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Period Doubling System Under Fractal Signal: Bifurcation in the Renormalization Group Equation

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Abstract—A model with a fractal signal having a two-scale Cantor-like phase portrait is introduced to describe real signals generated by dynamical systems at the onset of chaos. It is shown that the effect of such a signal on the period doubling system may lead to a bifurcation in RG equation, and in particular, to the confluence and stability exchange of its fixed points. In real dynamics it corresponds to change behaviour at the onset of chaos.

INTRODUCTION

Significant progress has been recently achieved in the field of nonlinear dynamics in connection with the study of scenarios of transition to chaos [1]. As a result, a lot of interesting examples were introduced of fractal attracting sets arising in the phase space of dynamical systems exactly at the borderline of chaos. Fractals are hierarchically organized repeating themselves in smaller scales (self-similarity), and are charcterized by a fractional value in the Hausdorff dimension [2,3]. The Feigenbaum attractor is a simple fractal example realized at critical points of transition to chaos via the period doubling sequence. It has a structure of Cantor set type with a Hausdorff dimension $D_F = 0.538045$.

A powerful tool for theoretically studying such situations is the renormalization group (RG) method, developed in nonlinear dynamics first by Feigenbaum [4] and then extended by many others. This approach consists of constructing a special renormalization procedure. For any given evolution operator of a dynamical system describing its behaviour over some temporal interval, this procedure allows one to define the evolution operator over a greater interval, as the dynamical variables are rescaled by some factor. Repetition of this procedure gives the operator sequence describing the behaviour over successively greater times. It corresponds to studying the dynamics in successively smaller scales in phase space. In critical situations, the operator sequence has some limit universal operator defining the structure of a fractal attracting set in phase space.

In understanding critical dynamics, the effects such movements have on each other arise naturally. The dynamics of the influencing system and that of the affected one are both hierarchically organized, and they involve a wide range of scales in phase space including arbitrarily small ones. Competition of movements with different scaling properties may, or may not, lead to a new critical behaviour of the affected system. As we shall show, a change in parameters, connected with the scaling properties of the influencing external signal, may cause transition from one critical situation to another.

The above phenomena appear in the framework of the RG approach as renormalization dynamics bifurcation. One may consider the RG transformation as a special type of dynamics in operator space ('renormalization dynamics'), the RG procedure step number being 'time' (i.e. 'renormalization time'). The universal limit operator is associated with the RG transformation fixed point, which may, in general, bifurcate when the control parameter changes. Such bifurcation is displayed in real dynamics as a change in critical behaviour type including reconstruction of fractal attracting set structures and modification of scaling factors.

FRACTAL SIGNAL

The signal we want to construct, in order to study its effect on the nonlinear system, must meet the following requirements. First, it must possess a fractal phase portrait and a hierarchy of scales; second, there must be parameters allowing to monitor the signal scaling properties, these changes being possibly able to cause bifurcation of renormalization dynamics.

We define fractal signal as a discrete time series y_n given by the recurrent relations

$$y_{2n} = b(1 + y_n),$$
$$y_{2n+1} = -a(1 + y_n), \tag{1}$$

where $a < 1$ and $b < 1$ – are some positive parameters. The initial iteration element is $y_0 = b/(1 - b)$.

It may be shown, that this signal is generated when the elements of the two-scale Cantor set [2] are taken in a definite order. This set is constructed from the initial segment $[-a/(1-b),\ b/(1-b)]$ under the following recurrent procedure. At the first step this segment is divided into three segments with the length ratio $a{:}1 - a - b{:}b$, while the middle one is removed. Each of the retained segments is subjected to the same procedure by removing the middle part and so on. The fractal dimension of the limit set D is defined by equation, see reference [2]:

$$a^D + b^D = 1. \tag{2}$$

Equal dimension lines in the parameter plane (a,b) are presented in Fig. 1(a).

It should be noted, that the above geometric interpretation and relation (2) are valid while $a + b < 1$, otherwise the segments by which the set is constructed begin to interlace. The initial definition (1) may be used in either case, but when the above relation is violated the commonly defined dimension becomes a unit.

With $a = 1/2.5029$ and $b = a^2$ the set constructed with the use of relations (1) approximates the Feigenbaum attractor appearing at the borderline of chaos via period doubling when the signal elements are numbered in a proper time succession. The Hausdorff dimension defined by the relation (2) is 0.5245, quite close to their D_F exact value.

Model (1) may be used in a similar way for description of the signals generated at the onset of chaos by the more common map $y_{n+1} = 1 - \lambda|y_n|^z$. To obtain an approximate signal we assume $a = 1/\alpha$, $b = 1/\alpha^z$, where α is a scale factor for any given z, see Ref. [5].

