Renormalization Chaos in Period Doubling Systems

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The present paper gives an account of a recent conception of renormalization chaos in application to some scaling properties of period doubling systems (global structure of the Feigenbaum attractor, ratios of spectral amplitudes, response under periodical perturbation).

1. INTRODUCTION. RENORMALIZATION DYNAMICS AND RENORMALIZATION CHAOS

Renormalization group approach is one of the powerful tools of modern theoretical physics for analyzing systems with coexisting patterns with wide interval of space and time scales. This approach arised in the quantum theory of field and was developed and applied later in the theory of phase transitions and critical phenomena /1/. Recently, beginning from FEIGENBAUM's works /2-4/, the renormalization group approach is actively used for investigating the behaviour of nonlinear dynamical systems near the onset of chaos.

What does the renormalization group analysis of problems of nonlinear dynamics consist in? The first step is usually a discretization of time. Then the dynamics is described by some evolution operator f_0 connecting the consequent states of the system: $x_{n+1} = f\left[x_n\right]$. The most important is the next step. This is a transition to some greater interval of time discretization accompanied by suitable rescaling or, more general, by some change of dynamical variables. At this stage the new evolution operator is introduced which evidently can be represented through the old one: $f_1 = R\left[f_0\right]$, where R is the operator of renormalization.

One can repeat the above procedure many times and so a sequence of evolution operators f_2, f_3,... may be defined. The whole construction is similar to the known picture of Kadanoff's blocks in phase transition theory /1/. It leads to rapid results in such situations, when the system is characterized by similar or approximately similar behaviour on different time scales including very large time scales. This is just the situation which has taken place at the onset of chaos through period doubling bifurcations or through quasiperiodical regimes.

The equation which expresses the evolution operator through the previous one, is the renormalization group equation. One can say that this equation defines some kind of dynamics in space of operators called the renormalization dynamics. The number of steps of renormalization may be determined as renormalization time.

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The traditional type of renormalization dynamics is a situation of a nonstable fixed point in operator space. This is just the case considered earlier in statistical mechanics and in the first investigations on renormalization group analysis of dynamical systems. However, the situations may exist in which the renormalization group equation leads to more complicated dynamics, such as periodic or chaotic dependence of evolution operators f_n on the renormalization time n. As the author knows, such a possibility was discussed first by OSTLUND ed al /5/ for a problem of transition from quasiperiodicity to chaos and independently by CHIRIKOV and SHEPELYANSKY /6,7/ for the analogous problem in Hamiltonian systems. They also introduced the terminology used here.

What are the structural peculiarities of the phase space and the parameter space of dynamical systems which are connected with each type of renormalization dynamics? The transition to larger time scales during the described renormalization is associated with the consideration of more and more small objects in the phase space and the parameter space. So the nontrivial renormalization dynamics corresponds to structures similar to the Russian toy "matreshka". Such a toy consists of a sequence of hollow wooden puppets, the smaller one put inside the larger one. Three different types of renormalization dynamics are illustrated in Fig.1. A situation, when all figures are similar to each other (scaling), is associated with a fixed point (Fig.1a). The case of repeating through defined number of steps M corresponds to a renormalization cycle of period M (Fig.1b). At last, in renormalization chaos there is no full repeating excluding the repeating in approximate, mean or statistical sense (Fig.1c).

It is clear that the connection must exist between characteristics of renormalization dynamics and scaling properties of structures of the "matreshka"-type in the phase space and the parameter space. For the case of a fixed point this connection is well known. Each nonstable direction of the fixed point is associated

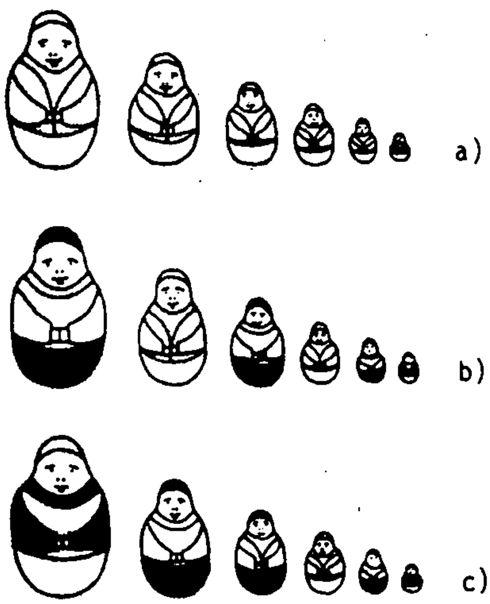


Figure 1. Russian toy "matreshka" as an illustration of three different types of renormalization dynamics.

with a relevant parameter of the system. The factor of scaling connected with this parameter is defined by the eigenvalue of linearized renormalization transformation. The case of renormalization cycle of some period M can be reduced to a case of fixed point by introducing a new renormalization group transformation consisting of M steps of the initial one. For renormalization chaos, however, a further investigation of a connection between its quantitative characteristics and scaling properties is needed.

