On the mechanism for the onset of quasiperiodic oscillations in coupled Feigenbaum systems

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The mechanism of quasiperiodic oscillations in two identical symmetrically coupled Feigenbaum systems is investigated. A method is proposed for describing discrete mappings near the point of bifurcation by transforming from discrete to continuous time. This method makes it possible to analyze efficiently the dynamics of coupled systems. It is shown that quasiperiodic regimes should exist in certain regions of the nonlinearity and coupling parameters, irrespective of the type of coupling, and within these regions the frequency of the new oscillatory component is a function of the nonlinearity and the coupling. The threshold frequency of the quasiperiodic oscillations satisfies the following similarity laws: When the time scale of the fundamental oscillations is doubled the threshold frequency remains unchanged in the case of dissipative coupling and is rescaled by a factor of $|a|/2$ in the case of inertial coupling, where $a = -2.5029$ is the well-known Feigenbaum scale factor.

INTRODUCTION

The idea of using simple systems which exhibit chaos as elements for constructing composite models of more complicated systems is now widely dissimenated in the literature. In particular, systems constructed from subsystems in which the transition to chaos occurs through a sequence of period-doubling bifurcations were studied in Refs. 1-4. In this paper we shall confine our attention to situations when the subsystems are at the threshold of bifurcation. In this case the description of the individual subsystems simplifies and reduces, with the help of a transition from discrete to continuous time, to some canonical differential equation. For a composite model there correspondingly arises a system of such equations, which can be analyzed by the traditional methods of the qualitative theory. This approach can be extended to systems of different structure, including distributed systems. Here as a specific illustration of this approach we shall study and explain the phenomenon, observed in numerical and physical experiments, of quasiperiodic oscillations in coupled Feigenbaum systems.

1. DYNAMICS OF AN INDIVIDUAL SYSTEM AT THE THRESHOLD OF PERIOD-DOUBLING BIFURCATION

We first study the simplest model mapping that exhibits transition to chaos through a cascade of period-doubling bifurcations:

$$X_{n+1} = \lambda - X_n^2, \tag{1}$$

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where X is the dynamical variable, n is discrete time, and λ is the control parameter, and we shall explain for this example the idea of transferring to a description of the dynamics in continuous time.

Let the value of the control parameter in Eq. (1) be close to λ_n – the point of the doubling bifurcation of a 2^n cycle, and X_0 is an element of this cycle. Consider the behavior of a small perturbation of the dynamical variable near the point X_0 as a function of the supercriticality $\Lambda = \lambda - \lambda_n$. We perform the mapping (1) 2^{n+1} times with the initial condition $X_0 = X_0 + \xi_0$ and we study the result of this procedure as a function of the initial element 2^{n+1}. Expanding the obtained function in a series in the small quantities Λ and ξ_0 we arrive at the approximate mapping after 2^{n+1} steps of discrete time

$$\xi_{2^{n+1}} = \xi_0 + A\Lambda\xi_0 + B\xi_0^3. \tag{2}$$

Here

$$A = \frac{\partial^2 X_{2^{n+1}}}{\partial X_0 \partial\lambda}, \quad B = \frac{1}{6}\frac{\partial^3 X_{2^{n-1}}}{\partial X_0^3}$$

are derivatives calculated at the point $X_0 = X_0$; $\lambda = \lambda_n$. Near the bifurcation the change in the perturbation over two periods of the starting cycle is small. For this reason it is possible to transfer from the expression (2) to a differential equation in which the time is measured in units of 2^{n+1},

$$\dot{\xi} = A\Lambda\xi + B\xi^3. \tag{3}$$

The coefficients A and B can be found numerically with the help of a simple algorithm. Differentiating the mapping (1) we arrive at a chain of recurrence relations:

$$\begin{aligned}
'X &= \lambda_n - X^2, \quad [X]_{2^n} = [X]_0;\\
'X_{X_0} &= -2XX_{X_0}, \quad [X_{X_0}] = 1;\\
'X_\lambda &= 1 - 2XX_\lambda, \quad [X_\lambda]_{2^n} = [X_\lambda]_0;\\
'X_{\lambda X_0} &= -2XX_{\lambda X_0} - 2X_\lambda X_{X_0}, \quad [X_{\lambda X_0}]_0 = 0;\\
'X_{X_0X_0} &= -2XX_{X_0X_0} - 2X_{X_0}^2, \quad [X_{X_0X_0}]_0 = 0;\\
'X_{X_0X_0X_0} &= -2XX_{X_0X_0X_0} - 6X_{X_0}X_{X_0X_0}, \quad [X_{X_0X_0X_0}]_0 = 0.
\end{aligned} \tag{4}$$

Here the initial or boundary conditions that the mapping must satisfy are given next to the mapping for each variable. We note that the coefficient B depends on which of the elements of the cycle is taken as the starting element. The values of A and B for all n from 1 to 7 are presented in Table I in the two left-hand columns for the case when the cycle element closest to the extremum is chosen as the initial element.

We now discuss the properties of Eq. (3). For $\Lambda < 0$ it has one stationary point $\xi = 0$, which becomes unstable at $\Lambda = 0$, after which there appear two stable symmetric positions of equilibrium $\xi = \pm(-A\Lambda/\Lambda B)^{1/2}$. This bifurcation corresponds to doubling of the period of the 2^n-cycle of the starting mapping (1). Figure 1a shows a a plot of the dependence of the positions of equilibrium ξ on the supercriticality Λ. This is a small piece of the well-known "Feigenbaum tree"[5] in a neighborhood of the bifurcation point. The temporal dynamics of Eq. (3) for $\Lambda = 0.01$ for the case of a 2-cycle is illustrated in Fig. 1b (solid line). The dashed line in the same figure shows the dependence of the solution of the starting mapping (1) on the discrete time n. Comparing these two plots shows that the obtained approximate canonical equation (3) adequately describes the dynamics of the quadratic mapping and hence also the characteristic universal dynamics near a period-doubling-bifurcation point. The universal properties of the transition to chaos are manifested for Eq. (3) in the scaling properties of the coerffficients: As one can see from Table I, the coefficient A is scales by the factor $\delta = 4.6692$ at a transition to a cycle with doubled period and the coefficient B is scaled by a factor a^2, where $a = -2.5029$ and δ and a are the well-known Feigenbaum constants.

2. STARTING EQUATIONS OF COUPLED FEIGENBAUM SYSTEMS

We now analyze two coupled systems. In Ref. 4 it was shown, with the help of a renormalization group analysis, that for the problem at hand a system of two logistic maps can be used as the universal model:

$$\begin{aligned}
X_{n+1} &= \lambda - X_n^2 + \varepsilon(K_1(X_n - Y_n) + K_2(X_n^2 - Y_n^2)),\\
Y_{n+1} &= \lambda - Y_n^2 + \varepsilon(K_1(Y_n - X_n) + K_2(Y_n^2 - X_n^2)),
\end{aligned} \tag{5}$$

where X and Y are the dynamical variables of the first and second subsystems; λ and ε are the nonlinearity and coupling parameters; and, the coefficients $K_{1,2}$ determine the character of the coupling.

Dissipative coupling obtains $K_1 = 0$ and $K_2 = 1$ and inertial coupling obtains for $K_1 = 1$ and $K_2 = -0.088$. According to the results of renormalization group analysis there are no other significant types of coupling.

In the absence of coupling ($\varepsilon = 0$) the system (4) exhibits the usual cascade of period-doubling bifurcations that ends with a transition to chaos. The difference from a single quadratic mapping lies in the fact that for a composite system each state of period 2^n can be realized in 2^n different ways, differing by the phase shift of the oscillations in the subsystems by one step of discrete time. The introduction of coupling modifies these regimes, but it is still possible to classify