With $a = b = 1/3$ the y_n elements run over the points of the classic Cantor set, disposed at the segment $(-1/2, 1/2)$. It is natural to call this signal a Cantor signal.

With $a = 1$, $b = 0$ the y_n elements take values 0 and -1 only, the order of which (in time succession) correspond to that of R and L in a symbolic dynamics orbit at the borderline of chaos via period doubling, see Ref [6].

Thus, a lot of nontrivial fractal signals, arising naturally in nonlinear dynamics or constructed artificially, are united in one wide class, since they all may be obtained, at least approximately, with the use of the general scheme (1) for different parameter values a and

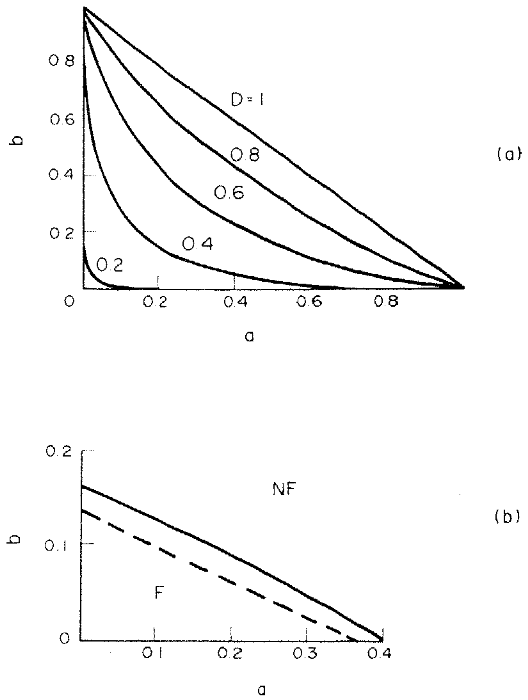

Fig. 1. Fractal signal parameters plane: (a) equal Hausdorff dimension lines; (b) the Feigenbaum's dynamics region F and that of non-Feigenbaum's one NF for system (3).

b. Although the suggested construction may look somewhat far-fetched, it seems to be useful for a deeper understanding of the nature of critical phenomena in nonlinear systems and, in particular, renormalization dynamics bifurcations.

FEIGENBAUM SYSTEM DYNAMICS UNDER THE FRACTAL SIGNAL

Approximate RG analysis

Suppose now a system, demonstrating transition to chaos via period doubling, is subjected to the influence of fractal signal (1). The situation is described by the model equation

$$x_{n+1} = 1 - \lambda x^2 + c y_n, \tag{3}$$

where x, λ and c are the dynamical state variable, control parameter of the initial system, and intensity parameter of the external signal, respectively. Note, that system (3), obviously, may have several attractors for some parameter values (multistability): for small enough c the system may exhibit movements with different phase shift in regards to the fractal signal. In this paper we shall restrict ourselves to the situation when the system begins its iterations from the vicinity of the origin at $n = 0$ (see the definition (1)). In some sense, this attractor may be regarded as the most typical of the system (3).

We begin with the approximate RG analysis in accordance with Ref. [8], allowing one to obtain RG transformation in explicit form and to discuss the qualitative features of the renormalization dynamics.

Let us construct the evolution operator over two discrete time steps. Considering only the vicinity of the origin and a small external signal intensity we shall neglect the terms containing x^4 and c^2:

$$x_{n+2} = \lambda(1 - \lambda x_n^2 + c y_n)^2$$
$$\approx 1 - \lambda + 2\lambda^2 x_n^2 + (y_{n+1} - 2\lambda y_n)c. \tag{4}$$

We can now use the definition of the fractal signal (1) to express y_n and y_{n+1} via $y_{n/2}$. Performing the variable change

$$x_n \to [1 - \lambda - (a + 2\lambda b)c]x_{n/2} \tag{5}$$

and parameter change

$$\lambda_1 = 2\lambda^2[\lambda - 1 + (a + 2\lambda b)c],$$
$$c_1 = c(a + 2\lambda b)/[\lambda - 1 + (a + 2\lambda b)c]. \tag{6}$$

As a result we obtain the map

$$x_{n+1} = 1 - \lambda_1 x_n^2 + c_1 y_n. \tag{7}$$

Thus, the two-time-step evolution operator is brought to the initial form (3). This means that the dynamics of two systems with parameters λ, c and λ_1, c_1 related via the equations (6) are similar, differing only in the time scale by a factor of two. The above procedure may be applied now to the map (7), to obtain the four-time-step evolution operator and so on. As a result of an m-fold renormalization, we come to the map (3) with parameters λ_m and c_m which may be obtained from the recurrent equations

$$\lambda_{m+1} = 2\lambda_m^2[\lambda_m - 1 + (a + 2\lambda_m b)c_m],$$
$$c_{m+1} = c_m(a + 2\lambda_m b)/[\lambda_m - 1 + (a + 2\lambda_m b)c_m]. \tag{8}$$

The values (λ_m, c_m) completely define the evolution operator in the framework of the given approximation, so that it may be regarded as a representation of this operator. Therefore, equations (8) are just equations of renormalization dynamics.