For understanding the generalization of renormalization chaos it is desirable, of course, to have as many examples of it as possible. The situations of transition from quasiperiodicity to chaos /5-7/ are sufficiently complicated. Simpler examples can be presented by periodic-doubling systems and will be considered in this paper. Section 2 is devoted to the discussion of global scaling properties of the Feigenbaum attractor from the viewpoint of renormalization chaos. The approximate approach developed here casts a new light on some known results and is associated in some aspects with thermodynamical formalism /5,7/. In Sect.3 the conception of renormalization chaos is applied to problems of periodical perturbation and spectral properties of period doubling systems. This material is based particularly on our works with PIKOVSKY.

2. GLOBAL SCALING PROPERTIES OF THE FEIGENBAUM ATTRACTOR

2.1. The Generalized Renormalization Scheme

Let us consider the well-known one-dimensional map

$$x' = 1 - Ax^2 \qquad (1)$$

and associated with it the geometrical image of the Feigenbaum tree or a graphical representation of x versus A (Fig.2). Each road up to the tree may be coded by a binary sequence. At each branching we use the following rule: the branch having the greatest deflections from the parent one is denoted by symbol 1 and the other by symbol 0. One can introduce also a code of moving down from the tree obtained by reading the previous code in backward order. For cycles of finite period 2^n each element has its own code consisting of n symbols. It is a remarkable obser-

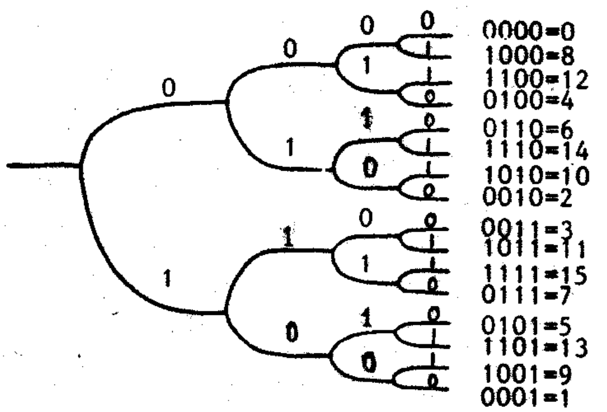


Figure 2. The Feigenbaum tree and a rule of coding of roads in it.

vation that the code of moving down from the tree correspondent to any element of the cycle is at the same time the binary representation of the number of this element in their natural time order. For the limit object formed by the infinite sequence of period doublings (the Feigenbaum attractor), the elements are coded by infinite binary sequences.

Let us remember the traditional Feigenbaum's renormalization group analysis. Performing twice an initial map f(x), one obtains a map f(f(x)) having a characteristic form with two humps (Fig.3a). For further details the consideration, the central extremum of this map is selected. After rescaling of x by factor $a < 0$, the new function is obtained: $f_1(x) = af(f(x/a))$. The described procedure is repeated many times. Exactly in the critical point of period doubling accumulation, the procedure leads to a fixed point of renormalization group equation, i.e. to the universal function denoted by g(x). This function satisfies functional equation $g(x) = ag(g(x/a))$ solved by FEIGENBAUM.

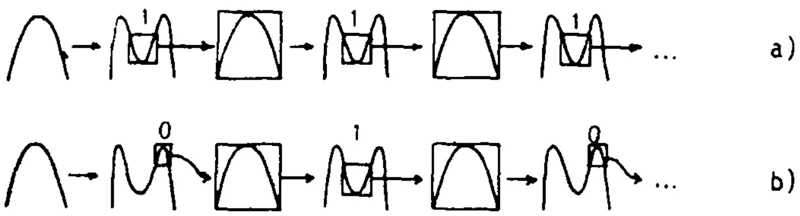


Figure 3. An illustration of the traditional Feigenbaum's renormalization transformation (a), and a generalized renormalization scheme (b).

It can be mentioned, however, that on each step of renormalization there are two possibilities: both the central and the right extremums of function f(f(x)) may be selected for further consideration. (The left extremum is not suitable because its vicinity is not visited by a system during its dynamics on attractor.) For the second variant the coordinate change must contain a shift of origin to a point of a new extremum, not only rescaling. We shall mark the first variant of transformation by symbol 1 and the second by symbol 0. Each binary sequence gives rise to its own renormalization scheme. Using such a construction we can achieve a hit of any element of 2^n cycle into a vicinity of the selected for detail consideration region of x in n-th time renormalized function. For this purpose the code of renormalization scheme must coincide with the code of the road up to the tree to the the selected element.

Thus, with taking into account the whole set of possible binary sequences, one receives information about global scaling properties of the Feigenbaum tree while the traditional approach describes only a single branch coded by 11111... . This is an example of the most simple renormalization dynamics of a fixed point or a cycle of period 1. Other periodic codes give rise to more complicated renormalization cycles. For instance, a code 01010... corresponds to a cycle of period 2, code 011011... to a cycle of period 3 and so on. The stochastic binary sequences are associated with renormalization chaos.

