9. S. Szatmari and F. P. Schäfer, Appl. Phys. B, 33, 219 (1984).
10. S. Watanabe, A. Endoh, et al., Opt. Lett., 13, 580 (1988).
11. A. Endoh, M. Watanabe, and S. Watanabe, Opt. Lett., 12, 906 (1987).
12. S. A. Akhmanov, A. M. Val'shin, et al., Kvantovaya Elektron., 13, 1992 (1986).

CRITICAL PHENOMENA IN FEIGENBAUM SYSTEMS WITH ONE-WAY COUPLING

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Transition to chaos in two Feigenbaum systems with one-way coupling is investigated as a function of three parameters (the controlling parameters of the subsystems and the coupling constant). It is demonstrated that there exists a hierarchy of types of critical behavior with respect to the increasing codimension: There exist Feigenbaum critical surfaces, limited by tricritical lines and intersecting along the bicritical line. These lines in turn intersect at a multicritical point of a new type. The dynamical regimes near the indicated critical situations are discussed. In particular, hyperchaos is realized near the bicritical line and multistability is realized near the tricritical line. A table of universal critical indices is given.

In the context of problems in nonlinear dynamics critical behavior is understood to be a collection of phenomena arising at the transition of dynamical systems to chaos and characterized by some properties of scale invariance (scaling) in the space of controlling parameters. The simplest and best-studied type of critical behavior is associated with transition to chaos, according to Feigenbaum's scenario, through a sequence of period-doubling bifurcations, and it is characterized by one-parameter scaling with a universal scaling constant $\delta = 4.66920$ [1].

It turns out that the problem of the behavior of two Feigenbaum systems in the presence of one-way coupling between them (the first system acts on the second system but the second system does not affect the first system) is extremely interesting from the viewpoint of critical phenomena [2]. In this paper we shall show that global analysis of the space of all significant parameters of the problem opens up an entire hierarchy of types of critical behavior, which are characterized by one-, two-, and three-parameter scaling. This analysis makes it possible to understand the characteristics of the coexistence and mutual arrangement of the critical points, lines, and surfaces of different type in the parameter space. In turns out that some of the types of critical behavior studied are associated with multistability, and this makes it possible to study and classify multistable states on the basis of the theory of critical phenomena.

From the viewpoint of physical applications, systems with one-way coupling are associated, in particular, with the problem of turbulence developing downstream [3]. They can also be built artificially, for example, in radioelectronics and optics, in order to create devices with new functional possibilities (noise generators, stable and multistable memory elements, etc.).

1. Model Equations. Double Feigenbaum Point. As is usually done in the study of critical phenomena, we shall consider a model system, which in our case consists of two logistic maps

$$x_{n+1} = 1 - \lambda x_n^2, \qquad y_{n+1} = 1 - Ay_n^2 - Bx_n^2, \tag{1}$$

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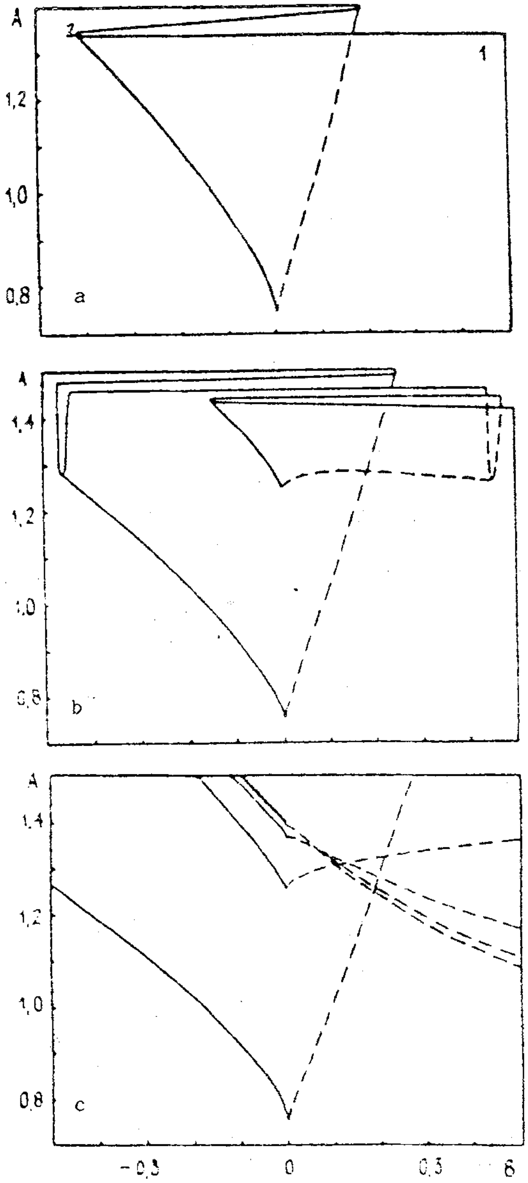