Assuming $c = 0$, we find the fixed point of equations (8) to be $\lambda_F = (\sqrt{3} + 1)/2$. This may be naturally denoted as a Feigenbaum point, since it is the solution defining the critical behaviour of the system while removing the external influence. There is also another, non-Feigenbaum point

$$\lambda_{NF} = 1/(\sqrt{a^2 + 4b} + a),\ c_{NF} = 1 + 2\lambda_{NF} - 2\lambda_{NF}^2. \tag{9}$$

Linearizing the map (8) near the fixed points, we may define their stability properties and obtain their corresponding eigenvalues. For the Feigenbaum fixed point they are

$$\delta_F^{(1)} = 4 + \sqrt{/3} \approx 5.73,\ \delta_F^{(2)} = 2\lambda_F(a + 2\lambda_F b), \tag{10}$$

and for the non-Feigenbaum one

$$\delta_{NF}^{(1,2)} = 1 - \lambda_{NF} + 2\lambda_{NF}^2(1 + bc_{NF}) \pm$$
$$\pm \sqrt{[1 - \lambda_{NF} + 2\lambda_{NF}^2(1 + bc_{NF})]^2 - 2\lambda_{NF}(3\lambda_{NF} - 2)}. \tag{11}$$

Under the condition $a + 2\lambda_F b < 1/2\lambda_F$ the first point is a saddle, and the second is the unstable node. The points collide when $a + 2\lambda_F b = 1/2\lambda_F$. If $a + 2\lambda_F b > 1/2\lambda_F$ then the Feigenbaum point becomes an unstable node and the non-Feigenbaum one a saddle. Thus, for parameter values satisfying the relation $a + 2\lambda_F b = 1/2\lambda_F$ confluence and stability exchange of fixed points take place, as is well known from bifurcations theory, see Ref. [9].

Figure 2 shows the renormalization dynamics in the (λ, c)-plane, defined by equations (8) for different a and b parameter values—before and after bifurcation. The line joining the two fixed points is a stable manifold of the saddle. Parameter values of the initial map corresponding to the points of this line are (within the approximation) the critical parameter values of transition to chaos. Indeed, if the (λ, c)-point belongs to the critical line, multiple repetition of RG transformation (8) will lead to a shift along this line, as it is shown by the arrows. Therefore, the critical phenomena at the border of chaos before the bifurcation are defined by the Feigenbaum fixed point, and after the bifurcation by the non-Feigenbaum one. In Fig. 1(b) the dashed straight line $a + 2\lambda_F b = 1/2\lambda_F$ in the fractal signal parameter (a, b)-plane is shown separating regions of Feigenbaum (F) and non-Feigenbaum (NF) critical dynamics.

Analysis allows one to obtain approximate values of the critical indices. Scaling factor in the parameter plane is defined as the greatest in modulus eigenvalue of the saddle fixed point:

$$\delta = \begin{cases} \delta_F^{(1)} = 5.732, & a + 2\lambda_F b < 1/2\lambda_F, \\ \delta_{NF}^{(2)}(a,b) & a + 2\lambda_F b > 1/2\lambda_F. \end{cases} \tag{12}$$

Scaling factor in phase space α may be found from the relation (5): $\alpha = \lambda - 1 + (a + 2\lambda bc)$. Substituting λ and c values corresponding to the saddle point for given a and b, we obtain

$$\alpha = \begin{cases} \alpha_F = 1/2\lambda_F = 2.732, & a + 2\lambda_F b < 1/2\lambda_F, \\ \alpha_{NF} = 2/(\sqrt{a^2 + 4b} + a), & a + 2\lambda_F b > 1/2\lambda_F. \end{cases} \tag{13}$$

Plots of scaling factors δ and α vs fractal signal parameter a are shown with dashed lines in Figs. 3 and 4 for two values $b = 0$ and 0.1, respectively. All of them have a typical break

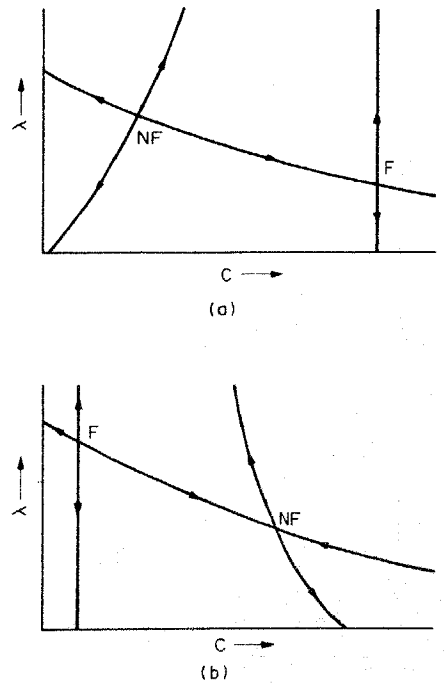

Fig. 2. Renormalization dynamics fixed points location before (a) and after (b) the bifurcation ($a = 0.3$ and 0.5, $b = 0$).