2.2. The Approximate Description of Renormalization Dynamics

Let us concentrate out attention on the consideration of global scaling properties of the Feigenbaum attractor. For this purpose we shall consider here only the critical value of parameter in Eq.(1), A = 1,401155... . Then the shape of function $f^{2^n}(x)$ near its central extremum is described by the Feigenbaum function $g(x)$ (for sufficiently large n). The key moment for further consideration is the fact that the shapes of this function near the other extremums are obtained from the central one by changes of the variable.

After the transition of origin to a point of extremum and rescaling it, provided the function at this point equals to 1, the change of variable must be close to identity. One can put

$$x \to (1-u)x + ux^2 \ , \tag{2}$$

where u is a parameter dependent on selected extremum. This parameter is proposed to be small and to be taken into account only in the first order. From the point of view of renormalization dynamics, u is a dynamical variable.

Performing the change (2) in the map $x' = g(x)$, one obtains

$$x' = g_u(x) = g(x) + u\left(g'(x)x^2-(g(x))^2\right) - u\left(g'(x)x-g(x)\right) \ . \tag{3}$$

Let us undertake a step of renormalization transformation for the map (3). We perform it twice and transform the resulting map by an appropriate change of variables to the primary form with a new value of u. This is accomplished by two different ways: we can select the vicinity of the central (code 1) or side (code 0) extremum for consideration. The result is the recurrent equation

$$u' = \begin{cases} u/a & \text{(code 1)} \\ p + su & \text{(code 0)} \end{cases} \tag{4}$$

and the rule of rescaling of the variable:

$$\Delta x' = \begin{cases} a(1-u+u/a)\Delta x & \text{(code 1)} \\ (1+uc)(1-c)^{-1}\Delta x & \text{(code 0)} \end{cases} \tag{5}$$

where a = -2.5029, p = 0.659, s = 0.155, c = 0.8323. Constants in Eqs.(4), (5) are defined through the function g(x) and its derivatives. The known polynomial approximation of g found in /2/ was used for calculations.

Taking an arbitrary code of the road up to the tree consisting of n symbols, we can find the representation of function $f^{2^n}(x)$ in the normalized form in the vicinity of the correspondent extremum. Starting from $u_0 = 0$ (for the initial map

(1)), we must iterate Eq.(4) selecting one or another formula depending on symbols of code sequence. The shape of the function is given by (3) where parameter u is a result of iterations. The ratio of scales of initial and renormalized variables is obtained by the product of coefficients in (5).

Thus, in our approximation the renormalization dynamics is described by one-dimensional map plotted in Fig.4. The points showing results of computations are in good agreement with (4). Let us briefly explain the idea of the computation. All interesting extrema of function $f^{2^n}(x)$ may be numbered by m such as the variable change $x' = f^m(x)$ transforms this extremum into a central one. Correspondent codes are determined by binary representation of m reading from the right to the left. Performing the computations for n and n+1, we find two values u' for each u. One can see from Fig.4 that u is always smaller than 0.1; so the above proposition is justified.

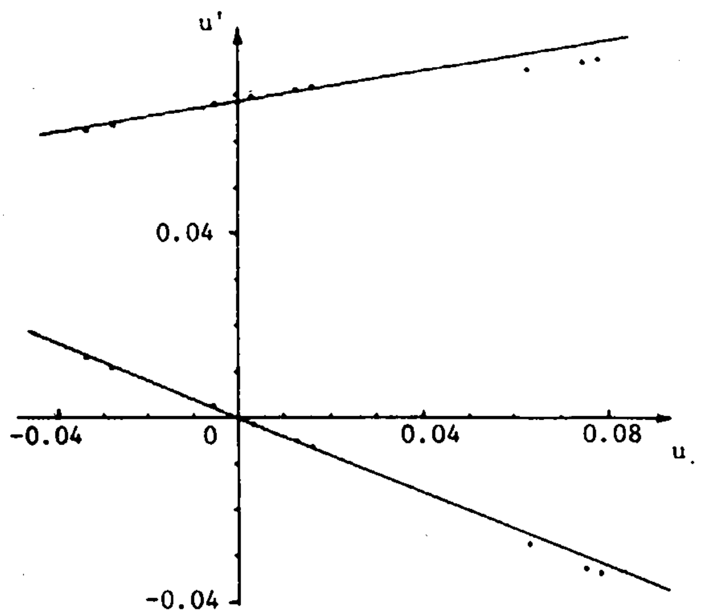


Figure 4. The one-dimensional map describing approximately the renormalization dynamics of parameter u. The exact numerical data are shown by points.

It is clear from (4) and Fig.4 that initial perturbations of u decrease under successive iterations. In other words, a value of u and so a form of $g_u(x)$ near some extremum are determined by the last symbols of the code. This is just a property of universality in a renormalization chaos. The quantitative characteristic of it is a negative Lyapunov exponent of the map (4). For the random code it is equal to $(1/2)\ln|(s/a)| = -1.39$.

