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Three-parameter scaling for one-dimensional maps

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Abstract

Universality and scaling properties are investigated in three-parameter families of one-dimensional period doubling maps in the context of a "multiparameter criticality" concept which considers critical situations at the border of chaos in order of increasing codimension.

A commonly recognized research program of bifurcation and catastrophe theories is based on the idea of consecutive consideration of phenomena starting with those of low codimension and then moving towards higher and higher codimensions. It includes also classification of the phenomena and the establishment of laws of subordination: how are the bifurcations (catastrophes) of lower codimension enclosed in the parameter space of bifurcations (catastrophes) of higher codimensions. We develop in this paper an analogous approach to the problem of transition to chaos in one-dimensional maps. Since the dynamics at the border of chaos is often spoken of as critical behaviour of nonlinear systems, this research direction may be referred to as *a theory of multiparameter criticality*.

It is known that in *one-parameter families* of one-dimensional maps with a quadratic extremum, the period doubling bifurcations obey universality and scaling laws, described by Feigenbaum [1] with the help of the renormalization group (RG) approach. Particularly, the accumulation rate of period doubling bifurcations is determined by the universal factor $\delta = 4.669201...$.

For *two-parameter families* of maps having two quadratic extrema (*bimodal maps*) a curve may exist in the parameter plane, defined by a condition that one extremum is mapped to another one after an iteration. On this curve the double iterated map has, apparently, a fourth order (quartic) extremum. Thus, the period doubling cascade (if it is observed on this curve) obeys a specific law, which differs from Feigenbaum's. Located at the border of chaos the accumulation point of this cascade is a codimension-two point which was called *tricritical* and denoted by a symbol T [2]. In the parameter plane the tricritical points appear as the terminal points of Feigenbaum's critical lines. In a small vicinity of the tricritical point a two-parameter vector scaling exists [2,3]: the topography of the dynamical regimes reproduces itself under the rescaling along the appropriately chosen axes ("scaling coordinates") by the factors of $\delta_1 = 7.284686...$ and $\delta_2 = 2.857124...$, respectively [#1].

[#1] In this paper we do not consider other types of codimension-two points which exist at the borderline of chaos and are associated with more complex solutions than the RG equation fixed points (see Ref. [4]).

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Now we turn to *three-parameter* families of one-dimensional maps $x \to f(x)$. Four distinct situations may appear typically along some curve lines in the three-dimensional parameter space:

(i) The function $f(x)$ has vanishing second and third derivatives at the extremum point (Fig. 1a).

(ii) The function $f(x)$ has both a quadratic extremum and a cubic inflection point; the quadratic extremum is mapped to the cubic point (Fig. 1b).

(iii) The function $f(x)$ has both a quadratic extremum and a cubic inflection point; the cubic point is mapped to the quadratic extremum (Fig. 1c).

(iv) The function $f(x)$ has three quadratic extrema, the first extremum being mapped exactly to the second one, and the latter in turn to the third one (Fig. 1d).

If we take a point on curve (i), then the function $f(x)$ has a fourth order extremum. On curves (ii) and (iii) the second iteration of the map, $f(f(x))$, has a sixth order extremum. Finally, on curve (iv) the third iteration, $f(f(f(x)))$, has an eighth order extremum. So, if we are moving along one of curves (i)–(iv), then the observed period doubling cascade will be analogous to the case of the map $x \to 1 - \lambda |x|^m$ with appropriate $m = 4$, 6, or 8. The period doubling cascade in situation (i) leads to the tricritical behaviour. In cases (ii)–(iv) other types of criticality arise. To distinguish them we introduce symbols S, S′, and E, respectively (the first letters of the words "six" and "eight").