TABLE I

n	A	B	B_1	Dissipative coupling			n	Inertial coupling		
				$F=F_1$	H	H_1		$F=F_1$	H	H_1
1	8.0	−11.716	−68.284	−28.0	−9.657	1.657	1	−17.536	−8.807	1.511
2	35.793	−72.028	−347.503	−52.495	−17.572	3.642	2	40.782	19.940	−4.133
3	165.756	−449.477	−2254.23	−105.288	−34.343	6.848	3	−102.900	−51.260	10.221
4	772.507	−2813.25	−14093.97	−210.629	−69.431	13.859	4	257.066	127.950	−25.540
5	3605.60	−17620.41	−88334.22	−421.213	−138.667	27.661	5	−643.592	−320.337	63.899
6	16833.83	−110379.3	−553353.9	−842.445	−277.411	55.336	6	1610.40	801.598	−159.897
7	78599.14	−691470.1	−3468526	−1684.88	−554.804	110.667	7	−4031.43	−2006.57	400.252

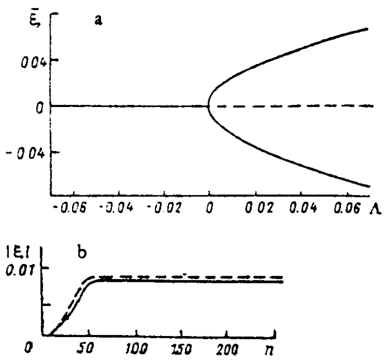

FIG. 1. The positions of equilibrium of Eq. (3) as a function of the supercriticality (a) and the temporal dynamics of the perturbation of a 2-cycle (b) for a quadratic mapping.

them according to the magnitude of the shift of the oscillations of the uncoupled sybsystems.[4,6] In Refs. 4, 6, and 7 it is shown that for regimes which are characterized by a shift of the phase of the oscillations of the subsystems by one-half the period quasiperiodic motions can arise in a certain range of values of the parameters λ and ε.

3. DYNAMICS OF COUPLED SYSTEMS NEAR THE BIFURCATION POINT OF AN INDIVIDUAL SUBSYSTEM

Extending the arguments of Sec. 1 to the case of coupled Feigenbaum systems (5), it is not difficult to obtain a system of differential equations describing the dynamics of the composite system in a neighborhood of the bifurcation point:

$$\begin{aligned}\dot\xi &= A\Lambda\xi + B\xi^3 + F\varepsilon\xi + H\varepsilon\eta,\\ \dot\eta &= A_1\Lambda\eta + B_1\eta^3 + F_1\varepsilon\eta + H_1\varepsilon\xi.\end{aligned} \tag{6}$$

Here ξ and η are the perturbations of the elements X and Y of the period-2^n cycle and the coefficients F and H are given by the expressions

$$\begin{aligned}F &= \frac{\partial^2 X_{2^{n+1}}}{\partial X_0 \partial\varepsilon}, \quad H = \frac{\partial^2 X_{2^{n+1}}}{\partial Y_0 \partial\varepsilon},\\ F_1 &= \frac{\partial^2 Y_{2^{n+1}}}{\partial Y_0 \partial\varepsilon}, \quad H_1 = \frac{\partial^2 Y_{2^{n+1}}}{\partial X_0 \partial\varepsilon},\end{aligned} \tag{7}$$

where the derivatives are calculated at the point $\varepsilon = 0$, $\lambda = \lambda_n$, $X_0 = \tilde{X}$, $Y_0 = \tilde{Y}_0$. The numerical procedure for determining the coefficients F and H is analogous to that described in Sec. 1. It is only necessary to take into account the following additional recurrence relations:

$$\begin{aligned}
'X_\varepsilon &= -2XX_\varepsilon + K_1(X - Y) + K_2(X^2 - Y^2), \quad [X_\varepsilon]_{2^n} = [X_\varepsilon]_0;\\
'Y_\varepsilon &= -2YY_\varepsilon + K_1(Y - X) + K_2(Y^2 - X^2), \quad [Y_\varepsilon]_{2^n} = [Y_\varepsilon]_0;\\
'X_{X_0\varepsilon} &= -2XX_{X_0\varepsilon} - 2X_{X_0}X_\varepsilon + K_1X_{X_0} + 2K_2XX_{X_0}, \quad [X_{X_0\varepsilon}]_0 = 0;\\
'X_{Y_0\varepsilon} &= -2XX_{Y_0\varepsilon} - K_1Y_{Y_0} - 2K_2YY_{Y_0}, \quad [X_{Y_0\varepsilon}]_0 = 0;\\
'Y_{Y_0\varepsilon} &= -2YY_{Y_0\varepsilon} - 2Y_{Y_0}Y_\varepsilon + K_1Y_{Y_0} + 2K_2YY_{Y_0}, \quad [Y_{Y_0\varepsilon}]_0 = 0;\\
'Y_{X_0\varepsilon} &= -2YY_{X_0\varepsilon} - K_1X_{X_0} - 2K_2XX_{X_0}, \quad [Y_{X_0\varepsilon}]_0 = 0.
\end{aligned}$$

Here we study the antiphase regim s, which give rise to quasiperiodic attractors. Table I gives the values of all coefficients F and H for n = 1-7 for the cases of intertial and dissipative coupling. For the first subsystem the element closest to the extremum is taken for X_0 and for the econd subsystem the element shifted in phase by one-half the period of the cycle is taken for Y_0.

We shall say a few words about the structure of Eqs. (6). It can be shown that for antiphase motions the relations $F = F_1$, $BH = -B_1H_1$ are satisfied. This makes it possible to simplify the system (6):

$$\begin{aligned}\dot\xi &= C\xi - \xi^3 - D\theta;\\ \dot\theta &= C\theta - \theta^3 + D\xi,\end{aligned} \tag{9}$$

where $\theta = (B_1/B_\eta)^{1/2}$, $C = -(A\Lambda + F\varepsilon)/B$, $D = -(\,|\,H_1H\,|\,)^{1/2}\varepsilon/B$.

This system of equations contains only two significant parameters C and D, which must be interpreted as the normalized supercriticality and the normalized coupling. Thus analysis of the dynamics of coupled mappings in a neighborhood of bifurcation points of an individual subsystem reduces to a two-parameter analysis, and in addition the analysis is of the same type for different types of couplings and for all bifurcation points.

Figure 2 shows a map of the dynamical regimes of the system (9) in the (supercriticality, coupling parameter) plane. The straight line $\Lambda = -F\varepsilon/A$ (l_1) and the broken lines $\Lambda = (2[2\,|\,HH_1\,|\,]^{1/2}\,|\,\varepsilon\,|\, - F\varepsilon)/A(l_2)$ divide the map into three characteristic regions, in which regimes of same·type are realized; bifurcations occur when the lines are intersected. Proceeding from bottom to top in this place we observe first a single stationary point – a stable focus with the coordinates $\bar\xi = 0$, $\bar\theta = 0$. On the line l_1 it becomes unstable owing to the Andronov–Hopf bifurcation, as a result of which a stable limit cycle with frequency $\omega_0 = D$ appears. As the supercriticality is further increased the period of the oscillations increases and four regions of condensation of the mapping points are formed on the limit cycle. On the bifurcation line l_2 the period of the oscillations becomes infinite and as a reuslt of a saddle–node bifurcation four more pairs of stationary points, which are either stable nodes or saddles,

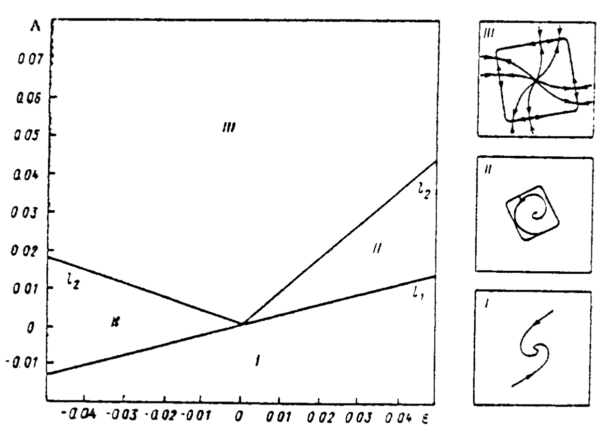

FIG. 2. Map of the dynamical regimes of the approximate canonical equations for two coupled Feigenbaum systems.