2.3. The Feigenbaum Scaling Function σ

FEIGENBAUM /3/ has proposed a function $\sigma(t)$ for the description of global scaling properties of the attractor in the critical point of onset of chaos. It is determined through the elements of the large period cycles existing in a critical point A_0:

$$\sigma(t) = \lim_{n,N\to\infty} \sigma\left(\frac{m}{2n}\right) = \lim \frac{x_m^{(2N)} - x_{m+N}^{(2N)}}{x_m^{(N)} - x_{m+N/2}^{(N)}} \quad , \quad N = 2^n \; , \tag{6}$$

where the superscripts denote the period of the cycle and subscripts the number of an element in time sequence. Let us show how to calculate this function from our renormalization dynamics.

Having some code moving down from the Feigenbaum tree a,b,c,... and reading it from the right to the left, we find from Eq.(4) the corresponding value of u. It makes sense for infinite codes as well as for finite ones because the main contribution is brought by several first symbols of the code a,b,c,... . Further we can consider the map (3) for this u and determine the distance between two elements of its cycle of period 2:

$$\Delta x(u) = 1.146(1+0.293u) \; . \tag{7}$$

Further, one can add symbol 1 or 0 to the code and find a next value of u' from Eq.(4). Then, substituting (7) into (6), with taking into account (5), we obtain the values of (t) for t = 0.01abc... and t = 0.00abc... :

$$\sigma(0.01abc\ldots) = (1+u-u/a)\frac{\Delta x(u')}{a\Delta x(u)} = \frac{1+0.989u}{a} \; , \tag{8a}$$

$$\sigma(0.00abc\ldots) = (1-c)(1-uc)\frac{\Delta x(u')}{\Delta x(u)} = 0.171(1-1.080u) \; . \tag{8b}$$

The results of calculation of function σ are compared with Feigenbaum's numerical data /3/ in Fig.5. The agreement is very good. Most important is the correct description of the fine (fractal) structure of the function. So the renormalization dynamics given by Eq.(4) is just the mechanism that provides this fine structure.

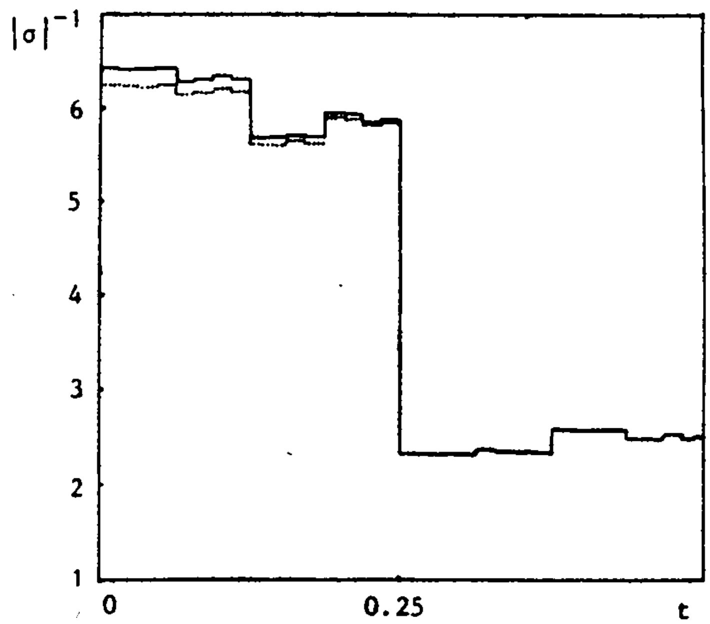


Figure 5. Feigenbaum's function σ. The dotted line corresponds to exact numerical data and the solid one to our approximate solution.

The most essential divergence takes place in the left part of the picture because this is just the region where u is maximal while the theory is constructed in the first order of accuracy in u.

2.4. Generalized Dimensions and Spectrum of Scaling Indices

The computations listed in the title of this section have recently been introduced in /10,11/ and are used for the description of multifractals or complicated sets which are imagined as interwoven variety of fractal (scaling invariant) sets. The Feigenbaum attractor is one of popular examples of multifractals. Spectra of generalized dimensions and scaling indices for this object are calculated in /11/. These characteristics as well as function σ give some form of a description of global scaling properties of the Feigenbaum attractor but they are presented by smooth functions.

