MODEL DESCRIPTION OF A CHAIN OF COUPLED DYNAMIC SYSTEMS NEAR ORDER-DISORDER PHASE TRANSITIONS

S. P. Kuznetsov

UDC 517.9

A method is suggested of phenomenological description of a chain of coupled systems, demonstrating during the variation of some parameter a transition from periodic to chaotic regimes through an infinite sequence of period doubling. The method consists of using models, chains of coupled, one-dimensional mappings. It is shown that from the point of view of chain behavior, following a large number of period doublings the coupling between the systems is a combination of two fundamental types of coupling; equality of the corresponding coupling coefficients for a chain and its models is the correspondence condition between them. A specific example is considered, a chain of parametrically excited nonlinear oscillators with dissipation.

1. The essence of the concept of a dynamic system consists of the fact that its state at the initial moment of time uniquely determines the state at any subsequent moment [1]. The state is a certain set of quantities referring to the system, for example, the deviation angle and the rate for a pendulum. It seems that for a change in some parameters many dynamic systems of diverse nature undergo a transition from periodic to chaotic behavior [2-10]. A common method of generating chaos is related to the existence of an infinite sequence of period doublings. In this case the value of the parameter λ_n at which doubling errors converges to a finite limit, the critical point λ_c [2-4]:

$$\lambda_n = \lambda_c - K\delta^{-n}. \tag{1}$$

Here λ_c and K depend on the specific system, while $\delta = 4.6692$ is a universal constant. As examples one can quote a nonlinear oscillator with dissipation, excited by periodic external action [5, 6], a number of hydrodynamic systems described by the Galerkin equation [7, 8], and several problems of chemical kinetics [9].

The simplest system demonstrating period doubling is the recurrent, one-dimensional mapping [2, 3]:

$$x_{n+1} = 1 - \lambda x_n^2 \equiv F(x_n). \tag{2}$$

Here x_n is a real variable, describing the state at the discrete moment of time n, and λ is a parameter. The values of the constants in Eq. (1) are: $\lambda_c = 1.40116$, $K = 0.7242$. For $\lambda < \lambda_c$ the system demonstrates periodic behavior, and for $\lambda > \lambda_c$ – the chaotic regime.*

As shown in [4], the system (2) can be considered as a model, describing the behavior of various real dissipative systems undergoing transition to chaos by means of period doubling. Initially one transforms from the differential equations to the Poincaré mapping [1]. After multiple iterations this mapping becomes equivalent to the one-dimensional one due to compression of phase volume in dissipative systems [4], and reduces to appropriate change of variables in the form (2). The dynamic model and the system correspond to each other if they are characterized by identical values of the quantity $(\lambda - \lambda_c)K^{-1}$.

The real problem consists of the study of generation of chaos in distributed systems [7, 8]. A common method in the theory of oscillations of studying transitions from simple to distributed systems consists of considering a chain or a medium consisting of weakly coupled

*For $\lambda > \lambda_c$ there exist, however, narrow regions of the parameter value, in which stable periodic regimes are realized [2].

Saratov State University. Translated from Izvestiya Vysshikh Uchebnykh Zavedenii, Fizika, No. 6, pp. 87-96, June, 1984. Revised article submitted November 10, 1983.

 0038-5697/84/2706-0522$08.50 © 1984 Plenum Publishing Corporation

coupled systems with properties under investigation. It is natural to pose the problem of behavior of a chain (medium) of systems demonstrating period doubling. The purpose of the present study is to show the correspondence conditions between the behavior of such a chain and its model, consisting of systems (2). The conditions found are illustrated by a specific example, which is of interest in solid state theory, nonlinear optics, and radio physics.

2. As a unit, on whose basis the chain is constructed, we use the system (2). We note that a small addition to the variable x is transformed per unit time according to the equation $\xi_{n+1} = F'(x_n)\xi_n$.

Transforming to a chain of coupled cells, we assume that Eq. (2) describes a spatially homogeneous solution, i.e., it determines the chain dynamics when x is independent of the cell number m. We consider a small perturbation of the spatially homogeneous solution $\xi_{n,m}$. To describe its evolution one must add to the perturbation equation a term linear in ξ:

$$\xi_{n+1,m} = F'(x_n)\xi_{n,m} - \varphi(x_n)(\xi_{n,m-1} - 2\xi_{n,m} + \xi_{n,m+1}), \tag{3}$$

where $\varphi(x)$ is a function whose shape depends on the specific method of introducing the coupling between the cells. Putting $\xi_{n,m} = \xi_n e^{i\beta m}$ (β is the wave number), and restricting ourselves to long-wave perturbations $\beta^2 \ll 1$, we obtain:

$$\xi_{n+1} = F'(x_n)\xi_n + \varphi(x_n)\beta^2\xi_n. \tag{4}$$

We verify Eqs. (2) and (4) by transforming its renormalization (doubling) [2, 3]: We express x_{n+2}, ξ_{n+2} in terms of x_n, ξ_n, neglecting terms of order β^4, and perform the replacement $x \to x/a_0$, $a_0 = 1/F(F(0))$:

$$x_{n+2} = a_0 F\left(F\left(\frac{x_n}{a_0}\right)\right),\ \xi_{n+2} = \left\{F'\left(F\left(\frac{x_n}{a_0}\right)\right)F'\left(\frac{x_n}{a_0}\right) + \left[F'\left(\frac{x_n}{a_0}\right)\times\right.$$
$$\left.\times\varphi\left(F\left(\frac{x_n}{a_0}\right)\right) + F'\left(F\left(\frac{x_n}{a_0}\right)\right)\varphi\left(\frac{x_n}{a_0}\right)\right]\beta^2\right\}\xi_n.$$

As a result of N-fold application of this procedure we have

$$x_{n+2^N} = F_N(x_n),\ \xi_{n+2^N} = F'_N(x_n)\xi_n + \beta^2\varphi_N(x_n)\xi_n, \tag{5}$$

where the functions $F_N(x)$ and $\varphi_N(x)$ satisfy the recurrence relations:

$$F_{N+1}(x) = a_N F_N\left(F_N\left(\frac{x}{a_N}\right)\right),\quad a_N = 1/F_N(F_N(0)), \tag{6}$$

$$\varphi_{N+1}(x) = F'_N\left(\frac{x}{a_N}\right)\varphi_N\left(F_N\left(\frac{x}{a_N}\right)\right) + F'_N\left(F_N\left(\frac{x}{a_N}\right)\right)\varphi\left(\frac{x}{a_N}\right). \tag{7}$$

Let $\lambda = \lambda_c$; then, according to [2, 3] there exist nontrivial limits $\lim\limits_{N\to\infty} f_N(x) = g(x)$ and $\lim\limits_{N\to\infty} a_N = a = -2.5029$. As seen from (6), the function g(x) must satisfy the functional equation [2]:

$$g(x) = ag\left(g\left(\frac{x}{a}\right)\right), \tag{8}$$

with $g(0) = 1$, $g'(0) = 0$, $g''(0) = 0$, $a = 1/g(1)$. As shown by Feigenbaum, the function g(x) can be found directly as a solution of Eq. (8). This function is universal (it is independent of the specifics of F(x)). A polynomial approximation of g(x) was given in [3], allowing to calculate it in the segment [−1, 1] with an accuracy of 10 decimal digits. The function g(x) is drawn in Fig. 1.

In view of that, relationship (7) at $\lambda = \lambda_c$ and large N acquires the form:

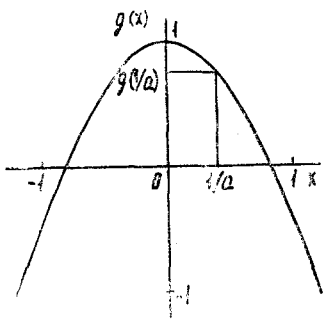

Fig. 1

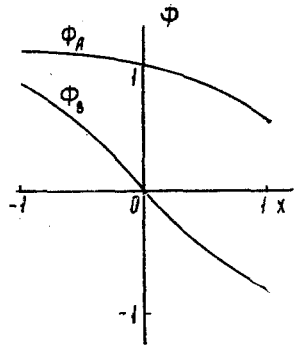

Fig. 2

$$\varphi_{N+1}(x) = g'\left(\frac{x}{a}\right)\varphi_N\left(g\left(\frac{x}{a}\right)\right) + g'\left(g\left(\frac{x}{a}\right)\right)\varphi_N\left(\frac{x}{a}\right).$$