appear in the cycle. The coordinates of these points are given by the expressions

$$\xi = \pm\sqrt{C - \left(\varkappa \pm \sqrt{\varkappa^2 + D^2}\right)/2},\qquad \eta = \pm\sqrt{C - \left(\varkappa \mp \sqrt{\varkappa^2 + D^2}\right)/2}, \tag{10}$$

where $\varkappa = (C + [C^2 - 8D^2]^{1/2})/2$ for the saddles and $(C - [C^2 - 8D^2]^{1/2})/2$ for the nodes. In Fig. 2 the qualitative phase portraits of the system re shown next to the parameter plane. The phase portraits obtained integrating the system (9) numerically with different values of C ($D = 0.15$), are presented in Fig. 3. A plot of one of the coordinates of the equilibrium position versus the supercriticality for fixed ε and $n = 1$ is presented in Fig. 4. The solid lines correspond to stable regimes and the dashed lines correspond to unstable regimes; in the region of the limit cycle the amplitude of the oscillations is plotted along the vertical axis. For the starting mapping (5) it follows from what has been said above that on the boundary l_1 the 2^n-cycle is replaced by a quasiperiodic attractor, which in turn becomes unstable on the line l_2, giving rise to two symmetric cycles whose period is equal to twice initial period.

The dynamics of the solution of the system (6) and the behavior of the perturbation of the mapping (4) in different regions of the parameters Λ and ε for $n = 1$-3 are compared in Fig. 5. It is obvious that the agreement between the model and the original system is good.

We now discuss the properties of universality and similarity of the[4] approximate canonical equations (6); these properties follow from the analogous properties of coupled Feigenbaum mappings. As one can see from Table I, as n varies the coefficients F and H are rescaled by a factor of two for purely dissipative coupling and by a factor a for inertial coupling. In the general case the coefficients F and H are a superposition of the known coefficients for inertial coupling (i) and dissipative coupling (d) $F = K_1F_i + K_2F_i$, $H = K_1H_i + K_2H_d$. An important consequence of these properties of the coefficients is the scaling of the threshold frequency of quasiperiodic motion $\omega_0 \sim (\mid HH_1 \mid)^{1/2}$. As n varies the threshold frequency remains unchanged in the case of dissipative coupling and is rescaled by a factor $\mid a \mid /2$ in the case of intertial coupling. Figure 6 shows a series of plots illustrating the change in the dependence of the frequency of quasiperiodic motion ω on the supercriticality Λ for successive period doublings. The results were obtained by direct numerical modeling of the dynamics of the mapping (4). The quantity $W = \omega/\varepsilon$ is plotted along the ordinate axis and the quantity $L = \beta(\Lambda - \Lambda_n)/\varepsilon$, where Λ_n is the value of the supercriticality for which bifurcation of creation of a torus from a 2^n-cycle occurs, is plotted along the abscissa axis; for dissipative coupling $\beta = (\delta/2)^{n-1}$ and in the case of inertial and mixed couplings $\beta = (\delta/\mid a \mid)^{n-1}$.

4. QUALITATIVE EXPLANATION OF THE APPEARANCE OF QUASIPERIODICITY

The mechanism of the appearance of quasiperiodic oscillations can be qualitatively understood from the following analysis. Let the value of the nonlinearity parameter be close to the bifurcation value for a cycle of some period (for example, period 2) of a separate uncoupled mapping. For $\varepsilon = 0$

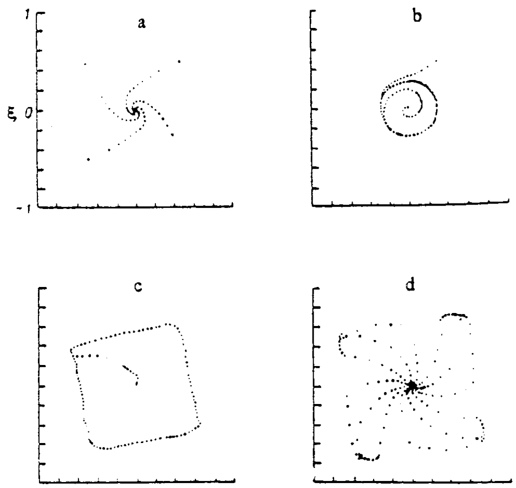

FIG. 3. Phase portraits of the system (9). a) 0.1; b) 0.3; c) 0.05; d) 0.43.