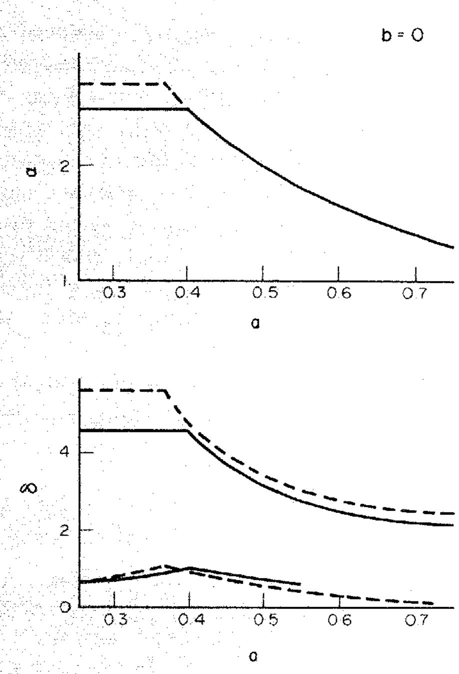

Fig. 3. Plots of scaling factors δ and α vs fractal signal parameter a for $b = 0$. The results of approximate analysis are shown with dashed lines and that of exact one—with solid lines.

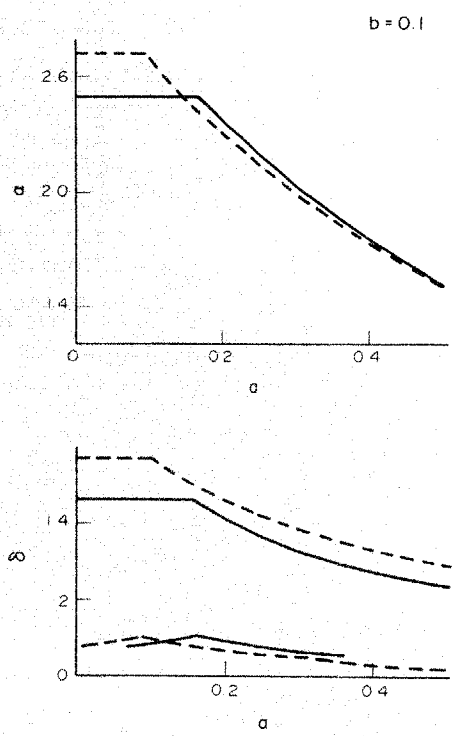

Fig. 4. Plots of scaling factors δ and α vs fractal signal parameter a for $b = 0.1$. Details as in previous figure.

at the bifurcation point associated with the critical dynamics change. Before the bifurcation one finds Feigenbaum behaviour with constant scaling factors δ_F and α_F. After the bifurcation one finds non-Feigenbaum behaviour with scaling factors depending on the parameters a and b.

EXACT RG ANALYSIS

We proceed now with exact RG analysis, confirming the qualitative results of our approximation and allowing us to precisely estimate the critical indices and bifurcation conditions at the parameter plane of the fractal signal (a,b).

We shall first obtain the exact RG equation. Denoting the right side function in equation (3) as $f_0(x,y)$ one can receive the map describing the change of state over two time steps:

$$x_{n+2} = f_0(f_0(x_n, y_n), y_{n+1}).$$

Rescaling the dynamical variable x by some factor $(-\alpha)$ and expressing y_n, y_{n+1} via $y_{n/2}$ in accordance to (1), we obtain a new map $f_1(x,y)$. Multiple repetition of this procedure leads to RG equation

$$f_{m+1}(x,y) = -\alpha f_m(f_m(-x/\alpha, b(1+y)), -a(1+y)). \quad (14)$$

The fixed points of this functional map are of particular importance. To solve for fixed points requires to find the scaling factor α and function $f(x,y)$ satisfying the equation

$$f(x,y) = -\alpha f(f(-x/\alpha, b(1+y)), -a(1+y)). \quad (15)$$

Substituting the solution f_m of equation (14) in a form $f_m(x,y) = f(x,y) + \delta^m h(x,y)$, where h is a small perturbation, we obtain in linear approximation:

$$\delta h(x,y) = -\alpha[f'(f(-x/\alpha, psb(1+y)), -a(1+y))h(-x/\alpha, b(1+y)) + \\ + h(f(-x/\alpha, b(1+y)), -a(1+y))]. \quad (16)$$

Eigenfunctions $h_s(x,y)$ and eigenvalues δ_s characterizing the dynamics in the vicinity of the fixed point may be found by solving equation (16). Only those eigenvalues are relevant for RG dynamics which are not connected with infinitesimal variable changes.