Fig. 1

where x and y are dynamical variables characterizing the states of the first and second subsystems, λ and A are the controlling parameters, and B is the coupling constant.

If we set $B = 0$, then the system (1) separates into two uncoupling Feigenbaum systems, which demonstrate period doubling and transition to chaos as a function of the corresponding controlling parameters λ and A. Let the values of these parameters be such that stable cycles of period 2^k are realized in both subsystems. Then the compound system has 2^k states, which differ by the shift of the oscillations in the subsystems relative to one another by an integer number of discrete-time units. These regimes are modified, but remain stable also when the coupling is switched on, if the magnitude of the coupling is sufficiently small.

Thus the system (1) exhibits multistability (though the mechanism, under discusion, of its appearance is not unique; see below). The formation of multistable states can be traced geometrically by studying the transformation of the parameter plane (A, B) as λ increases. When λ passes through the bifurcation value $\lambda_1 = 0.75$, the first subsystem undergoes the first period doubling. At this time there appears in the (A, B) plane an accumulation point $(\lambda_1, 0)$ into which two fold lines, which bound the region of coexistence of two different 2-cycles of the compound system, converge. The (A, B) plane can now be thought of as consisting of two partially overlapping sheets 1 and 2 (Fig. 1a), corresponding to the two cycles mentioned. A jump from the first sheet onto the second sheet occurs at the transition through

TABLE 1. Accumulation Points Converging to a Tricritical Point T_1 with $\lambda = 0.85$.

N	A_N	B_N	y_N	$\frac{A_\infty - A_N}{A_\infty - A_{2N}}$	y_N / y_{2N}
4	1,283293	0,891207	0,113696		
8	1,330607	0,716126	-0,070483	-0,466	1,613
16	1,314539	0,701132	0,043617	1,992	1,616
32	1,304689	0,701732	-0,026291	2,551	1,659
64	1,300657	0,702751	0,015662	2,740	1,679
128	1,299164	0,703224	-0,009287	2,809	1,686
256	1,298629	0,703407	0,005498	2,845	1,689
∞	1,298339	0,703509		2,85713	1,69030