It can be simplified by means of the replacement $\varphi_N(x) = g'(x)f_N(x)$:

$$f_{N+1}(x) = f_N\left(\frac{x}{a}\right) + f_N\left(g\left(\frac{x}{a}\right)\right) \equiv \hat{L}f_N(x), \tag{9}$$

where the identity following from (8) was used

$$g'(x) = g'\left(\frac{x}{a}\right) g'\left(g\left(\frac{x}{a}\right)\right).$$

The derivative $g'(x)$ has a single simple root in the segment $[-1, 1]$ at the point $x = 0$. We consider the function f, represented in the form $f(x) = \text{const}/x + f(x)$, where $f(x)$ satisfies the Lipschitz condition in the segment $[-1, 1]$. (For this it is sufficient to require twofold differentiability of the coupling function $\varphi(x)$ for $x \in [-1, 1]$.) Then, according to Theorem 3, formulated and proved in the Appendix, the function $f_N(x)$ is uniformly approximated by the expression $f_N(x) = C_A\psi(x)a^N + C_B\cdot 2^N + O\left(\left|\frac{2}{a}\right|^N\right)$, and consequently,

$$\varphi_N(x) = C_A\Phi_A(x)a^N + C_B\Phi_B(x)\cdot 2^N C + O\left(\left|\frac{2}{a}\right|^N\right), \tag{10}$$

where C_A and C_B are constants, depending on the initial function $\varphi(x)$; $\Phi_A(x) = g'(x)\psi(x)$, $\Phi_B(x) = g'(x)$, and the function $\psi(x)$ is introduced in the Appendix. The functions $\Phi_A(x)$ and $\Phi_B(x)$ are drawn in Fig. 2. Since the function $\varphi_N(x)$ characterizes the coupling between chain cells, we treat each of the two terms in (10) as a definite type of coupling, the A- and B-types, respectively. Thus, the problem of behavior of chains of the shape considered reduces after a large number of period doublings to the study of chains with coupling functions $C_A\Phi_A(x) + C_B\Phi_B(x)$, i.e., the problem is radically simplified in comparison with the original statement of the problem, containing the arbitrary function $\varphi(x)$.* The condition of identical behavior of two different chains at the critical point is the equality of the corresponding coefficients C_A and C_B. It hence follows that it is necessary to be able to find C_A and C_B for specific systems; one method of determining them is presented below.

3. As well known [2], the mapping (2) has an $\lambda = \lambda_c$ unstable cycles of period 2^N, $N = 0, 1, \ldots \infty$. We define the multiplier of the 2^N-cycle $\mu_N(\beta)$ as the quantity showing how much a small perturbation of shape $e^{i\beta m}$ changes during the period of a cycle on the background of the cycle of the spatially homogeneous solution corresponding to it. As seen from

*A chain with pure B-coupling was investigated in [10].

Eq. (5), the renormalized x value corresponding to the 2^N-cycle for large N is $x_* = 0.5439$, the root of the equation $x = g(x)$ [2]. Substituting $x = x_*$ into Eq. (5), using (10) and the definition of μ_N, we obtain:

$$\mu_N(\beta) = \mu_* + (c_A a^N + c_B \cdot 2^N)\beta^2 \tag{11}$$

(for sufficiently large N), where

$$\mu_* = g'(x_*) = -1.60119; \quad c_{A,B} = C_{A,B}\Phi_{A,B}(x_*).$$

If for some chain we calculate the multipliers of both cycles at the critical point, the coefficients c_A and C_B are then found from (11). One can then describe the chain model, consisting of the systems (2):

$$\begin{aligned} x_{n+1,m} = 1 - \lambda x_{n,m}^2 + 1.0979\, c_A(-x_{n,m-1} + 2x_{n,m} - x_{n,m+1}) + \\ + (0.8751\, c_B - 0.135\, c_A)(-x_{n,m+1}^2 + 2x_{n,m}^2 - x_{n,m+1}^2). \end{aligned} \tag{12}$$

The numerical coefficient values were selected in such a fashion that the multipliers of the 2^N-cycles were expressed in terms of the parameters c_A and c_B appearing in (12), according to Eq. (11). We note that the model (12) can also be used near the critical point if the correspondence with the model system is guaranteed in terms of the parameter $(\lambda-\lambda_c)K^{-1}$.

4. We demonstrate the applicability of the theory developed to a system of practical interest, and describe the calculations required to determine the model parameters. Consider a chain of parametrically excited nonlinear oscillators [5]:

$$\begin{aligned} \ddot{X}_m + \lambda\dot{X}_m + \left(1 + q\cos\frac{2\pi t}{T}\right)X_m + X_m^3 = \varepsilon\Psi(X_{m-1}, X_m, X_{m+1}, \\ \dot{X}_{m-1}, \dot{X}_m, \dot{X}_{m+1}), \end{aligned} \tag{13}$$

where X_m is the generalized coordinate of the m-th oscillator, λ, q, T are parameters, the function Ψ defines the coupling between oscillators, and ε is the coupling coefficient. We study three methods of introducing the coupling:

$$1)\ \Psi_1 = x_{m-1} - 2X_m + X_{m+1};\ 2)\ \Psi_2 = X_m^2(X_{m-1} - 2X_m + X_{m+1});$$
$$3)\ \Psi_3 = \dot{X}_{m-1} - 2\dot{X}_m + \dot{X}_{m+1}.$$

