyze this situation here. It might be noted that this question arises in several applied problems, e.g., in analysis of the operation of a traveling-wave tube near the transmission boundary of the waveguide system.[3]

2. We assume that the critical point for the first wave falls in the coupling region, $\omega_1'(\beta_k)=0$. Then

$$[\omega-\omega_0-\frac{\omega_1''(\beta_k)}{2}(\beta-\beta_k)^2][\omega-\omega_0-\omega_2'(\beta_k)(\beta-\beta_k)]=\pm\varepsilon^2. \qquad (3)$$

For definiteness we assume $\omega_1''(\beta_k)<0$ and $\omega_2'(\beta_k)>0$. Figure 1 is drawn in accordance with this choice of signs.

All the effects associated with the existence of a critical point occur in characteristic frequency and wave-number intervals determined by the relation $\Delta\omega\propto|\omega_1''|\Delta\beta^2\propto\varepsilon^{4/3}$. We therefore introduce the dimensionless frequency Ω and the dimensionless wave number *K* as follows:

$$\Omega=\left[\frac{|\omega_1''|}{2(\omega_2')^2}\right]^{1/3}\frac{\omega-\omega_k}{\varepsilon^{4/3}},\quad K=\left[\frac{2}{|\omega_1''|\omega_2'}\right]^{1/3}\frac{\beta-\beta_k}{\varepsilon^{2/3}} \qquad (4)$$

Transforming to dimensionless variables in (3), and discarding the term containing the small parameter ε, we find

$$\Omega=-K^2+\frac{1}{K+B} \qquad (5)$$

Here $B=\left[\frac{|\omega_1''|}{2\omega_2'}\right]^{1/3}\frac{\omega_2(\beta_k)-\omega_k}{\varepsilon^{2/3}}$ is a parameter which is a measure of the separation of the dispersion characteristics of non-interacting waves; the value *B* = 0 corresponds to their intersection exactly at the critical point (Fig.1). For the scales chosen here the dispersion characteristic of the second wave appears as a vertical line, *K* = –*B*. We recall, however, that this line is slightly inclined,[2] as in Fig. 1.

3. Let us find the boundary value of the parameter B at which the absolute instability disappears. This happens when the growth rate corresponding to the saddle point of the function $\Omega(K)$, which is responsible for the absolute instability vanishes

$$\operatorname{Im}\Omega(K_0) = 0, \ \frac{\partial\Omega(K_0)}{\partial K_0} = 0. \qquad (6)$$

For dispersion law (5) there is a unique value of the parameter B (B = 1.5) at which both conditions in (6) hold. At larger values of B, the point at which the two branches intersect lies far from the critical point and we can use the Sturrock [2] criterion. At large negative values of B (Fig.1) the instability is absolute, while at large positive values it is convective. An absolute instability thus occurs for B<1.5.

4. Let us assume that the system contains a source which is oscillating at a frequency Ω_0. Let us find the boundaries of the amplification and transmission regions in the Ω_0, B parameter plane. Waves with wave numbers determined by

$$\Omega_0 = \Omega(K) \qquad (7)$$

travel away from the source. *At* the boundary of the amplification region, the spatial growth rate of at least one if the waves vanishes:

$$\operatorname{Im}K(\Omega_0) = 0. \qquad (8)$$

Figure 2 shows all three of the lines which satisfy Eqs. (7) and (8) for B>1.5 and for dispersion law (5). At large positive values of B, according to Sturrock, an amplification band should exist near the dispersion characteristic for the second wave, while outside this band there should be a transmission region. These arguments, combined with a study of the asymptotic behavior of the lines determined by Eq. (7) and (8), lead to the assertion that the amplification region lies between the two solid curves, while the dashed line lies entirely in the transmission region.

5. Figure 2 shows the regions in which the system exhibits different types of behavior in the plane of the parameters Ω_0 and B. Here A represents an absolute instability, C represents a convective instability, and T represents transmission. Since we have not specified the physical nature of the interacting waves at any point, the results are universal.

We note in conclusion that we have actually used the following method: The parameter space was partitioned into regions by the surfaces in (6) and (7)-(8). The nature of the solution within each region was determined from physical considerations, e.g., the asymptotic behavior. This method is convenient since a determination of the boundaries of the regions requires only the necessary instability conditions, which can be determined very simply. It is also simpler to study the behavior of the system within each region than to test the lengthy instability con-

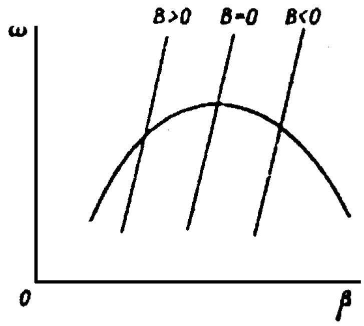

FIG. 1

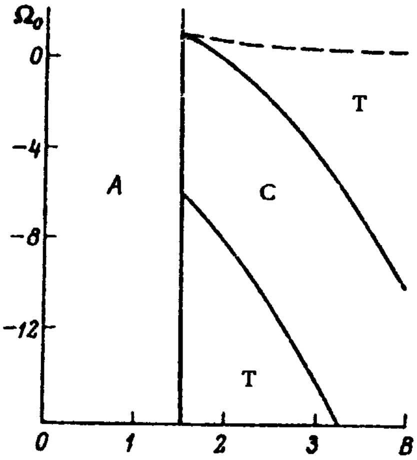

FIG. 2

[1] We will call this point (ω_k, β_k) the "critical point."

[2] This assertion is equivalent to a certain rule for circumventing the singularity in dispersion law (5) in an integration in the K plane.

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