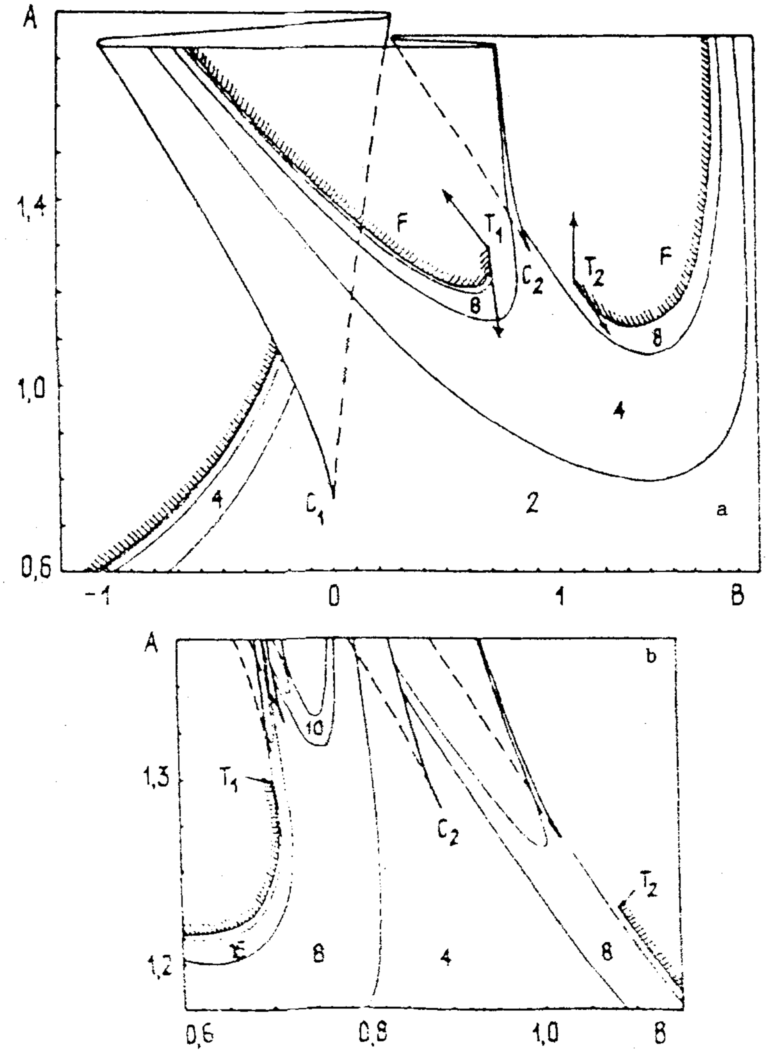

Fig. 2

the left-hand fold line and the opposite jump occurs at the transition through the right-hand fold line. After the next doubling bifurcation in the first subsystem each of the two sheets in turn bifucates. On each of them there appears an accumulation point (λ_2, 0), in which the newly arising sheets are connected with one another and the fold lines which are their edges converge (Fig. 1b). As the parameter λ is further increased the process of formation of new accumulations and bifurcation of sheets continues and at $\lambda_c = 1.401155$ the number of sheets and accumulation points is infinite. In the limit the accumulation points condense on the point (λ_c, λ_c, 0) in the parameter space (λ, A, B). We shall call this point the

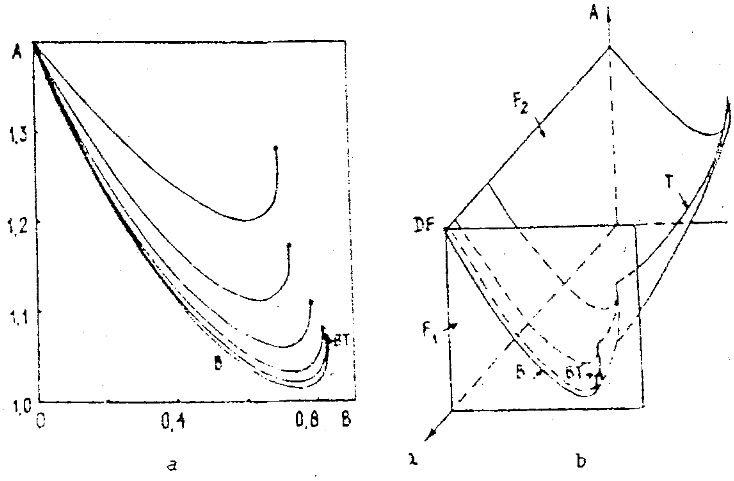

Fig. 3

double Feigenbaum point and designate it by DF. The above-described process of multiplication of multistable states is associated with a double Feigenbaum point in the same sense as the period doublings are associated with the critical point of an individual mapping.

Diverse regimes and bifurcations can be observed by moving along any of the sheets. In what follows we shall focus on studying one particular sheet, which forms the front side of the surfaces shown in Fig. 1 and which we shall call the in-phase sheet. The terminology is explained by the fact that as the parameters are varied in the direction of the point of zero coupling the regimes corresponding to this sheet transform into in-phase oscillations of the uncoupled subsystems.

The scaling properties of a neighborhood of a double Feigenbaum point on the in-phase sheet follow from the renormalization-group analysis performed in [3, 4]. The structure of the space of three parameters is invariant relative to changes of scales along the characteristic directions $\Lambda_1 = \lambda$, $\Lambda_2 = A + B$, $\Lambda_3 = B$, respectively, by factors of $\delta_1 = \delta_2 = 4.66920$ and $\delta_3 = 2$.

2. Tricritical Behavior. The above discussion and Fig. 1 give only a preliminary idea of the structure of the parameter space. We shall now undertake a more detailed analysis and we shall begin, once again, with the situation when a 2-cycle is realized in the first subsystem.