Let us remember the algorithm of calculations proposed earlier /11/. Firstly, the sequence is determined x_0, x_1, x_2, ... beginning from the point of extremum $x_0 = 0$ and consisting of iterations of this point by the map (1). One can consider the Feigenbaum attractor as a limit object of construction shown in Fig.6. It is similar to the construction of the Cantor set but the rule of decreasing of interval length is more complicated. At the n-th level of resolution we have 2^n intervals of different length l_i with equal probability of visiting $p_i = 2^{-n}$. The sum of values $p_i{}^q/l_i{}^\tau$ is considered depending on two parameters q and τ. Then the condition is adopted that this sum is equal to 1 and so an equation connecting q and τ is obtained in the limit of large n:

$$q = \lim \frac{1}{n} \log_2 \sum_{i=1}^{2^n} |x_i - x_{i+2^n}|^{-\tau} \quad . \tag{9}$$

Values D_q, f and α are determined through q and τ:

$$D_q = \frac{\tau}{q-1} , \quad \alpha = \frac{d\tau}{dq} , \quad f = \alpha q - \tau \quad . \tag{10}$$

The dependence D vs. q is determined as a spectrum of generalized dimensions, and

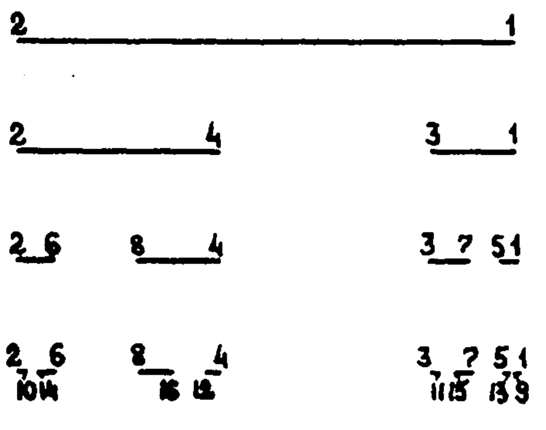


Figure 6. The construction of the Feigenbaum attractor.

f vs. α is determined as a spectrum of scaling indices of $f(\alpha)$-spectrum. Below we show how to obtain these spectra through our renormalization dynamics technique.

Let us consider the n-th level of resolution of attractor structure. It can be proved that the lengths of intervals l_i are

$$l_i = |f^{2^n}(\bar{x}_{i-1}) - f^{2^n}(f^{2^n}(\bar{x}_{i-1}))| \ , \tag{11}$$

where $\bar{x}_{i-1}$ is the point of extremum of a function f^{2^n}, for which the code of moving down from the Feigenbaum tree is just the binary representation of i-1. By constructing two sums

$$Z_n = \sum_{i=1}^{2^n} l_i^{-\tau} \text{ and } S_n = \sum_{i=1}^{2^n} u_i l_i^{-\tau} \ , \tag{12}$$

we can put a question: how do these sums change at the next level of resolution, i.e. for n+1? Instead of each interval l_i we have now two intervals with lengths

$$l_i\beta(1+\mu u_i) \ , \quad \beta = |a|^{-1} \ , \quad \mu = 1-a^{-2} \tag{13a}$$

for an added symbol 1 in the code and

$$l_i\gamma(1+\nu u_i) \ , \quad \gamma = (1-p/a)(1-c) \ , \quad \nu = -c + (1-s)/a \tag{13b}$$

for an added symbol 0. These formulae are obtained using the representation of a function $f^{2^n}(x)$ near the correspondent extremum by Eq.(3) and scaling rules (5) in the first order in u. Introducing (13) into (12) and neglecting the terms of the second power of u, we obtain the following matrix equation:

$$\begin{pmatrix} Z \\ S \end{pmatrix}_{n+1} = \begin{pmatrix} \beta^{-\tau}+\gamma^{-\tau} & -\tau(\mu\beta^{-\tau}+\nu\gamma^{-\tau}) \\ p\gamma^{-\tau} & (s-p\nu\tau)\gamma^{-\tau}+\beta^{-\tau}/a \end{pmatrix} \begin{pmatrix} Z \\ S \end{pmatrix}_n \tag{14}$$

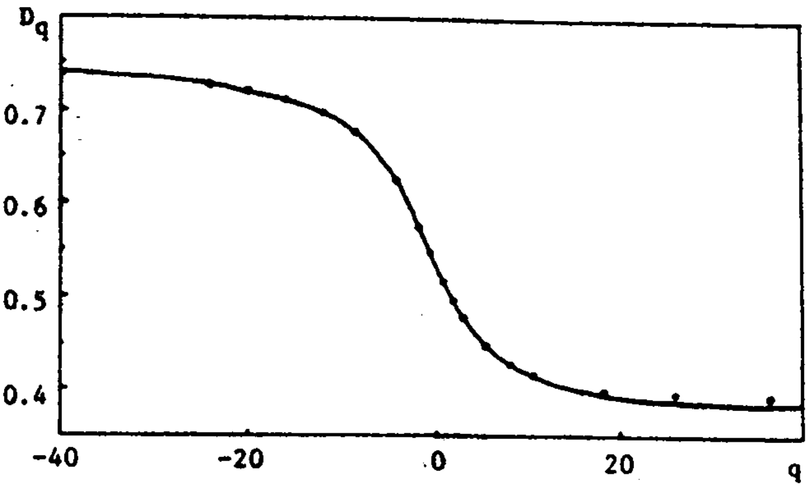


Figure 7. The spectrum of generalized dimensions D_q. The solid line corresponds to the exact numerical solution /11/ and dots to our approximate approach.

