Effect of a periodic external perturbation on a system which exhibits an order-chaos transition through period-doubling-bifurcations

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The effect of a periodic perturbation on the dynamics of a model system which exhibits a transition to chaos through period doubling is analyzed. A scaling law is found. The structure of the space parameter at small amplitudes of the perturbation is described.

The transitions of dynamic systems from periodic to random behavior upon changes in the parameters have recently attracted much interest.[1–8] A typical scenario for the transition from order to chaos involves an infinite sequence of bifurcations at each of which the period of the motions doubles; this sequence of bifurcations is described by the Feigenbaum similarity laws.[2,3] Such transitions have been observed experimentally and in numerical simulations in mechanical, electronic, and hydrodynamic systems.[4–7] One of the remarkable results derived by Feigenbaum[3] is that, regardless of the particular nature of the equations of the system, the behavior of the system near the transition point can be analyzed by a model of a one-dimensional recurrent mapping, e.g.,

$$x_{m+1} = \lambda - x_m^2 . \tag{1}$$

System (1) demonstrates a period doubling as the parameter λ is increased. The sequence of bifurcation values λ_n^0 is given by Feigenbaum's formula $\lambda_n^0 = \lambda_c^0 - K\delta^{-n}$ (which, strictly speaking, is valid at large n). Here $\delta = 4.6692$ is a universal constant, while $\lambda_c^0 = 1.40116$ and $K = 0.7242$ are particular constants of system (1).

Let us consider the effect of a small periodic perturbation on a system which, in the absence of the perturbation, demonstrates a transition to chaos through period doubling and which is thus described by model (1). According to Feigenbaum,[2] λ is the only important parameter of the model, so that we need to examine only the effect of the periodic change in this parameter. We thus analyze the model

$$x_{m+1} = \lambda + A\cos(2\pi m w + \theta) - x_m^2 , \tag{2}$$

where A and θ are the amplitude and initial phase of the perturbation, and w is the rotation number, which is determined by the ratio of the periods of the intrinsic motion and of the external perturbation. Rational values of w correspond to synchronization regimes, while irrational values correspond to beat regimes. The results reported below are a generalization of numerical results calculated for model (2), but they are just as general as Feigenbaum's results, as can be shown by a renormalization approach. The proof, however, lies outside the scope of this paper.

We known that in the interval between two bifurcation values of the parameter $\lambda_{n-1}^0 < \lambda < \lambda_n^0$ the unperturbed system has a stable regime of motion: a cycle of period 2^n. If we ignore the regions near the ends of the interval, then a motion occurs near the 2^n cycle (nearest, the smaller the perturbation) when a perturbation is applied. If the rotation number is rational, $w = p/q$, this is a periodic motion with a period N equal to the least common multiple of the numbers 2^n and q; if the number of rotations is instead irrational, the motion is a quasiperiodic motion. Of particular interest is the vicinity of the point λ_n^0, since it is there that we see a bifurcation: a transition from motion near the 2^n cycle of the unperturbed system to motion near the 2^{n+1} cycle.

We first assume that the rotation number is rational. We consider an N cycle of the perturbed system with $\lambda \approx \lambda_n^0$. Since the perturbation is small, one of the elements of this cycle, x_0, is near an element of the 2^n cycle of the unperturbed system, $\bar{x}_0$. We are interested in the parameter $\mu = \partial x_N / \partial x_0$, which is the multiplier of the N cycle, and we are also interested in a quantity associated with this parameter: the Lyapunov characteristic index $\gamma = (1/N)\ln|\mu|$. Values $|\mu| < 1$, $\gamma < 0$, evidently correspond to a stable cycle, while $|\mu| > 1$, $\gamma > 0$, correspond to an unstable cycle. From the equation $|\mu| = 1$ or $\gamma = 0$ we can find the surface in the space of parameters in which the bifurcation occurs.