Figure 2a shows a map of the dynamic regimes in the (A, B) plane with $\lambda = 0.85$ (in-phase sheet). The figure contains lines of period-doubling bifurcations, which condense on the Feigenbaum critical lines F — boundaries of appearance of chaos. The region of existence of 4-cycles contains a new accumulation point A2. Fold lines — the edges of the new sheets, on which the in-phase sheet bifurcated — converge at this point. In the region between the fold lines the system has two different stable regimes, while on the lines themselves a hard transition between these regimes occurs. The map fragment shown in Fig. 2b and portraying in an enlarged form a neighborhood of the point A2, demonstrates a set of accumulations and folds on the basis of cycles of period 8, 16, 32, ... The lines of period doubling of different order bend around different accumulation points and for this reason no longer converge to the Feigenbaum line, which, therefore, terminates at some point. This point is the limit of a definite sequence of accumulation points (Table 1) and is called a tricritical point. The tricritical points were introduced in [5] in an analysis of a one-dimensional two-parameter mapping of degree four. Our problem can be transformed into this form when the period of the oscillations in the first subsystem is equal to two.

Using twice the second equation (1), we find a map describing the change in the state of the second subsystem with two iterations:

$$y_{n+2} = [1 - Bx_1^2 - A(1 - Bx_0^2)] + 2(1 - Bx_0^2)\, A^2y_n^2 - A^3y_n^4.$$

Here $x_{0,1} = [1/2 \pm (\lambda - 3/4)^{1/2}]/\lambda$ are the elements of the 2-cycle of the first subsystem. After making the substitution of variables we have

TABLE 2. Convergence of the Sequence of Tricritical Points to the Multicritical Point BT

λ	A_N	B_N	$\frac{A_\infty - A_N}{A_\infty - A_{2N}}$
1,25	1,279894	0,690210	
1,368099	1,173074	0,721136	1,998
1,394046	1,108720	0,784254	2,506
1,399631	1,080086	0,817598	3,034
1,400829	1,071161	0,828522	2,729
1,401155	1,066	0,83505	2,654

$$y \to y[1 - Bx_1^2 - A(1 - Bx_0^2)^2]$$

$$y_{n+2} = 1 + ay_n^2 + by_n^4, \tag{2}$$

where

$$a = 2(1 - Bx_0^2)(1 - Bx_1^2 - A(1 - Bx_0^2)^2)A^2,$$

$$b = -[A(1 - Bx_1^2 - A(1 - Bx_0^2)^2)]^3. \tag{3}$$

Recomputing using the formula (3), we find that the tricritical points, which have in the (a, b) plane the coordinates (−2.81403, 1.40701) and (0, −1.59490) [5], fall in the (A, B) plane with $\lambda = 0.85$ at the points (1.29834, 0.70351) and (1.23075, 1.08446). In Fig. 2 they are designated as T1 and T2.

In accordance with the renormalization-group analysis developed in [5], near each tricritical point there exists two-parameter scaling with the scaling constants $\delta_1 = 7.28469$ and $\delta_2 = 2.85713$. The corresponding characteristic directions in the (A, B) plane are indicated by the arrows. The first one proceeds along the line on which the dynamics of the second subsystem for some choice of the variable is described by the map (2) with $a = 0$ while the second one proceeds along the Feigenbaum critical line. On the other side of the tricritical point, the sequence of accumulations (Table 1) converges along the same direction toward it by a geometric progression with the exponent δ_2; this can be used as a basis for an algorithm for searching for tricritical points.

We note that the term tricritical point is introduced by analogy to the theory of phase transitions: A tricritical point is a point in whose neighborhood first- and second-order phase transitions are realized. The first-order transitions are associated with hard bifurcations on the fold lines and the second-order transitions are associated with the soft path to chaos through period doubling.

Now let the value of the parameter of the first subsystem be such that it exhibits a cycle of period 4. Then the configuration of the regions in the (A, B) plane become more complicated, but no new types of critical behavior are observed. This is also true of situations when the cycle in the first system has period 8, 16, etc.

3. Bicritical Behavior and the Multicritical Point BT. Now let the controlling parameter λ of the first subsystem be equal to the critical value $\lambda_c = 1.401155$. Then as the controlling parameter A of the second subsystem is increased another type of critical behavior, termed in [2] bicritical, is observed on the boundary of chaos. The bicritical line in the (A, B) plane can be found by following the evolution of the Feigenbaum line in the second subsystem as $\lambda \to \lambda_c$. Figure 3a shows the arrangement of this line in the (A, B) plane for values of λ corresponding to the threshold of instability of 2-, 4-, 8-, and 16-cycles of the first subsystem. One can see from the figure that as the parameter λ approaches the critical value the Feigenbaum lines in the (A, B) plane accumulate on the bicritical line B while the tricritical points accumulate on some point BT.

