A RENORMALIZATION-GROUP ANALYSIS FOR TWO UNIDIRECTIONALLY COUPLED FEIGENBAUM SYSTEMS AT THE HYPERCHAOS THRESHOLD

S. P. Kuznetsov

UDC 534.015;537.86;519

The behavior of two unidirectionally coupled systems is considered, which show period doubling near the threshold for hyperchaos. The renormalization group equation is derived and solved. Universal functions have been found to describe the behavior near the critical point. The scale constants have the Feigenbaum values −2.5029 and 4.6692 in the first subsystem, while in the second they have the new universal values −1.50532 and 2.39274, which characterize the splittings in the cycle elements and the convergence rate for the doubling bifurcations.

1. Concepts such as renormalization group, universality, and scaling have led to considerable progress in elucidating the transition tochaos in a nonlinear system via period-doubling bifurcations, switching, and quasiperiodic modes [1, 2]. One naturally examines whether a similar approach and analogous trends can apply in more complicated situations, namely at the hyperchaos threshold, which is a compound dynamic state characterized by two positive Lyapunov parameters [3]. Here I consider a simple example of hyperchaos in a system composed of two subsystems capable of showing Feigenbaum period doubling in the presence of a unidirectional link between them (the first subsystem affects the second but the second does not influence the first). There are various types of system in which the transition to chaos occurs by period doubling: a nonlinear dissipative oscillator with external input, certain hydrodynamic systems, electronic oscillators, etc. [4-6]. The unidirectional coupling can occur between any two systems of that type and has attracted attention in particular in connection with turbulence developing downstream [7, 8].

The model system is taken as the coupled mappings

$$x_{n+1} = g_0(x_n), \quad y_{n+1} = f_0(x_n, y_n), \tag{1}$$

where

$$g_0(x) = 1-\lambda x^2, \quad f_0(x, y) = 1-Ay^2-Bx^2,$$

with x and y dynamic variables indicating the states of the first and second subsystems, while λ, A, and B are parameters. Figure 1 shows the dynamic states in the (λ, A) plane derived numerically with B = 0.375. System (1) demonstrates a special kind of critical behavior: two-parameter scaling near the bicritical point having coordinates $\lambda_c = 1{,}401155$, $A_c = 1{,}124981$, and near that point, there is a complicated configuration that includes periodic states (light regions), chaotic states each having one positive Lyapunov parameter (horizontal and vertical strokes), and hyperchaos (crosses). That type of critical behavior was first observed numerically [9] in a mapping system analogous to (1) and by experiment in a system of coupled nonlinear tuned circuits externally excited. To provide a theoretical basis and to evaluate their general significance, one needs a renormalization group analysis, which is considered here.

2. To derive the renormalization group equations, we follow Feigenbaum's method [2] and transfer to mappings that describe the state change in two iterations and alter the scale of the variables x and y by certain factors a and b, so

$$x_{n+2} = ag_0(g_0(x_n/a)), \quad y_{n+2} = bf_0(g_0(x_n/a), f_0(x_n/a, y_n/b)). \tag{2}$$

Saratov Branch, Radio Engineering and Electronics Institute, USSR Academy of Sciences. Translated from Izvestiya Vysshikh Uchebnykh Zavedenii, Radiofisika, Vol. 33, No. 7, pp. 788-792, July, 1990. Original article submitted January 24, 1989.

 0033-8443/90/3307-0578$12.50

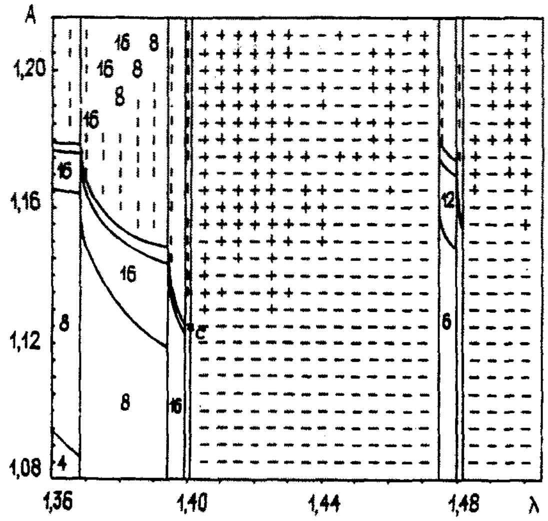

Fig. 1

The procedure is applied repeatedly to get the renormalization group equations

$$g_{m+1}(x) = ag_m(g_m(x/a)), \quad f_{m+1}(x, y) = bf_m(g_m(x/a), f_m(x/a, y/b)). \tag{3}$$

The bicritical point should correspond to the fixed point in (3) in the (g, f) function space:

$$g(x) = ag(g(x/a)), \quad f(x, y) = bf(g(x/a), f(x/a, y/b)), \tag{4}$$

which serves to define the scale constants a and b. With the normalization $g(0)=1$, $f(0,0)=1$, we have $a=(g(1))^{-1}$, $b=(f(1,1))^{-1}$.