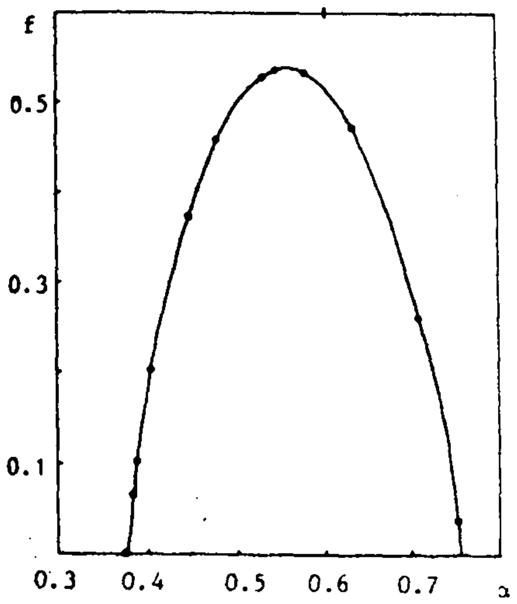


Figure 8. The spectrum of scaling indices. The solid line corresponds to the exact numerical solution /11/ and dots to our approximate approach.

For large n the sums Z_n and S_n will change proportionally to $\lambda(\tau)^n$, where λ is the larger eigenvalue of the matrix. So it follows from (9) that $q(\tau) = \log_2\lambda(\tau)$. Values D_q, α and f are then obtained from Eqs.(10).

Figures 7 and 8 show the spectra of generalized dimensions and scaling indices of the Feigenbaum attractor. The solid lines correspond to calculations by method described in /11/ and points - to our approximate solution. Some special values of the generalized dimensions are compared in Table 1. The good agreement of data in Figs.7 and 8 as well as in Table 1 means that the considered approximation for renormalization dynamics correctly reflects the principle of constructing the Feigenbaum attractor as a multifractal set.

Table 1

Dimension	Exact numerical /11/	Our approximation
Fractal D_0	0.53804	0.5375
Information D_1	0.51710	0.5161
Correlation D_2	0.49780	0.4964
Minimal D_∞	0.37776	0.3727
Maximal $D_{-\infty}$	0.75551	0.7555

3. FURTHER EXAMPLES OF RENORMALIZATION CHAOS

For a wider understanding of the conception of renormalization chaos we shall briefly consider its manifestations in two other problems also connected with period doubling systems.

3.1. Periodical External Perturbations of the System

Let us introduce an additional term into Eq.(1) corresponding to external periodical perturbation

$$x' = 1 - Ax^2 + b \cos 2\pi nw \quad , \tag{15}$$

where b is an amplitude, w is a frequency of the perturbation and n is the discrete time.

The empirical computer investigation of scaling properties of model (15) was undertaken in /13/. In /14/ the authors developed the renormalization group approach to the problem. An attempt of renormalization analysis was also undertaken by ARNEODO /15/. However, he did not take into account the principal moment of a change of the frequency parameter under renormalization which just leads to renormalization chaos. According to /14/, the map describing a system evolution through 2^m units of time has for small amplitudes of perturbation the following form

$$x' = f_m(x) + \frac{1}{2}b\ \phi_m(x)\exp(2\pi i n w_m) + \text{c.c.} \quad , \tag{16}$$

where $f_m(x)$ characterizes an evolution without external perturbation while the renormalization dynamics of function $\phi(x)$ and parameter w is determined by equations

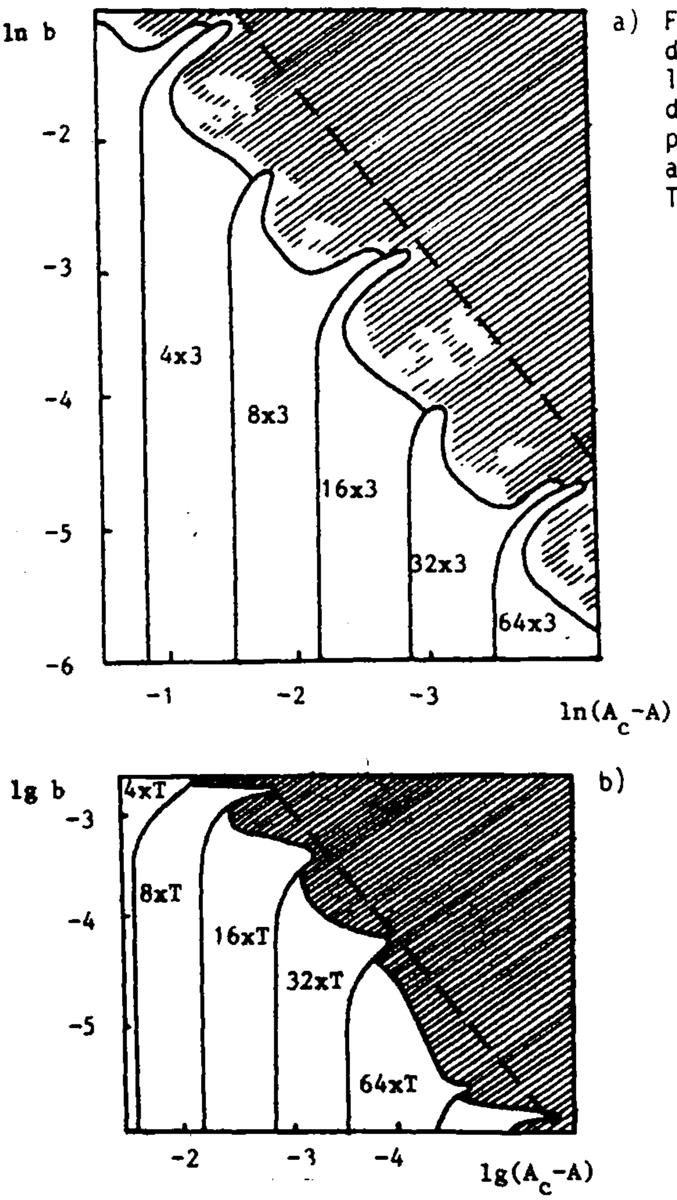
$$\phi_{m+1}(x) = a\ g'(g(\tfrac{x}{a}))\phi_m(\tfrac{x}{a}) + \phi_m(g(\tfrac{x}{a}))\exp(2\pi i w_m) \quad , \tag{17a}$$