TABLE 3. Numerical Characteristics of the Hierarchy of Critical Dynamics of Unidirectionally Coupled Feigenbaum Systems

Type of critical point	n	m	δ	α	μ	γ, dB	$\varkappa$	χ
Feigenbaum F	1	1	4,66920	-2,50291	-1,6012	13,35	0,69	0,4498
Tricritical T	2	1	7,28469 2,85712	-1,69030	-2,0509	10,40	0,40	0,3491 0,6603
Bicritical B	2	2	4,66920 2,39272	-2,50291 -1,50532	-1,6012 -1,1789	13,35 7,98	0,69 0,92	0,4498 0,7945
Multicritical BT	3	2	4,66920 2,65465 1,54172	-2,50291 -1,24166	-1,6012 -1,3980	13,35 6,85	0,69 0,82	0,4498 0,7100 1,6012
Double Feigenbaum DF	3	2	4,66920 4,66920 2,00000	-2,50291 -2,50291	-1,6012 -1,6012	13,35 13,35	0,69 0,69	0,4498 0,4498

Nomenclature: n is the number of significant parameters (codimension), m is the number of significant dynamical variables, δ is the scaling factor in the parameter space, α is the scaling factor in the phase space, μ is the multiplicator of 2^n cycles at the critical point, γ and κ are constants which characterize the spectrum of oscillations at the critical point (difference between the subharmonics of different level and the nonuniformity of the amplitudes within a given level), and χ is the critical index for the Lyapunov exponent.

Near the bicritical line in the space (λ, A, B) there exists two-parameter scaling with the constants $\delta_1 = 4.66920$ and $\delta_2 = 2.392724$. The first characteristic direction is the λ axis; Feigenbaum critical lines of the second subsystem approach along the second direction the bicritical line in the sections (λ, A).

The point BT is a multicritical point of a new type, whose neighborhood is characterized by three-parameter scaling with the scale constants $\delta_1 = 4.66920$, $\delta_2 = 2.654654$, $\delta_3 = 1.541720$. The characteristic direction associated with the constant δ_1 is the λ axis, for δ_2 the characteristic direction is the line along which the tricritical points converge to BT, and the characteristic direction for δ_3 is the direction along the bicritical line (Table 2).

4. General Discussion of Critical Dynamics. We shall now sum up and give a general idea of the geometry of the parameter space of unidirectionally coupled Feigenbaum systems from the viewpoint of the types of critical behavior realized in them. The space (λ, A, B) contains two Feigenbaum tricritical surfaces F1 and F2 (Fig. 3b). The first one is the plane $\lambda = 1.401155$ and the second one is a complicated surface which is the boundary of chaos in the second system.* These surfaces intersect along the bicritical line B. The surface F2 has a boundary — the line of tricritical points T. The tricritical and bicritical lines converge and terminate at the multicritical point BT. The other end of the bicritical line is a double Feigenbaum point DF. We note that any neighborhood of the tricritical line and the points BT and DF contain regions of multistability and hard bifurcations.

To each of the types of critical behavior enumerated above there corresponds a definite set of quantitative characteristics (number of significant parameters and dynamical variables, scaling constants in the parameter space and in the phase space, critical multiplicator of 2^k cycles, etc.); they are summarized in Table 3.

The spectrum of oscillations of the second subsystem in different critical situations — Feigenbaum, tricritical, bicritical, and at the point BT — have a clearly pronounced hierarchical organization with respect to the levels of the amplitudes of the subharmonics and can be approximately described by the recurrence relation

*The surface F2 in reality consists of a set of pieces. For simplicity Fig. 3b only shows one piece, which, in Fig. 2, corresponds to the section of the tricritical line from B = 0 up to the first tricritical point.

$$S(\omega)\ \frac{\alpha^2 + \beta^2 \pm 2\alpha\beta\cos(\pi\omega/2)}{4\alpha^2\beta^2} \longrightarrow \begin{cases} S(\omega/2), & +\text{ sign} \\ S(1-\omega/2), & -\text{ sign} \end{cases} \tag{4}$$

This is a generalization of the well-known result obtained by the same method in [6, 7] for a Feigenbaum spectrum. In the formula (4) α is a scale factor for the dynamical variable y in the corresponding critical situation (see Table 3), $\beta = \alpha^2$ is the scale factor for the Feigenbaum and bicritical points, and $\beta = \alpha^4$ for the tricritical point and the point BT.