In our calculations the function $f(x,y)$ was represented with the Tchebyshev polynomial expansion

$$f(x,y) = \sum_{m,n=0}^{M,N} u_{m,n} T_{2m}(x) T_n(y) \quad (17)$$

the function chosen even with respect to the first argument.

Newton's technique was used to numerically locate the fixed points of equation (14). In accordance with this method, a new approximation for the function $f(x,y)$

$$f_{new}(x,y) = f(x,y) + h(x,y)$$

is obtained by solving the equation

$$f(x,y) + h(x,y) = -\alpha f(f(-x/\alpha, b(1+y)), -a(1+y)) + \hat{L}\, h(x,y) \quad (18)$$

where $\hat{L}$ is a linear operator shown in the right hand side of equation (16). With expansion (17), equation (18) may be reduced to the set of $M \times N$ nonlinear algebraic equations, matrix elements of operator $\hat{L}$ in appropriate basis being calculated with the use of Tchebyshev polynomials orthogonality on the set of points in the plane (x,y) produced by zeros in the product $T_{2m}(x)T_n(y)$. Values of M and N in our calculations were limited by

10. Table 1 presents the resulting scaling factors and relevant $\hat{L}$ operator eigenvalues for several parameter values a and b.

At the bifurcation point one of the linear operator $\hat{L}$ relevant eigenvalues crosses the unit circle. To simplify the problem we shall consider the Feigenbaum fixed point stability in the perturbation subspace defined by the relation $h_r(x,y) = h_0(x) + yh(x)$ with arbitrary $h_0(x)$ and $h(x)$. It may be shown from equation (16) that this subspace is invariant under operator $\hat{L}$, so the bifurcation point we are going to find remains valid for the complete space. In this case equation (16) may be reduced to the next eigenvalue problem

$$\delta h(x) = -\alpha a[(b/a)g'(g(-x/\alpha))h(-x/\alpha)(-h(g(-x/\alpha))] = -\alpha a\hat{L}_r h(x). \quad (19)$$

The usual Feigenbaum eigenvalue problem [4] arises, but it is of no interest to us for the time being.

For particular $b/a = \xi$ we may find the largest eigenvalue ν of a linear operator $\hat{L}_r$ in equation (19) by direct numerical iteration of some initial function $h(x)$. Assuming then $a = -1/\alpha\nu$, $b = -\xi/\alpha\nu$ we obtain a parametrically defined curve in the parameter plane (a, b). It is obvious that the points of this curve correspond to the unit eigenvalue modulus and, hence, it is the bifurcation line we are searching for. This bifurcation is shown in Fig. 1(b) by the solid line.

The scaling factors δ and α obtained by numerically solving equation (15) as functions of fractal signal parameter are presented in Figs. 3 and 4. One may compare them with those obtained by the approximate analysis, shown by dashed lines.

HOW RG EQUATION BIFURCATION DISPLAYS IN OBSERVABLE SYSTEM BEHAVIOUR

We are now going to discuss the change in critical dynamics of the initial system (3), accompanying the RG equation bifurcation. It is the modification of scaling properties and reconstruction of the fractal structure arising in phase space at the onset of chaos.

The bifurcation tree—orbit in phase space, as a function of parameter λ—and a plot of Lyapunov exponents for the Feigenbaum situation (before the bifurcation) are presented in

Table 1. Scaling factors and relevant eigenvalues for the non-Feigenbaum RG equation fixed point

b	a	α_{NF}	$\delta_{NF}^{(1)}$	$\delta_{NF}^{(2)}$
0	0.28571	3.50000	9.451	1.232
	0.38462	2.60000	5.03506	1.03642
	0.43416	2.30329	3.99075	<1
	0.5	2.00000	3.15823	<1
0.05	0.1	3.62884	10.08665	1.41811
	0.15	3.27345	8.08463	1.30943
	0.2	2.95726	6.52502	1.19965
0.1	0	3.16228	7.42181	1.36605
0.1	0.1	2.74442	5.57055	1.14818
	0.3	2.04806	3.33476	
	0.4	1.17701	2.75770	<1
	0.5	1.53259	2.39421	<1
0.15963	0.39954	1.54945	2.44700	<1
0.2	0.1	2.03762	3.36808	
	0.2	1.83305	2.93239	<1
	0.3	1.62654	2.57723	<1
0.3	0.1	1.68773	2.71621	<1