$$w_{m+1} = 2w_m \ , \quad \text{mod } 1 \ . \tag{17b}$$

The dynamics described by Eq.(17b) may be periodical (for frequencies w represented by periodical binary fractions) and chaotical (for typical irrational w). Consequently, the scaling properties of dynamical regimes on (A,b)-plane are also determined by binary representation of w.

The first example of a picture of dynamical regimes on (A,b) -plane is shown in Fig.9a. It corresponds to a case of rational $w = 1/3 = 0.010101...$ and represents a renormalization cycle of period 2. White regions denote periodical regimes while shaded ones are domains of chaos. A form of each second region repeats periodically the previous form. Constants of scaling for A and b are correspondently 21.8 (the Feigenbaum's constant δ squared) and 58.96 which is found in /14/ by numerical solution of (17) as an eigenvalue of renormalization transformation (17b) through the period of renormalization cycle for $w = 1/3$. For other rational w coded by period periodical binary fractions the repeating of forms takes place through each p steps where p is a period of binary fraction.

For typical irrational w the renormalization chaos is realized and there is no exact repeating of forms (see Fig.9b taken from /16/). One can, however, understand a special form of scaling in a statistical sense. It is determined by the mean scaling index $\langle \ln|\phi_{m+1}/\phi_m| \rangle \approx 1.83$. The approximate repetition of forms must be observed under condition



a) Figure 9. The domains of different dynamics in (A,b)-plane in the logarithmical scale for period-doubling system under external periodical perturbation for: a) $w = 1/3$ and b) $w = (5^{1/2}-1)/2$. The second one is taken from /16/.

$$b \sim (\Delta A)^{-\chi} , \quad \chi \simeq 1.83/\ln \delta \simeq 1.19 , \tag{18}$$

giving in the logarithmic scale a straight dotted line in Fig.9b. One can see that the border between quasiperiodical and chaotical regimes actually lies along this line. Using Eq.(18) one can evaluate a number of "torus doubling bifurcations" /16, 17/ observed in a system when A is increasing with constant b ($n \sim -\frac{1}{\chi} \log_\delta b$).

3.2. Spectrum of the Feigenbaum System and Renormalization Chaos

It was noted by PIKOVSKY that a similar to (18) renormalization group equation may may be used for the analysis of scaling laws in spectrum generated by a period doubling system. For this aim the next system of two maps may be considered

$$x' = f(x) , \quad C' = C + \phi(x)\exp(2\pi i n w) , \tag{19}$$

where the initial function $\phi(x) = 1$. It is clear that for sufficiently large n the