The spatially homogeneous solution is described by the equation

$$\ddot{X} + \lambda\dot{X} + \left(1 + q\cos\frac{2\pi t}{T}\right)X + X^3 = 0. \tag{14}$$

Small perturbations of this solution of the form $\xi e^{i\beta m}$ for the three methods mentioned of introducing coupling obey at $\beta^2 \ll 1$ the linear equation:

$$\ddot{\xi} + \lambda\dot{\xi} + \left(1 + q\cos\frac{2\pi t}{T}\right)\xi + 3X^2\xi = \begin{cases} -\varepsilon\beta^2\xi \\ -\varepsilon\beta^2 X^2\xi. \\ -\varepsilon\beta^2\dot{\xi} \end{cases} \tag{15}$$

For fixed q, T the solution (14) undergoes period doubling for decreasing λ. Following successive doublings, one can find the critical point λ_c. In particular, for $q = 6$, $2\pi/T = 2.04$ we obtained $\lambda_c = 0.97723$, $K = -0.1505$. To search the period cycle $2^N T$ at $\lambda = \lambda_c$ Eq. (14) is solved numerically on the segment $[0, 2^N T]$ with initial conditions $(X(0), \dot{X}(0))$. The refined position of the initial position at the cycle is determined by the equation

$$X_{\text{ут}}(0) = X(0) + [X(2^N T) - X(0)](1 - \mu_*)^{-1},$$
$$\dot{X}_{\text{ут}}(0) = \dot{X}(0) + [\dot{X}(2^N T) - \dot{X}(0)](1 - \mu_*)^{-1};$$

TABLE 1

	F_1			F_2			F_3		
N	$\varkappa_N$	c_A/ε	c_B/ε	$\varkappa_N$	c_A/ε	c_B/ε	$\varkappa_N$	c_A/ε	c_B/ε
1	3,570			1,982			3,431		
2	−1,931	−0,805	0,778	−2,602	−0,583	0,262	11,29	0,392	2,208
3	18,56	−0,795	0,762	11,21	−0,582	0,261	11,17	0,404	2,189
4	−19,11	−0,796	0,759	−18,80	−0,584	0,257	51,11	0,407	2,195

when by necessity the iteration is repeated. After the cycle has been found, Eqs. (14) and (15) are simultaneously solved on the same time segment, while two linearly independent solutions of (15) are sought with initial conditions $\xi = 1$, $\dot{\xi} = 0$ and $\xi = 0$, $\dot{\xi} = 1$. From the quantities $\xi(2^N T)$ and $\dot{\xi}(2^N T)$, corresponding to these solutions, one constructs a 2×2 matrix, whose eigenvalues are the multipliers. For large N one of the multipliers tends to zero, and the second is represented in the form $\mu_N = \mu_* + \varepsilon\varkappa_N\beta^2$. Table 1 shows the $\varkappa_N$ values calculated for the three methods of introducing the coupling, as well as the coefficients c_A/ε, c_B/ε, found by means of (11) for each pair $\varkappa_N$, $\varkappa_{N+1}$. The approximate constancy of the coefficients in each column implies that the introduced A- and B-type couplings make sense for the chain (13). For a choice of c_A and c_B according to the data of Table 1 the model (12) must correctly describe the oscillator chain near the critical point.*

The analysis provided concerned the dynamics of small perturbations of spatially homogeneous solutions of the problems. Therefore it gives only necessary conditions of correspondence between the model and the system. Numerical experiments, however, show that quantitative correspondence between the dynamics of the chain (13) and the model (12) is also maintained for finite perturbations.

APPENDIX

In the following considerations we use without reservation several properties of the Feigenbaum function g(x), being the solution of the functional equation (8). These properties were established in [2-4]. In particular, in the segment [−1, 1] the function g(x) is bounded and has a bounded derivative. For any $x \in [-1, 1]$ and $i > 0$ it is true that $g^i(x) \in [-1, 1]$. Here and below we use the notation $g^i(x) = \underbrace{g(g(\ldots g}_{\text{i times}}(x)\ldots))$. By $\hat{L}$ we denote the linear operator defined by relationship (9).