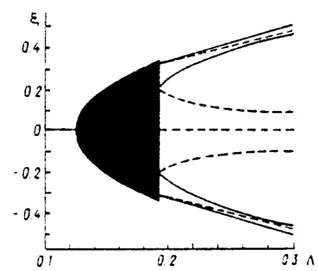

FIG. 4. The coordinates of the equilibrium position versus the supercriticality for the approximate canonical equations of two coupled systems.

period-2 regimes will exist in both subsystems. Consider the situation when the subsystems oscillate in antiphase. In Fig. 7 the dots denote successive times and the numbers 1 and 2 denote the states of the subsystems at these times. We now perturb the second subsystem (the upward pointing arrow at the initial moment of discrete time in the figure). Over one period of the cycle this perturbation changes sign (flips), since the cycle multiplier is equal to approximately -1. Now let the coupling between the subsystems be different from zero. The action of the second subsystem at the kth step will result in the appearance of a small perturbation in the first subsystem at the $(k + 1)$-st step. The sign of this perturbation will depend on the sign of ε. For definiteness, let it be directed in the same direction. It is obvious that the perturbation of the first subsystem will grow in time owing to the constant influence of the second subsystem, until, finally, it itself starts to exert a back effect. Because of the symmetry of the coupling the new perturbation of the second susbystem is directed in the same direction as the perturbation of the first subsystem giving rise to it, i.e., it is directed opposite to the initial perturbation. For this reason, the perturbation in the second subsystem will, in time, decrease to zero. In the process, the perturbation in the first subsystem will reach a maximum. The process will then repeat, since we have returned to the starting state, with the only difference that the subsystems have exchanged roles. If the cycle multiplier is greater than one, then the process described above is accompanied by

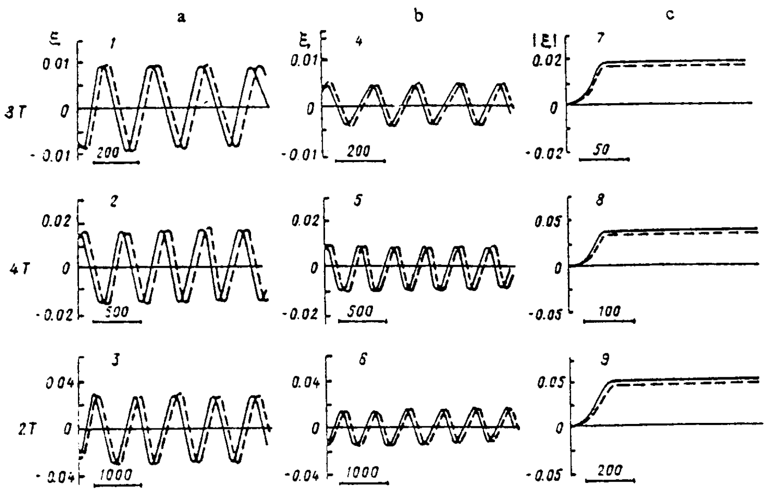

FIG. 5. Temporal dynamics of the perturbations of antiphase cycles of coupled systems in different regions of the parameters Λ and ε. a, b) 0.02; c) 0.0022; Λ: 1, 4) 0.001454; 2, 5) 0.003401; 3, b) 0.008; 7) 0.02; 8) 0.004901; 9) 0.01.