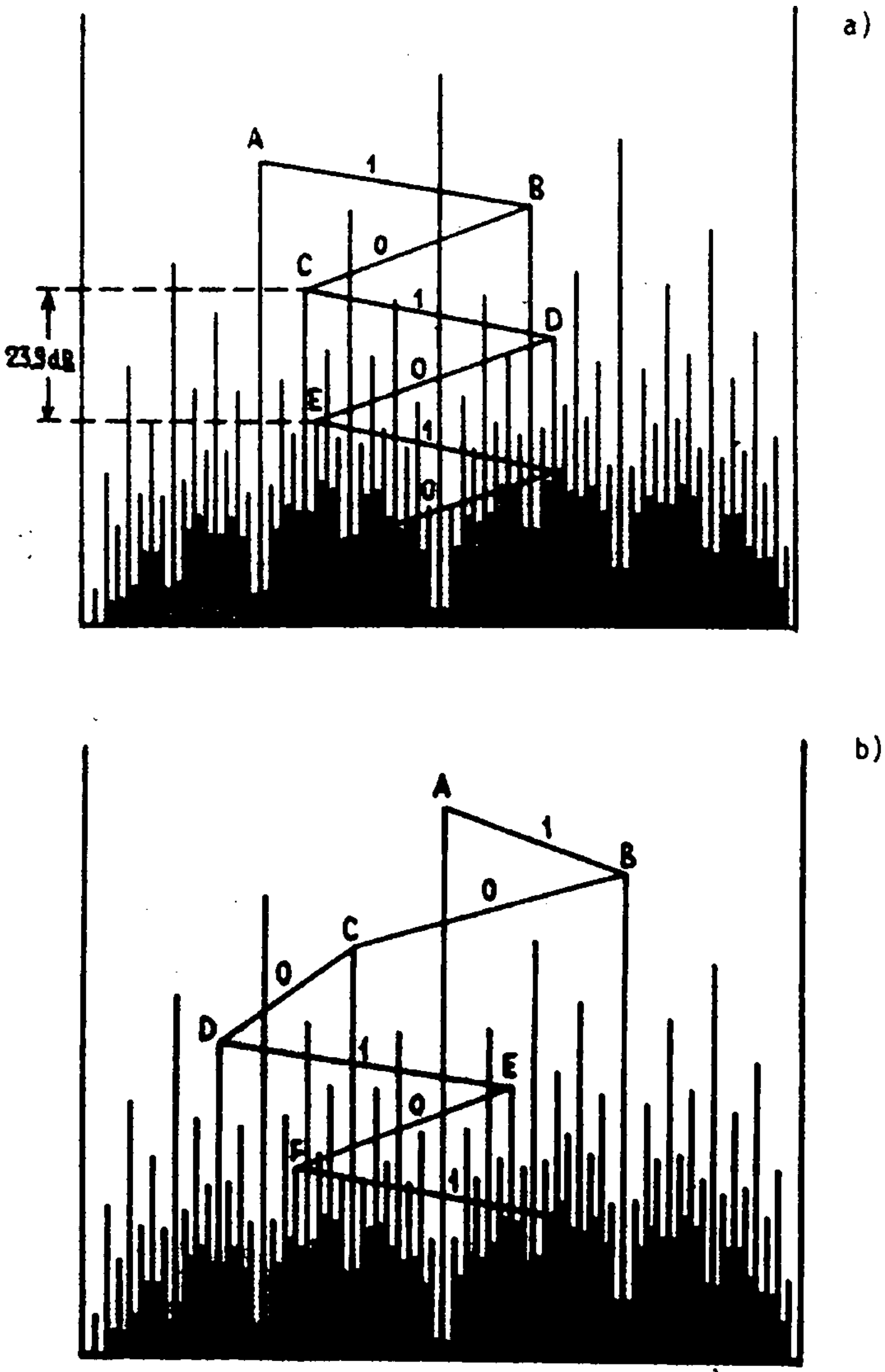


Figure 10. The renormalization cycle (a) and renormalization chaos (b) in spectral amplitudes of period-doubling system obtained for the sequences of binary shifts of the frequencies w = 0.0101010 and w = 0.1010011, correspondingly.

value C_n will be proportional to the amplitude of the spectral component with frequency w. Performing the Feigenbaum's doubling procedure to Eqs.(19) many times, we obtain the following recurrent renormalization group equations

$$\phi_{m+1}(x) = \frac{1}{2}\,\phi_m(x/a) + \phi_m(g(x/a))\exp(2\pi i w_m) \quad , \qquad (20a)$$

$$w_{m+1} = 2w_m \; , \quad \mathrm{mod}\ 1 \; . \qquad (20b)$$

Analyzing these equations allows us to make conclusions about ratios of amplitudes of spectral components. These ratios are found to be determined by the structure of w as a binary fraction.

Let us take any binary fraction w = 0.abcd...e, and consider a sequence of w's obtained by binary shifts: w_1 = 0.bcd...e, w_2 = 0.cd...e etc. Then we proclaim that the ratios of spectral amplitudes over frequencies of this sequence change periodically in a case of periodical structure of code abcd...e and chaotically for random structure of it. Figure 10 illustrates both situations. For the renormalization cycle of period 2 (Fig.10a) the amplitudes of spectral components decrease by 0.0638 times or -23.9 dB per period of the cycle. The same value was obtained in /24/ by numerical solution of (20) for w = 1/3 as an eigenvalue of renormalization transformation through the whole cycle. For random codes (as in Fig.10b), one can determine a mean scaling factor. According to calculations of /14/ it is equal to

$$\langle 20 \lg|\phi_{m+1}/\phi_m|\rangle \approx -13.9 \text{ dB} .$$

group. Illustrations of chaotical renormalization dynamics in this paper are simpler than those considered earlier in /6-7/ and so may be useful for a better understanding of this subject as well as of the approximate approach to the description of renormalization dynamics developed in the present paper.

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