<u>LEMMA.</u> Let x, y $\in$ [−1, 1], with k and i being integers, $\kappa \geqslant 0$, $0 < i \leqslant 2^\kappa$. Then

$$\Delta_\kappa^i - |g^i(x/a^\kappa) - g^i(y/a^\kappa)| < \frac{|x-y|}{|a|^{\kappa-1}} \leqslant \frac{2}{|a|^{\kappa-1}}.$$

<u>Proof.</u> For k = 0, i = 1 we have:

$$|g(x) - g(y)| < \max_{|x|\leqslant 1} |g'(x)|\,|x-y| = |g'(1)|\,|x-y| = |a|\,|x-y| \leqslant 2|a|.$$

We assume that the statement of the lemma is valid for some k. We show that its validity then follows for k + 1.

a) Let i be even, $i = 2m$, $0 < m \leqslant 2^\kappa$. Using (8), we have:

$$\Delta_{\kappa+1}^{2m} = \left| g^{2m}\left(\frac{x}{a^{\kappa+1}}\right) - g^{2m}\left(\frac{y}{a^{\kappa+1}}\right)\right| = \left|\frac{1}{a} g^m\left(\frac{x}{a^\kappa}\right) - \frac{1}{a} g^m\left(\frac{y}{a^\kappa}\right)\right| < \frac{|x-y|}{|a|^\kappa} \leqslant \frac{2}{|a|^\kappa}.$$

b) Let i be odd, $i = 2m+1$, $0 \leqslant m < 2^\kappa$. We denote $g^{2m+1}\left(\frac{x}{a^{\kappa+1}}\right) = \xi$, $g^{2m+1}\left(\frac{y}{a^{\kappa+1}}\right) = \eta$, and estimate the difference $\Delta_{\kappa+1}^{2m+1} = |\xi - \eta|$. We note that $\xi = g\left(g^{2m}\left(\frac{x}{a^{\kappa+1}}\right)\right) = g(X)$, where

*We note that according to the results of the calculations c_A and c_B depend on the problem parameters q, T.

$X=\frac{1}{a}g^m\left(\frac{x}{a^\kappa}\right)\in\left[-\frac{1}{|a|}\ \frac{1}{|a|}\right]$, since $\left|g^m\left(\frac{x}{a^\kappa}\right)\right|\leqslant 1$. Then $g\left(\frac{1}{a}\right)\leqslant\xi\leqslant 1$ (Fig. 1), and similarly $g\left(\frac{1}{a}\right)\leqslant\eta\leqslant 1$, $g\left(\frac{1}{a}\right)=0.7589$. According to point (a)

$$|g(\xi)-g(\eta)|=\Delta_{\kappa+1}^{2m+2}<\frac{|x-y|}{|a|^\kappa}\leqslant\frac{2}{|a|^\kappa}. \tag{A.1}$$

On the other hand

$$|g(\xi)-g(\eta)|>\min|g'(\xi)||\xi-\eta|>|\xi-\eta|, \tag{A.2}$$

since $\min|g'(\xi)|=\left|g'\left(g\left(\frac{1}{a}\right)\right)\right|\approx 2.10$. Comparing (A.1) and (A.2), we have:

$$\Delta_{\kappa+1}^{2m+1}=|\xi-\eta|<|g(\xi)-g(\eta)|<\frac{|x-y|}{|a|^\kappa}\leqslant\frac{2}{|a|^\kappa}.$$

By induction the lemma is valid for all k.

We investigate the multiple action of the operator $\hat{L}$ on several classes of functions.

THEOREM 1. If the function f(x) is uniformly bounded on the segment [−1, 1]; $|f(x)|<C$, then $|\hat{L}^k f(x)|<C\cdot 2^k$.

Proof. We write down the result of k-fold action of the operator $\hat{L}$ on the function f(x):

$$\hat{L}^\kappa f(x)=\sum_{i=0}^{2^\kappa-1}f\left(g^i\left(\frac{x}{a^\kappa}\right)\right). \tag{A.3}$$

We estimate the absolute value of the right-hand side of (A.3):

$$\left|\sum_{i=0}^{2^\kappa-1}f\left(g^i\left(\frac{x}{a^\kappa}\right)\right)\right|\leqslant\sum_{i=0}^{2^\kappa-1}\left|f\left(g^i\left(\frac{x}{a^\kappa}\right)\right)\right|<C\cdot 2^\kappa.$$