Fig. 5(a). A critical point of transition to chaos is defined as a point where the Lyapunov exponent crosses the zero value. In contrast to typical figures illustrating period doubling scenarios there are no clear-cut bifurcations: they are washed out by the fractal signal influence, containing components with various subharmonic frequencies. The picture of a series is presented in Figs. 5(b–d) showing bifurcation trees and Lyapunov exponent plots in the vicinity of the critical point under the applied successively refined procedure. At each step, in accordance with Feigenbaum scaling, the horizontal resolution is increased by the factor $\delta_F = 4.669\ldots$ while the vertical resolution is rescaled by the factor $\alpha_F = 2.5029\ldots$ for trees, and two for Lyapunov exponents. Comparison of the figures shows the usual universal patterns of bifurcation tree and Lyapunov exponents at deeper levels of resolution. At the critical point the classic Feigenbaum attractor arises with Hausdorff dimension D_F. Pressure of the external signal displays only in critical parameter value λ dependence on the influence intensity c.

The second, non-Feigenbaum situation is represented by Fig. 6. Just like in the previous case, the small-scale structure of the bifurcation tree and Lyapunov exponents plot near the critical point of transition to chaos is illustrated by a succession of figures listed in order of increasing resolution. Rescaling rules correspond now to the scaling defined by the non-Feigenbaum RG equation fixed point for given a and b. At the second and third steps the patterns stop changing already. The presence of expected scaling is confirmed, which means that the system dynamics at the onset of chaos is defined by a non-Feigenbaum fixed point. A new fractal structure is forming at the critical point of transition to chaos: total external signal influence on all hierarchy levels leads to a transformation of the Feigenbaum attractor into a new attracting set. The Hausdorff dimension of this attractor depends now on the fractal signal parameters a and b (see Fig. 7).

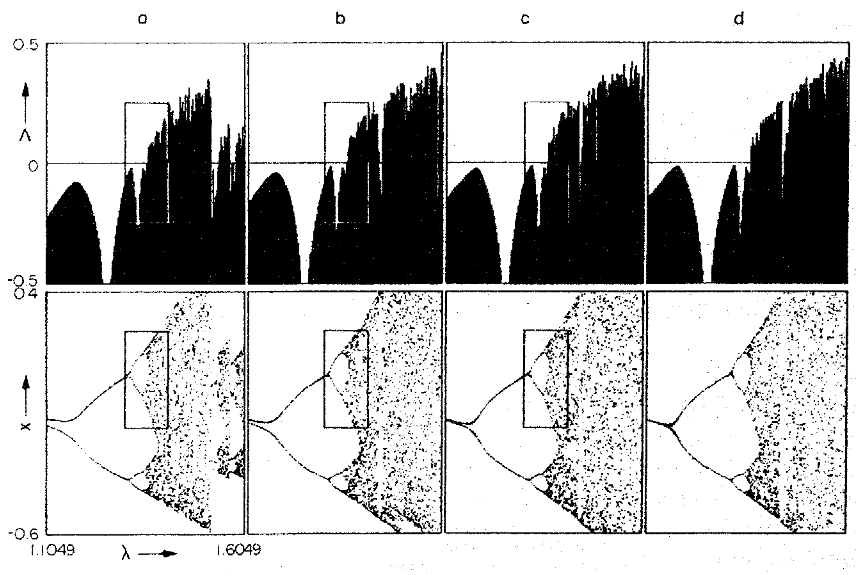

Fig. 5. Bifurcation trees and Lyapunov exponent plots for $a = 0.15$, $b = 0$, $c = 0.5$. The patterns within small rectangular regions in parts a, b, c are reproduced in full scale in parts b, c, d, respectively. At each step of refinement the horizontal resolution is increased by factor of 4.6692 while the vertical one is rescaled by factor of -2.5029 for trees and 2 for Lyapunov exponents.