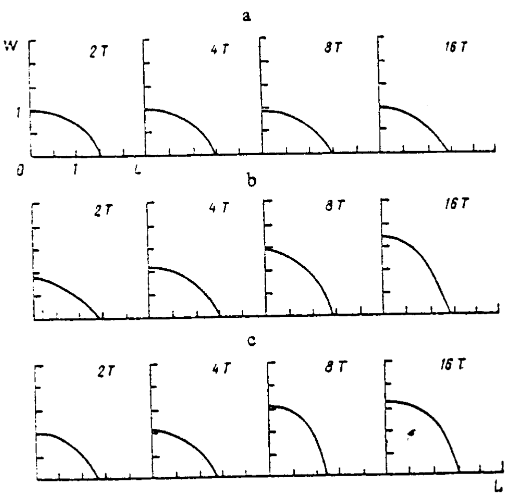

FIG. 6. The frequency of quasiperiodic motion versus the supercriticality. a) Dissipative coupling, b) inertial coupling, and c) mixed coupling: K_1 = 1, K_2 = 0.

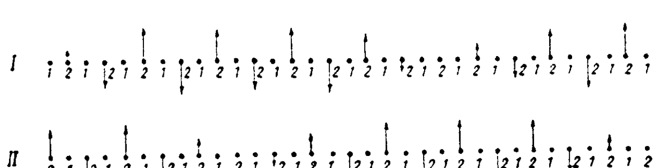

FIG. 7. Qualitative explanation of the appearance of quasiperiodicity. I and II are the first and second subsystems, respectively.

an increase in the amplitude of the oscillations. Nonlinearity will stabilize the amplitude at some level. As follows from the arguments presented above (and in agreement with Eq. (9)), the period of the renewed oscillations is proportional to the coupling parameter.

CONCLUSIONS

The method, developed in this work, for transferring from a discrete model to canonical differential equations near the bifurcation point makes it possible to analyze effectively the dynamics of coupled systems. It was shown, in particular, that in certain regions of the nonlinearity and coupling parameters quasiperiodic regimes will exist irrespective of the form of the coupling between the subsystems, and in addition the frequency of the new oscillatory component is directly proportional to the coupling parameter and, for fixed ε, it decreases as the supercriticality increases. This means that in the general case it is not in rational ratio with the discrete-time step in the starting mapping (4). For this reason, in the region of existence of quasiperiodicity the regions of synchronization of the frequencies – periodic motions and strictly quasiperiodic oscillations – should have a complicated structure. Such structure is indeed observed in a physical experiment for the example of two dissipatively coupled nonlinear circuits under an external periodic perturbation.[8]

[1]J.-H. Guan, M. Jung, and L. Nardussi, *Phys. Rev. A* 28, 1662 (1983).
[2]S. P. Kuznetsov, *Izv. Vyssh. Uchebn. Zaved., Radiofiz.* 24, 888 (1986).
[3]B. P. Bezruchko, Yu. V. Gulyaev, S. P. Kuznetsov, and E. P. Seleznev, *Dokl. Akad. Nauk SSSR* 287, 619 (1986) [*Sov. Phys. Dokl.* 31, 258 (1986)].
[4]S. P. Kuznetsov, *Izv. Vyssh. Uchebn. Zaved., Radiofiz.* 28, 991 (1985).
[5]M. Feigenbaum, *Los Alamos Sci.* 1, 4 (1980).
[6]V. V. Astakhov, B. P. Bezruchko, V. I. Ponomarenko, and E. P. Seleznev, *Izv. Vyssh. Uchebn. Zaved., Radiofiz.* 31, 627 (1988).
[7]V. V. Astakhov, B. P. Bezruchko, Yu. V. Gulyaev, and E. P. Seleznev, *Pis'ma Zh. Tekh. Fiz.* 15(3), 60 (1989) [*Sov. Tech. Phys. Lett.* 15, 105 (1989)].
[8]V. V. Astakhov, B. P. Bezruchko, S. P. Kuznetsov, and E. P. Seleznev, *Pis'ma Zh. Tekh. Fiz.* 14, 37 (1988) [*Sov. Tech. Phys. Lett.* 14, 16 (1988)].

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