THEOREM 2. If the function f(x) satisfies the Lipschitz condition $|f(x)-f(y)|<K|x-y|$, where x, y ∈ [−1, 1], and K is a fixed constant, then there exists constants C and P, such that

$$|\hat{L}^\kappa f(x)-C\cdot 2^\kappa|<P\left|\frac{2}{a}\right|^\kappa.$$

Proof. According to (A.3)

$$\hat{L}^{\kappa+1}f(x)=\sum_{i=0}^{2^\kappa-1}\left[f\left(g^i\left(\frac{x}{a^{\kappa+1}}\right)\right)+f\left(g^i\left(\frac{y}{a^{\kappa+1}}\right)\right)\right],$$

where $y=a^\kappa g^{2^\kappa}(x/a^{\kappa+1})$. By means of (8) one can show that $y=g(x/a)\in[-1,1]$. We estimate $\Delta=|\hat{L}^{\kappa+1}f(x_1)-2\hat{L}^\kappa f(x_2)|$, where $x_1, x_2\in[-1,1]$. Denoting $y_{1,2}=g(x_{1,2})$, we have

$$\Delta=\left|\sum_{i=0}^{2^\kappa-1}\left\{\left[f\left(g^i\left(\frac{x_1}{a^{\kappa+1}}\right)\right)-f\left(g^i\left(\frac{x_2}{a^{\kappa+1}}\right)\right)\right]+\left[f\left(g^i\left(\frac{y_1}{a^\kappa}\right)\right)-\right.\right.$$
$$\left.\left.-f\left(g^i\left(\frac{y_2}{a^\kappa}\right)\right)\right]\right\}\Bigg|<K\sum_{i=0}^{2^\kappa-1}\left[\left|g^i\left(\frac{x_1}{a^{\kappa+1}}\right)-g^i\left(\frac{x_2}{a^{\kappa+1}}\right)\right|+\left|g^i\left(\frac{y_1}{a^\kappa}\right)-g^i\left(\frac{y_2}{a^\kappa}\right)\right|\right].$$

Using the lemma:

$$\Delta<K\cdot 2^\kappa(|x_1-x_2||a|^{-\kappa}+|y_1-y_2||a|^{-\kappa-1})<Q\left|\frac{2}{a}\right|^\kappa,$$
$$Q=2K(1+|a|). \tag{A.4}$$

Putting 1) $x_1=x_2=x$, 2) $x_1=x$, $x_2=0$, 3) $x_1=0$, $x_2=0$, we find the following inequalities:

$$|\Phi_{\kappa+1}(x)-\Phi_{\kappa}(x)|<Q|a|^{-\kappa}, \tag{A.5}$$

$$|\Phi_{\kappa+1}(x)-\Phi_{\kappa}(0)|<Q|a|^{-\kappa}, \tag{A.6}$$

$$|\Phi_{\kappa+1}(0)-\Phi_{\kappa}(0)|<Q|a|^{-\kappa}, \tag{A.7}$$

where the notation $\Phi_k(x) = 2^{-k}\hat{L}^k f(x)$ was introduced. It follows from (A.7) that by the Cauchy criterion [11, p. 40] the sequence Φ_k (10) has the limit $\lim_{\kappa\to\infty}\Phi_\kappa(0)=C$. It is seen from (A.6) that the same number C is the limit of the sequence $\Phi_k(x)$ for any $x \in [-1, 1]$.

Let $\Delta_k(x) = \Phi_{k+1}(x) - \Phi_k(x)$, then $\sum_{k=m}^{m+N-1}\Delta_\kappa=\Phi_{m+N}-\Phi_m$. The limit of the right-hand side exists for $N \to \infty$, and consequently $\sum_{\kappa=m}^{\infty}\Delta_\kappa = C-\Phi_m$. According to (A.5) $|\Delta_k(x) < Q|a|^{-k}$, therefore

$$|\Phi_m-C|=\left|\sum_{\kappa=m}^{\infty}\Delta_\kappa\right|\leqslant\sum_{\kappa=m}^{\infty}|\Delta_\kappa|<\sum_{\kappa=m}^{\infty}Q|a|^{-\kappa}=P|a|^{-m},$$

where $P = Q[1 - |a|^{-1}|^{-1}$. Hence follows the assertion of Theorem 2.

<u>THEOREM 3.</u> Let $f(x) = 1/x + \tilde{f}(x)$, where $\tilde{f}(x)$ is a function satisfying the Lipschitz condition $|\tilde{f}(x) - \tilde{f}(y)| < K|x - y|$, $x, y \in [-1, 1]$, where K is a fixed constant. There exist then constants C, P and a function $\psi(x)$, such that for any $x \in [-1, 1]$ the following inequality is valid

$$|\hat{L}^\kappa f(x)-a^\kappa\psi(x)-C\cdot 2^\kappa|<P\left|\frac{2}{a}\right|^\kappa. \tag{A.8}$$

We prove a number of statements.