3. The first equation in (4) is not dependent on the second. Feigenbaum solved it numerically to derive $a=-2{,}502907$ and obtained a representation of g(x) as a polynomial containing even powers of x, which enables one to compute it in the [0, 1] interval to the tenth decimal place [2].

The task is thus to solve the second equation in (4). The method is especially considered because there is a difference from other functional equations encountered in renormalization group analysis for nonlinear dynamics [1, 2] in that the unknown function f is dependent on two arguments, not one, which is a fundamental point and is associated with our considering a chaotic state having two positive Lyapunov parameters.

Figure 2* shows the method of representing f(x, y) in the region $x, y \in [0, 1]$. It is specified as a table at $(M + 1)^2$ nodes on the square net in the (x, y) plane and a defined procedure for interpolation between those nodes. The region is split up into $M^2/8$ triangular elements, in each of which one uses a distinct interpolating polynomial of fourth degree in x and y. Within each triangle and its boundaries, there are 15 nodes, which is the same number as that of the coefficients in the interpolating polynomial, i.e., the latter is defined uniquely. The polynomials corresponding to adjacent triangles are equal at the boundaries between them, since there they are converted to polynomials of fourth degree in one variable, which coincide at the five interpolation nodes.

As we have a table of values for f_m, an interpolation procedure, and a known method for calculating g, one can derive numerically values for the expression on the right in the second equation in (4) at the nodes in this net and obtain a table for f_{m+1}. Here b may be taken as $(f_m(1,1))^{-1}$. Then f_{m+1} is derived in the sense in which f_m is defined. These functional iterations may be repeated as desired, but the resulting f_m sequence diverges. To provide convergence to the fixed point in the renormalization group equation, the following iteration scheme was chosen empirically:

*It can be shown that f(x, y) should be even on both its arguments.

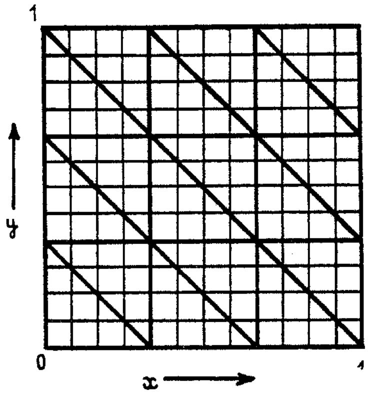

Fig. 2

$$f(x, y) \leftarrow 0{,}93\,Rf + 0{,}39R^2f - 0{,}32R^3f\,, \tag{5}$$

in which R^k denotes k-fold functional iteration.

4. The calculations were performed for a 21 × 21 node net. The initial approximation was the function in (1) at the bicritical point. About 20 iterations on (5) gave f to the sixth figure after the decimal point that satisfied the second dquation in (4),together with b = −1.50532. Figure 3 shows the universal function f(x, y).

The mapping

$$x_{n+1} = g(x_n), \quad y_{n+1} = f(x_n, y_n)\,, \tag{6}$$

describes the behavior at the bicritical point and has the fixed point $x_* = 0{,}549305$, $y_* = 0{,}528067$. However, then (4) clearly implies a two-cycle, one of the elements being $x_*/a,\ y_*/b$. The existence of the 2-cycle implies a 4-cycle with element $x_*/a^2,\ y_*/b^2$ and so on to infinity. All these cycles have identical multipliers μ_1 = −1,601191 (for x perturbations according to [2]) and $\mu_2 = -1{,}17886$ (for perturbations in y obtained as the derivative $\partial f(x\,y)/\partial y$ at x_*, y_*).

5. We now examine the effects from perturbations in the initial mapping that cause the system to deviate from the bicritical point. We consider only perturbations that correspond to unidirectional coupling betwen the subsystems. Formally speaking, the treatment consists in examining the evolution of small perturbations in the fixed point (g, f) in the (3) functional mapping. We put $g_m(x) = g(x) + h_m(x)$ and $f_m(x) = f(x, y) + \varphi_m(x, y)$, in which $h,\ \varphi \ll 1$, to get from (4) in the approximation linear in h and φ:

$$h_{m+1}(x) = a\left[g'\left(g\left(\frac{x}{a}\right)\right)h_m\left(\frac{x}{a}\right) + h_m\left(g\left(\frac{x}{a}\right)\right)\right],$$

$$\varphi_{m+1}(x) = b\left[f'_y\left(g\left(\frac{x}{a}\right), f\left(\frac{x}{a}, \frac{y}{b}\right)\right)\varphi_m\left(\frac{x}{a}, \frac{y}{b}\right) + \varphi_m\left(g\left(\frac{x}{a}\right), f\left(\frac{x}{a}, \frac{y}{b}\right)\right)\right] + bf'_x\left(g\left(\frac{x}{a}\right), f\left(\frac{x}{a}, \frac{y}{b}\right)\right)h_m(x)\,. \tag{7}$$

Feigenbaum [2] solved the first equation in (7) and found the first eigenvalue δ_1 = 4.669201 and the polynomial approximation for the corresponding eigenfunction $h_{(1)}(x)$.