To find μ and γ we write x_N as a power series in $\Lambda = \lambda - \lambda_n^0, \xi = x_0 - \bar{x}_0$ and A. The coefficients of the series (the partial derivatives of x_N at the point $\Lambda = 0$, $\xi = 0$, $A = 0$) can be calculated from recurrence relations found by differentiating (2):

$$x_{m+1} = \lambda_n^0 - x_m^2; \quad x_{m+1,\Lambda} = -2x_m x_{m,\Lambda} + 1;$$

$$x_{m+1,A} = -2x x_{m,A} + \cos(2\pi m w + \theta);$$

$$x_{m+1,\xi} = -2x_m x_{m,\xi}; \quad x_{m+1,A\xi} = -2(x_m x_{m,A\xi} + x_{m,\xi} x_{m,A}); \tag{3}$$

$$x_{m+1,AA} = -2(x_m x_{m,AA} + x_{m,A}^2);$$

$$x_{m+1,\Lambda\xi} = -2(x_m x_{m,\Lambda\xi} + x_{m,\Lambda} x_{m,\xi});$$

$$x_{m+1,AA\xi} = -2(x_m x_{m,AA\xi} + x_{m,\xi} x_{m,AA} + 2x_{m,A} x_{m,A\xi}),$$

where $x_N = x_0$, $x_{N,\Lambda} = x_{0,\Lambda}$, $x_{N,A} = x_{0,A}$, $x_{N,AA} = x_{0,AA}$, $x_{0,\xi} = 1$, $x_{0A\xi} = 0$, $x_{0,AA\xi} = 0$, $x_{0\Lambda\xi} = 0$. Using (3), we can show that if $q \neq 2^k$, then $x_{N,A} = 0$, while $x_{N,AA}$ does not depend on the phase[1] θ. Noting that we have $\mu = -1$ for $\Lambda = 0$ and $A = 0$, we find

$$\mu \approx -1 + x_{N,\Lambda\xi}\Lambda + x_{N,AA}\frac{A^2}{2} = -1 - C_n\Lambda + D_n(w)A^2;$$

$$\gamma \approx \frac{C_n\Lambda - D_n(w)A^2}{N}. \tag{4}$$

Consequently, μ remains real when the perturbation is applied. The bifurcation occurs at

$$\mu = -1, \text{ i.e., } \Lambda_n = D_n(w)A^2/C_n \text{ or } \lambda_n = \lambda_n^0 + D_n(w)A^2/C_n. \tag{5}$$

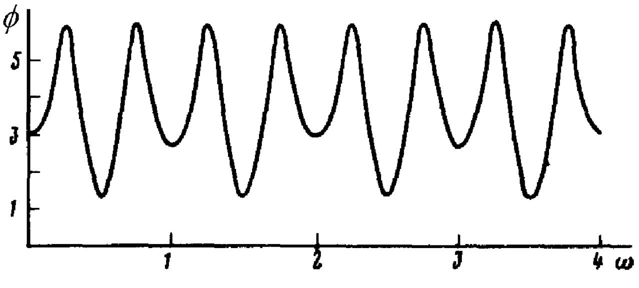

FIG. 1. The function $\phi(w)$.

The transition of the multiplier through -1 is known[9] to be accompanied by a period-doubling bifurcation, in this case the creation of a $2N$ cycle.

How do the coefficients in Eq.(4) depend on n and w? From the solution of the unperturbed problem[8] we have $C_n \sim \delta^n$, where $\delta = 4.6692$. To determine the behavior of the coefficient $D_n(w)$, we examine the behavior of the ratios $\Delta_n = D_{n+1}(w)/D_n(w)$. Numerical calculations show that the following similarity law holds at sufficiently large values of n (in practice, at $n \gtrsim 2$): As the rotation number is changed by a factor of 2^k, the sequence $\Delta_n(w)$ shifts k steps; i.e., $\Delta_n(2^k w) = \Delta_k(w)$. It follows that if the rotation number is a binary fraction with a period k, then the sequence Δ_n also has the period k: $\Delta_{n+k}(w) = \Delta_n(w)$.

Figure 1 shows the w dependence of Δ_n found numerically in terms of the coordinates $\omega = 2^n w, \phi = \ln\Delta_n$. At $n \gtrsim 2$ the points corresponding to different values of n and w conform to a common curve[2)] $\phi = \phi(\omega)$. This result suggests that ϕ and Δ_n are also meaningful for irrational values of w.