1. The following inequality holds

$$\left|f_\kappa(x)-\frac{a^\kappa}{x}\right|<M|a|^\kappa, \tag{A.9}$$

where $f_k(x) = \hat{L}^k f(x)$; $x \in [-1, 1]$, and M is a constant.

For k = 0 the statement is valid if we put M > 2K.

We assume that (A.9) is satisfied for some k. We then have for k + 1:

$$\left|f_{\kappa+1}(x)-\frac{a^{\kappa+1}}{x}\right|=\left|f_\kappa\left(\frac{x}{a}\right)+f_\kappa\left(g\left(\frac{x}{a}\right)\right)-\frac{a^{\kappa+1}}{x}\right|\leqslant\left|f_\kappa\left(\frac{x}{a}\right)-\right.$$
$$\left.-\frac{a^\kappa}{(x/a)}\right|+\left|f_\kappa\left(g\left(\frac{x}{a}\right)\right)-\frac{a^\kappa}{g(x/a)}\right|+\left|\frac{a^\kappa}{g(x/a)}\right|\leqslant\left[2M+\frac{1}{g(1/a)}\right]|a|^\kappa,$$

where we used the fact that $\min_{|x|\leqslant 1} g(x/a)=g(1/a)$. We require that the expression in the square brackets be less than $M|a|$. This is guaranteed for $M>\frac{1}{(|a|-2)\,g(1/a)}\approx 2.62$. Then $|f_{\kappa+1}(x) - a^{\kappa+1}/x|<M|a|^{\kappa+1}$. We conclude by induction that inequality (A.9) is true for any k if one chooses M > max [2.63; 2K].

2. The limit $\lim_{\kappa\to\infty}[\hat{L}^\kappa f(x)/a^\kappa]=\psi(x)$ exists.

We introduce the notation $\psi_\kappa(x)=\hat{L}^\kappa f(x)/a^\kappa=f_\kappa(x)/a^\kappa$, and consider the absolute value of the difference

$$|\psi_{\kappa+1}(x)-\psi_\kappa(x)|=|a|^{-\kappa-1}|f_{\kappa+1}(x)-af_\kappa(x)|=|a|^{-\kappa}|\hat{L}^\kappa[f_1(x)-af(x)]|.$$

The function $f_1(x) - af(x)$ is uniformly bounded:

$$|f_1(x)-af(x)|=|f(x/a)+f(g(x/a))-af(x)|=$$
$$=|\tilde{f}(x/a)+a/g(x/a)+\tilde{f}(g(x/a))-a\tilde{f}(x)|<(2+|a|)K+|a/g(1/a)|=D.$$

Therefore by Theorem 1 $|f_{\kappa+1}(x)-af_\kappa(x)|<D\cdot 2^\kappa$ and $|\psi_{\kappa+1}(x)-\psi_\kappa(x)|<\frac{D}{|a|}\left|\frac{2}{a}\right|^\kappa$. The right-hand side of the latter expression tends to zero for $k\to\infty$. By the criterion of uniform convergence of a sequence of functions follows the existence of the limit $\lim\limits_{\kappa\to\infty}\psi_\kappa(x)=\psi(x)$, where the function $\psi(x)$ is the eigenvalue of the operator $\hat{L}$ with eigenvalue a. Indeed, for large k $\psi_{k+1}(x)\approx\psi_k(x)\approx\psi(x)$, i.e., $\hat{L}\psi=a\psi$.

Putting $f(x) = 1/x$, we obtain:

$$\psi(x)=\lim_{\kappa\to\infty}\frac{\hat{L}^\kappa f(x)}{a^\kappa}=\lim_{\kappa\to\infty}\left\{\frac{1}{x}+\sum_{i=1}^{2^\kappa-1}[a^\kappa g^i(x/a^\kappa)]^{-1}\right\}.$$

Choosing large, but finite, k, one can use this representation of the function $\psi(x)$ for its evaluation.

3. The function $\tilde{\psi}(x) = \psi(x) - 1/x$ satisfies the Lipschitz condition

$$|\tilde{\psi}(x)-\tilde{\psi}(y)|<S|x-y|, \tag{A.10}$$

where $x, y \in [-1, 1]$, and S is an appropriately chosen constant.