We solved the second equation in (7) by the use of f previously derived as a table. The derivative f'_y was calculated by analytic differentiation of the interpolating polynomial. A table was found for $\varphi(x, y)$ as for f(x, y), together with interpolating polymonials of fourth degree in the triangular elements into which the region was split up (Fig. 2). Simple iteration for a sufficient number of steps with all the trial initial perturbations gave rise to the same eigenfunction $\varphi_2(x, y)$ with eigenvalue $\delta_2 = 2{,}39274$. Figure 4 shows this graph. No other significant eigenfunctions for the (7) linear operator were observed (certain eigenfunctions correspond to perturbations that are eliminated by infinitesimal change in the variables, while the others had eigenvalues less in modulus than one and are damped for m → ∞).

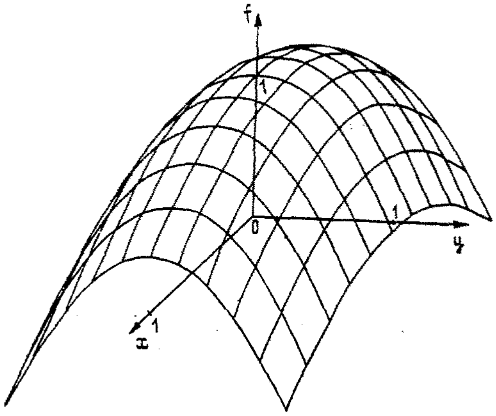

Fig. 3

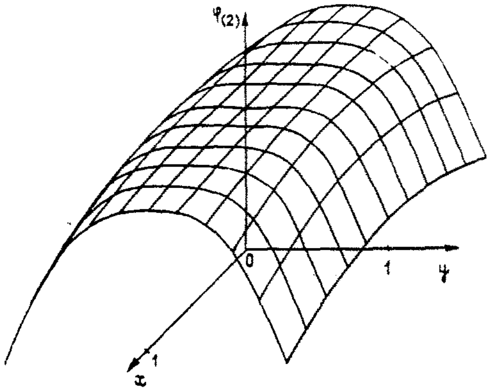

Fig. 4

For large m, the (7) solution is a linear combination of two eigenvectors $(h_{(1)}, \varphi_{(1)})$ and $(0, \varphi_{(2)})$, in which the coefficients are major parameters. The behavior near the bicritical point is described by

$$x_{n+1} = g(x_n) + \Lambda_1 h_{(1)}(x_n), \qquad (8)$$

$$y_{n+1} = f(x_n, y_n) + \Lambda_1 \varphi_{(1)}(x_n, y_n) + \Lambda_2 \varphi_{(2)}(x_n, y_n).$$

With the (3) transformation corresponding to description over two steps in discrete time, the additional terms in the first and second equations are multiplied correspondingly by δ_1 and δ_2, so in the plane of the parameters (Λ_1, Λ_2), there will be a universal scale-invariant pattern of regions that is converted into itself when one changes the scales for the coordinates Λ_1 and Λ_2 correspondingly by factors of δ_1 and δ_2. This agrees with existing numerical and experimental data [9].

6. As the solutions to (4) and (7) can be constructed without reference to the detailed forms of the seed functions g_0 and f_0, the functions f and $\varphi_{(2)}$ and the constants b, μ_2, and δ_2 are universal, as also are the Feibenbaum functions and constants $g, h_1, a, \mu_1, \delta_1$. Therefore, the dynamic behavior near the bicritical point in such a system showing period doubling is universal and is independent of the coupling strength and of whether one describes the subsystem dynamics by means of mappings or differential equations. We thus have a renormalization group basis for the universality and scaling occurring in a new type [9] of critical behavior for coupled systems. This is a nontrivial example of scaling behavior in dynamic systems at the hyperchaos threshold.

LITERATURE CITED

1. G. Schuster, Deterministic Chaos [Russian translation], Mir, Moscow (1988).
2. M. J. Feigenbaum, J. Stat. Phys., 21, No. 6, 669 (1979).
3. O. Rossler, Phys. Lett., 71A, 155 (1979).
4. P. S. Linsay, Phys. Rev. Lett., 47, No. 19, 1349 (1981).
5. A. Libchaber, S. Fauve, and C. Laroche, Physica, 7D, No. 1-3, 73 (1983).
6. V. S. Anishchenko, Stochastic Oscillations in Radio Physics Systems, Parts 1 and 2 [in Russian], Saratov State Univ. (1985 and 1986).
7. A. V. Gaponov-Grekhov, M. I. Rabinovich, and I. M. Starobinets, Pis'ma Zh. Éksp. Teor. Fiz., 38, No. 12, 561 (1984).
8. V. S. Anishchenko, I. S. Aranson, D. E. Postnov, and M. I. Rabinovich, Dokl. AN SSSR, 286, No. 5, 1120 (1986).
9. B. P. Bezruchko, Yu. V. Gulyaev, S. P. Kuznetsov, and E. P. Seleznev, Dokl. AN SSSR, 287, No. 3, 619 (1986).