Let us consider a sequence of periodic binary fractions, $w_i = p_i/q_i$, which converge to an irrational limit w. Corresponding to each rational approximation w_i is a cycle of period N_i, which undergoes a bifurcation at $\lambda_n(w_i)$ [see (5)]. Using (3), we can show that in the limit $w_i \rightarrow w$ the following quantities have finite values: $\lambda_n(w_i), \Delta_n(w_i), \phi(w_i), \gamma(w_i)$. We assign corresponding limiting values to the functions λ_n, Δ_n, ϕ, and γ at the irrational point w. We note that we have $\gamma(w) < 0$ at $\lambda < \lambda_r(w)$ and $\gamma(w) > 0$ at $\lambda > \lambda_n(w)$. This means that at $\lambda = \lambda_n(w)$ there is a bifurcation of the quasiperiodic motion localized around the 2^n cycle of the unperturbed problem. This motion loses its stability, and another stable quasiperiodic regime sets in, localized near the 2^{n+1} cycle. It was apparently "torus-doubling" bifurcations of this type which were observed in Ref. 5.

It is now a simple matter to describe the structure of the space of parameters (λ, A, w). The bifurcation surfaces are described by Eq. (5), where $D_n/C_n \sim \delta^{-n}\prod_{i=0}^{n-1}\Delta_n(w) \sim \delta^{-n}\exp\sum_{i=0}^{n=1}\phi(w\cdot 2^i)$. For rational values of w, which are binary fractions of period k, the configuration of the regions in the (λ, A) plane converts into itself upon a change in the scale with respect to the point $(\lambda_c^0, 0)$ by a factor of δ^k along the λ axis and by a factor of $\exp 1/2\Sigma_{i=0}^{k-1}\phi(w\cdot 2^i)$ along the A axis (Fig. 2a). For irrational values of w, there is no scale invariance in the usual sense (Fig. 2b). For numbers w, which belong to a set of measure 1 on the interval (0,1), however, we can

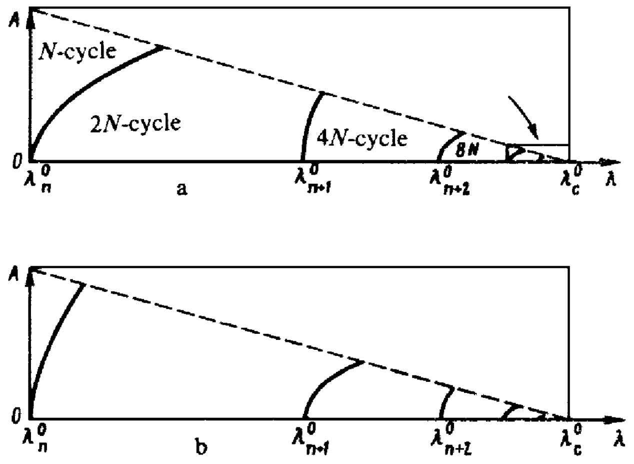

FIG. 2. Sketch of the (λ, A) parameter plane. a: For rational w (a binary fraction of period 3). b: For irrational w. Solid curves—Period-doubling bifuractions of cycles (a) and "torus-doubling" bifurcations (b); dashed lines—nominal applicability boundary of Eqs. (4) and (5). Part a is reproduced in smaller scale in the rectangle shown by the arrow (not drawn to scale).

say that there is a scale invariance of regions in a statistical sense. Instead of the constants which characterize the change in scale along the A axis upon doubling bifurcations, we are now dealing with random numbers with definite statistical characteristics. These characteristics can be found by working with the function ϕ, since we know that for a point w chosen at random the fractional part of the number $2^n w$ is distributed with a uniform probability over the entire interval (0,1). Consequently, the average logarithm of the scale factor Δ_n, for example, is simply $\langle \ln\Delta \rangle = 1/2^k \int_0^{2k} \phi(w) dw \approx 3.66$ (a universal constant).

[1] The case $q = 2^k$ is of no particular interest, since the effect of the perturbation, after a k-fold doubling of the period, reduces to a constant correction to λ, and the transition to chaos occurs in the Feigenbaum manner.

[2] From Fig. 1 we see that the dependence $\phi(\omega)$, which is nearly periodic, can be approximated by a function of period 2^k. The error of this approximation falls off rapidly with increasing k.

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