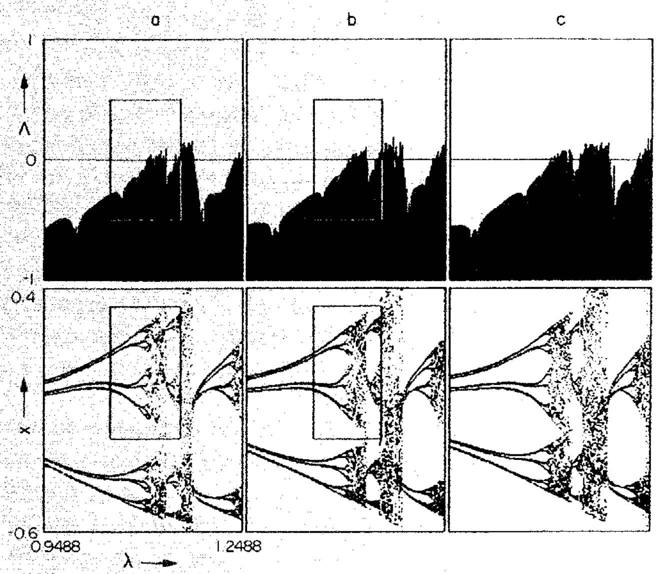

Fig. 6. Bifurcation trees and Lyapunov exponent plots for $a = 0.2$, $b = 0.2$, $c = 0.5$. The patterns within small rectangular regions in parts a, b are reproduced in full scale in parts b, c, respectively. At each step of refinement the horizontal resolution is increased by factor of 2.9324 while the vertical one is rescaled by factor of -1.8330 for trees and 2 for Lyapunov exponents.

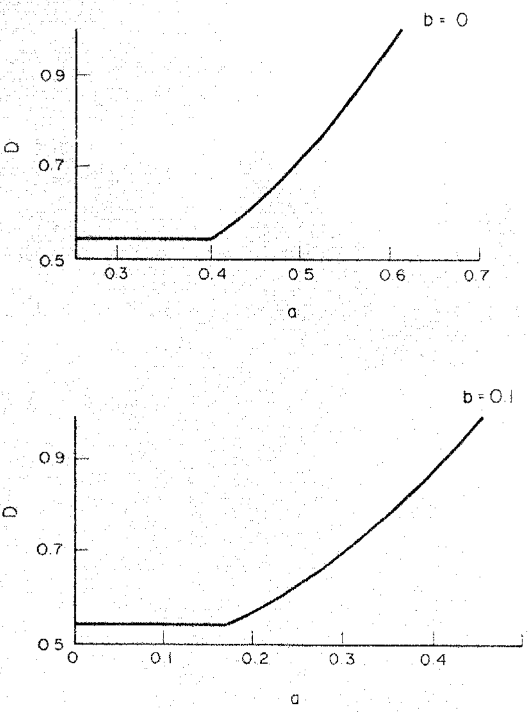

Fig. 7. Hausdorff dimension of the attractor D versus fractal signal parameter a for $b = 0$ (a) and $b = 0.1$ (b).

CRITICAL DYNAMICS IN THE VICINITY OF RG EQUATION BIFURCATION POINT QUASI-SCALING

It is well known that in real dynamics the development of processes in time becomes slower as one approaches the bifurcation point. Just the same occurs in renormalization dynamics, provided that the parameter values a and b are chosen close enough to the bifurcation curve (see Fig. 1b). The closer we are to the bifurcation point, the slower the renormalization dynamics. In the approximate RG analysis framework, the slow dynamics equation may be derived by assuming $\lambda_{m+1} = \lambda_m \approx \lambda_F$ in equation (8). Then from the first equation we obtain

$$\lambda_m = \lambda_F - c(a + 2\lambda_f b)/(2 - 1/\lambda_F),$$

and substituting it into the second gives

$$c_{m+1} = 2\lambda_m(a + 2\lambda_F b)c_m - \frac{a + 4\lambda_F b}{2\lambda_F - 1} c_m^2. \tag{20}$$

Now we may pass from discrete to continuous variable time

$$\mathrm{d}c/\mathrm{d}m = \varepsilon c - \frac{a + 4\lambda_F b}{2\lambda_F - 1} c^2, \tag{21}$$

where $\varepsilon = 2\lambda\,(a + 2\lambda_F b) - 1$ is the bifurcation value excess.

In the real dynamics of system (3), approaching the RG equation bifurcation point requires one to go deeper into resolution levels in order to observe the scaling, corresponding to the relevant RG equation fixed point. Needed level depth may be evaluated by the relation

$$m \approx \mathrm{const}/\varepsilon. \tag{22}$$

derived from equation (21).

A particular situation arises at a close vicinity of the RG equation bifurcation point which we call quasi-scaling. Consider, for example, the point in parameter plane disposed in the Feigenbaum's dynamics region near the bifurcation curve. We observe that the bifurcation tree and Lyapunov exponents plot are nearly unchanged when passing from one resolution level to another under Feigenbaum's rescaling, although the figures are quite different from Feigenbaum's universal patterns (see example in Fig. 5). A very slow pattern of evolution indeed takes place defined by the continuous renormalization time value. It becomes Feigenbaum's pattern asymptotically at $m \to \infty$ (see Fig. 8).