If we show that an inequality of type (A.10) is satisfied for the function $\psi_k(x) = f_k(x)/a^k$ for any k, then it is also valid for the function $\psi(x)$. For k = 0 the inequality holds by the conditions of Theorem 3. We assume it is valid for some k. We then have for k + 1:

$$|\tilde{\psi}_{\kappa+1}(x)-\tilde{\psi}_{\kappa+1}(y)|=\left|\left[\frac{f_\kappa(x/a)-f_\kappa(y/a)}{a^{\kappa+1}}-\frac{1}{x}+\frac{1}{y}\right]+\right.$$
$$+\frac{1}{a}\left[\frac{f_\kappa(g(x/a))-f_\kappa(g(y/a))}{a^\kappa}-\frac{1}{g(x/a)}+\frac{1}{g(y/a)}\right]+$$
$$\left.+\frac{1}{ag(x/a)}-\frac{1}{ag(y/a)}\right|<S\left|\frac{x-y}{a}\right|+\frac{S}{|a|}\left|g\left(\frac{x}{a}\right)-\right.$$
$$\left.-g\left(\frac{y}{a}\right)\right|+\frac{1}{|a|}\left|\frac{1}{g(x/a)}-\frac{1}{g(y/a)}\right|.$$

Using the inequality

$$|g(x/a)-g(y/a)|\leqslant\max_{|x|\leqslant 1}\left|g'\left(\frac{x}{a}\right)\right|\left|\frac{x}{a}-\frac{y}{a}\right|=\left|\frac{x}{a}-\frac{y}{a}\right|\left|g'\left(\frac{1}{a}\right)\right|,$$
$$|1/g(x/a)-1/g(y/a)|\leqslant|g'(1/a)|\left|\frac{x}{a}-\frac{y}{a}\right|\left[\min_{|x|\leqslant 1}\left|g\left(\frac{x}{a}\right)\right|\right]^{-2}=$$
$$=|g'(1/a)|\left|\frac{x}{a}-\frac{y}{a}\right|[g(1/a)]^{-2},$$

we obtain

$$|\tilde{\psi}_{\kappa+1}(x)-\tilde{\psi}_{\kappa+1}(y)|\leqslant\left[S\left(\frac{1}{|a|}+\frac{|g'(1/a)|}{a^2}\right)+\left|\frac{g'(1/a)}{a(g'(1/a))^2}\right|\right]|x-y|.$$

The right-hand side is less than $S|x - y|$, if one puts

$$S > |g'(1/a)||a|(g(1/a))^2(1 - 1/|a| - |g'(1/a)|/|a|^2)^{-1} \approx 2.0184.$$

Consequently, (A.10) is satisfied if $S > \max[K; 2,1]$.

4. Using the results obtained, the validify of Theorem 3 is easily verified. Obviously, the function $F(x) = f(x) - \psi(x)$

$$|F(x) - F(y)| \leqslant |\tilde{f}(x) - \tilde{f}(y)| + |\tilde{\psi}(x) - \tilde{\psi}(y)| = (K + S)|x - y|.$$

According to Theorem 2, there exist C and P such that

$$|\hat{L}^{\kappa}(f(x) - \psi(x)) - C \cdot 2^{\kappa}| < \left|\frac{2}{a}\right|^{\kappa} \cdot P.$$

As was shown, $\hat{L}\psi = a\psi$. Therefore, hence follows (A.8).

The author is grateful to D. I. Trubetskov for a useful discussion, and to A. P. Kuznetsov and A. G. Rozhnev for their help.

LITERATURE CITED

1. Yu. I. Neimark, The Method of Point Transformations in the Theory of Nonlinear Oscillations [in Russian], Nauka, Moscow (1972).
2. M. J. Feigenbaum, J. Stat. Phys., 19, 25 (1978).
3. M. J. Feigenbaum, J. Stat. Phys., 21, 669 (1979).
4. M. J. Feigenbaum, Commun. Math. Phys., 77, 65 (1980).
5. F. M. Izrailov, M. I. Rabinovich, and A. D. Ugodnikov, Phys. Lett., A86, 321 (1981).
6. J. Testa, J. Perez, and C. Jeffries, Phys. Rev. Lett., 48, 74 (1982).
7. Strange Attractors [Russian translation], Mir, Moscow (1981).
8. J. D. Eckman, Rev. Mod. Phys., 53, 643 (1981).
9. T. Kai, Phys. Lett., A6, 263 (1981).
10. S. P. Kuznetsov, Izv. Vyssh. Uchebn. Zaved., Radiofizika, 25, 1364 (1982).
11. L. D. Kudryavtsev, Mathematical Analysis [in Russian], Vysshaya Shkola, Moscow (1973).