As one can see from Eq. (4), the quantity $\gamma = (1/4\alpha^2 + 1/4\beta^2)$ characterizes the average difference between the neighboring levels of the subharmonics while the coefficient $\kappa = |2\alpha\beta/(\alpha^2 + \beta^2)|$ characterizes the degree of nonuniformity of the distribution of the amplitudes of the subharmonics at a given level of the hierarchy. The numerical values of γ and κ are also presented in Table 3.

The Lyapunov exponents L near the critical points satisfy scaling relations of the form

$$L \longrightarrow L/2, \qquad \Lambda \longrightarrow \Lambda/\delta, \tag{5}$$

where Λ is a parameter measured along some characteristic direction and δ is a scale factor corresponding to this direction. Hence there follows a relation for the envelope of the Lyapunov exponent

$$L \propto \Lambda^{\chi}, \tag{6}$$

where the critical index χ is defined as $\chi = \ln 2/\ln \delta$. The quantities χ are presented in Table 3. We note that when in the parameter space the bicritical point is crossed two Lyapunov exponents become positive at the same time, i.e., bicritical behavior corresponds to the threshold of appearance of hyperchaos.

5. Conclusions. It is well known that one of the key ideas in the theory of bifurcations, theory of catastrophes, and in part the theory of phase transitions is the idea of "motion along codimension." In accordance with this idea, situations which are typical at first in one-parameter families and then in families with two, three, etc. parameters, are successively introduced and analyzed. The results of this work show clearly that an analogous approach is also frutiful in the theory of critical phenomena at the threshold of chaos. From this standpoint, the basic problems of the theory consist of searching for and classifying typical variants of the critical dynamics as a function of the number of significant parameters, determining the universality and scaling properties inherent to them, and finding canonical models describing each critical situation. It is also necessary to analyze the laws of coexistence of types of critical behavior in the parameter space of dynamical systems and methods for finding and identifying them experimentally.

At the present time a large number of systems exhibiting Feigenbaum paths to chaos are now known. The prevalence of this path is a result of the fact that the Feigenbaum critical point is typically encountered in systems with one controlling parameter (n = 1, Table 3). What can be said about the possibility of realization of other types of critical dynamics discussed here?

As regards tricritical behavior, such behavior becomes typical when two controlling parameters are present and, evidently, it can be observed in many real systems. The situation, studied here, of unidirectionally coupled systems is in this sense only one possible example. It can be realized in practice, in particular, on the basis of two nonlinear oscillatory circuits, excited by an external period signal [2]. As far as we know, the question of the experimental observation of tricritical behavior, which has been discussed thus far only in a formal mathematical model [5], has still not been posed.

Bicritical behavior and the multicritical point BT are apparently characteristic only for flow systems, since the introduction of the back effect of the second subsystem on the first subsystem destroys these types of critical dynamics. The experimental observation of bicritical behavior is described in [2]. It should be noted that critical phenomena of this type can also be observed in chains consisting of three, four, and more elements with one-way coupling, if the parameters of the subsystems can be controlled independently. In order to realize a bicritical situation two parameters must be selected while for the BT point three parameters must be selected.

LITERTURE CITED

1. M. Feigenbaum, Usp. Fiz. Nauk, 141, No. 2, 343 (1983).
2. B. P. Bezruchko, Yu. V. Gulyaev, et al., Dokl. Akad. Nauk SSSR, 287, No. 3, 619 (1986).
3. I. S. Aranson, A. V. Gaponov-Grekhov, and M. I. Rabinovich, Physica D, 33, No. 1-3, 1 (1988).
4. S. P. Kuznetsov, Izv. Vyssh. Uchebn. Zaved., Radiofiz., 28, No. 8, 991 (1985).
5. S.-J. Chang, M. Wortis, and J. A. Wright, Phys. Rev. A, 24, No. 5, 2669 (1981).
6. R. Huberman and A. Zisook, Phys. Rev. Lett., 26, No. 10, 626 (1981).
7. M. Nauenberg and J. Rudnik, Phys. Rev. B, 24, No. 1, 493 (1981).