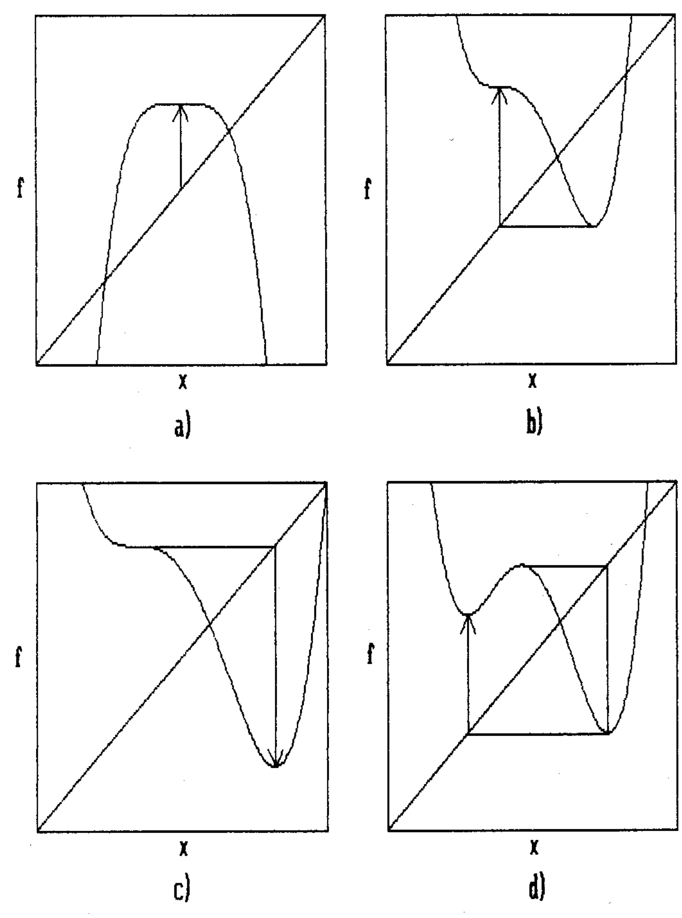

Fig. 1. Four characteristic configurations of one-dimensional maps giving rise to distinct types of three-parameter criticality: (a) T type, (b) S type, (c) S′ type, (d) E type.

To reveal the universality and scaling properties for three-parameter criticality we should appeal to the Feigenbaum theory extension on the maps with a non-quadratic extremum [5,6]. The basic equation which permits one to find the universal map g describing large-time-scale dynamics at the onset of chaos is Feigenbaum's RG equation

$$g(x) = ag(g(x/a)) , \tag{1}$$

where a is a scaling factor which must be defined during the solution process. The function $g(x)$ should be sought in the form of an expansion containing terms 1, $|x|^m$, $|x|^{2m}$, $|x|^{3m}$, ..., where m is the extremum order. The subject of investigations in Refs. [5,6] was the solution dependence on the value of m which might be an arbitrary real number. Here we are interested in special cases of $m = 4$, 6, and 8 only. We have reproduced the solutions of Eq. (4) with high enough accuracy. Scaling factors and polynomial approximations for functions g are presented in Tables 1 and 2.

The next step of the analysis is to consider the eigenvalue spectrum for the linearized RG equation

$$\delta \cdot h(x) = a[g'(g(x/a))h(x/a) + h(g(x/a))] . \tag{2}$$

Those δ eigenvalues which are greater than unity in modulus are of interest since they are essential for the multiple iteration behaviour of the RG equation solution near a fixed point.

The largest eigenvalues for T, S, and E critically types, δ_{T1}, δ_{S1}, and δ_{E1} can be found numerically from Eq. (2). Other eigenvalues exceeding unity in modulus are integer powers of a up $m-1$ excluding 1. The eigenvalue $\delta = a$ is not relevant because it relates to a perturbation of the RG equation solution (eliminated) by an infinitesimal shift of the origin for the

Table 1
Scaling factors for codimension-three critical points

Type	Orbit scaling factor a	Parameter space scaling factors		
		δ_1	δ_2	δ_3
T	−1.6903029714	7.284686217	−4.829405415	2.857124135
S	−1.4677424503	9.296246833	4.640870187	2.154267900
S′	−1.4677424503	9.296246833	−3.161910446	2.154267900
E	−1.3580172791	10.94862427	3.401113956	1.844210930

Table 2
Polynomial approximation for solutions of the RG equation (1)