EXPLANATION OF CRITICAL BEHAVIOUR FOR TWO UNIDIRECTIONALLY COUPLED FEIGENBAUM SYSTEMS

In the paper Ref. [10] a new type of critical behaviour, called bicritical, was numerically found for a model system of two unidirectionally coupled logistic maps and experimentally obtained for a system of unidirectionally coupled periodically driven nonlinear oscillators. It occurs at the particular point in the subsystems control parameters plane, where simultaneously the border to chaos in both subsystems occurs. This means that both subsystems are successively brought to their own border to chaos. This is possible because of the unidirectional character of coupling. Chaotic behaviour arises from arbitrarily small increases in the control parameters in either subsystem. A universal pattern in the parameter plane exists at the vicinity of this point, featured by a two-parametric scaling: it

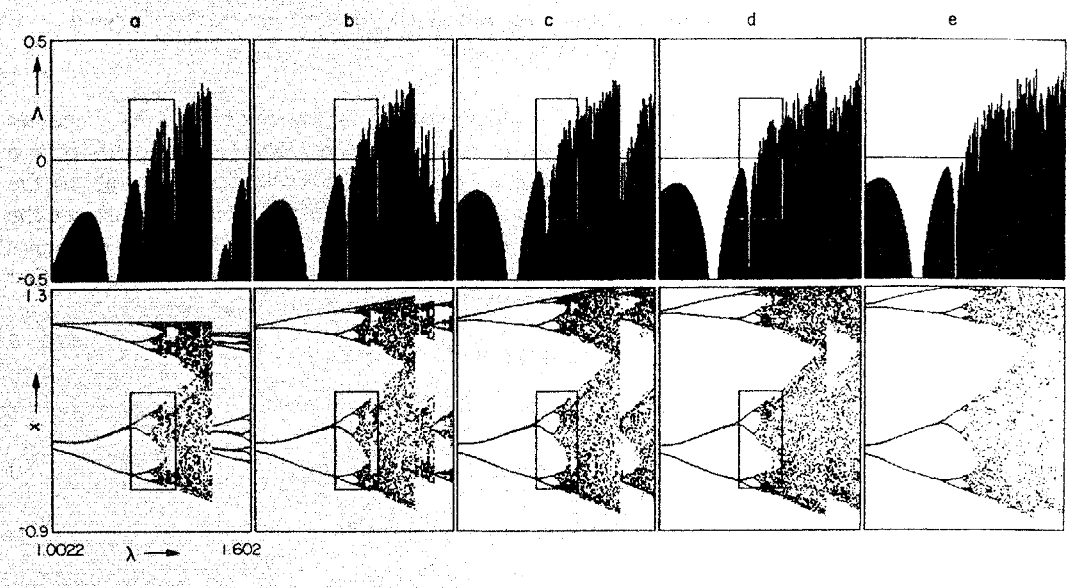

Fig. 8. Bifurcation trees and Lyapunov exponent plots for $a = 0.3$, $b = 0$, $c = 0.5$. Details as in Fig. 6.

is invariant under rescaling of coordinates by factors $\delta^{(1)} = 4.6692$ and $\delta^{(2)} = 2.39$, the scaling factors in the phase space being $\alpha^{(1)} = 2.5029$ and $\alpha^{(2)} = 1.51$.

These results may be explained in the framework of the wider approach, developed in this paper. If the first subsystem is just at the borderline to chaos, the fractal signal will be generated, described approximately by the model (1) with appropriate parameter values $a \simeq 0.4$ and $b \simeq 0.16$ (see Fig. 1b). This parameter plane point lays above the bifurcation curve, therefore the behaviour of the second subsystem at the onset of chaos would be defined by the non-Feigenbaum fixed point with scaling factors, presented for this situation in Table 1. They are in good accordance with the exact factor values $\alpha^{(2)}$ and $\delta^{(2)}$ found in Ref. [10].

CONCLUDING REMARKS

In previous six sections we have revealed some features of the Feigenbaum system behaviour under the fractal signal influence depending on parameter values a and b defining the scaling properties of such a signal. For small a and b the transition to chaos in system (3) obeys Feigenbaum scaling laws with classic scaling factors $\delta = 4.6692$ and $\alpha = 2.5029$ independent of a and b. After crossing some critical line in the (a,b) plane the scaling properties become functions of the parameters a and b. The process resembles a situation in phase transitions theory whereby the space dimension when regarded as a continuous parameter crosses the critical value $d = 4$. Equation (21) of slow renormalization dynamics near the bifurcation point coincides with the coefficient form of the classic Wilson–Fisher equation [7].

In the above analogy, one may think that, an appropriate control parameter for our problem would be the fractal dimension of an external signal. However, this is not the case: the bifurcation curve in the (a,b) plane does not coincide with the equal fractal dimension line. This analogy is only of methodological significance. Indeed, including phase transition problems into the wider class (containing non-physical problems as well) is fruitful for understanding and approximate description. Analysing the effect that an artificially constructed signal such as in (1) has on a Feigenbaum system sheds some light on the behavioural features of such systems, under a realistic fractal signal influence.

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