$$g_T(x)=1-1.83410790700x^4+0.01296222601x^8+0.31190173945x^{12}-0.06201465160x^{16}-0.03753928670x^{20}$$
$$+0.01764731251x^{24}+0.00193826520x^{28}-0.00282047096x^{32}+0.00011545666x^{36}+0.00039947082x^{40}-0.00002479251x^{44}$$
$$-0.00012164143x^{48}+0.00007043385x^{52}-0.00001797963x^{56}+0.00000190870x^{60}\,.$$
$$g_S(x)=1-1.907736962548x^6-0.332883482858x^{12}+0.712701624335x^{18}+0.035178927147x^{24}-0.272460026974x^{30}$$
$$+0.025549567361x^{36}+0.095651810755x^{42}-0.023675376504x^{48}-0.011912112668x^{54}-0.041730389115x^{60}+0.089665302592x^{66}$$
$$-0.083627645830x^{72}+0.049185410745x^{78}-0.019538741778x^{84}+0.004900692140x^{90}-0.000587039134x^{96}$$
$$g_E(x)=1-1.89735300202x^8-0.73884402381x^{16}+0.98978292252x^{24}+0.44569064453x^{32}-0.58599845479x^{40}$$
$$-0.28196762332x^{48}+0.39494697971x^{56}-0.03215266990x^{64}+0.39148984068x^{72}-1.21607366528x^{80}+1.45859462729x^{88}$$
$$-0.97341263480x^{96}+0.38717922254x^{104}-0.08672779729x^{112}+0.00847797841x^{120}$$

x variable. So, we obtain the following sets of eigenvalues,

for $m=4$: $\delta_{T1}, a_T^{\frac{2}{T}}, a_T^{\frac{3}{T}}$,

for $m=6$: $\delta_{S1}, a_S^{\frac{2}{S}}, a_S^{\frac{3}{S}}, a_S^{\frac{4}{S}}, a_S^{\frac{5}{S}}$,

for $m=8$: $\delta_{E1}, a_E^{\frac{2}{E}}, a_E^{\frac{3}{E}}, a_E^{\frac{4}{E}}, a_E^{\frac{5}{E}}, a_E^{\frac{6}{E}}, a_E^{\frac{7}{E}}$.

In accordance with the general principles of the RG analysis this means that an essential control parameter has to be associated with each member of these sets. In other words, the number of parameters needed typically for a family of systems to demonstrate the various types of critically is equal to three, five, and seven, respectively.

However, at the beginning of the paper we provided reasonable arguments that the tricriticality may appear as a codimension-two phenomenon, and the S and E critically types occur as codimension-three phenomena. The reason for the apparent contradiction is that those arguments relate to one-dimensional maps being a particular, specific class of nonlinear systems. It can be said that the situations when an extremum of order $m=4$, 6, or 8 occurs in iterated one-dimensional maps are characterized by some kind of "hidden symmetry". Owing to this symmetry only some of the eigenfunctions of the linearized RG equation arise due to an arbitrary perturbation of the initial map. This results in existence of definite *selection rules*. For example, there are two eigenvalues δ_T and $a_T^{\frac{2}{T}}$ while observing codimension-two tricriticality in bimodal one-dimensional maps; the third eigenvalue $a_T^{\frac{3}{T}}$ is ruled out. However, for tricriticality arising in the three-parameter situation (i) all three eigenvalues δ_T, $a_T^{\frac{2}{T}}$ and $a_T^{\frac{3}{T}}$ are engaged. For cases (ii) and (iii) selection rules do not coincide: in the S case the relevant eigenvalues are δ_S, $a_S^{\frac{2}{S}}$, $a_S^{\frac{4}{S}}$, while in the S′ case they are δ_S, $a_S^{\frac{2}{S}}$, $a_S^{\frac{3}{S}}$. Finally, in the E type situation (iv) the list of essential eigenvalues is δ_E, $a_E^{\frac{2}{E}}$, $a_E^{\frac{4}{E}}$. To substantiate these rules one might consider an arbitrary perturbation of the initial map $f(x)$ and examine the Taylor expansion of the perturbation for the multiply iterated map. Among eigenfunctions obtained from Eq. (2) only those functions have to be taken which contain appropriate terms in their Taylor expansions. In Table 1 a summary of the eigenvalues is presented taking into account the selection rules.

In a small neighborhood of a codimension-three critical point the normalized map describing evolution of the x variable over a great number of 2^k steps belongs to the three-dimensional unstable manifold of the RG equation fixed point, namely

$$g_k(x)=G(x, C_1\delta_1^k, C_2\delta_2^k, C_3\delta_3^k) . \quad (3)$$

Here δ_i are the relevant eigenvalues of the linear operator (2), the factors C_i depend on the perturbation removing the map from the fixed point $g(x)=G(x, 0, 0, 0)$. The values C_i may be taken as the most suitable parameters to analyze dynamics near the critical point. Eq. (3) leads to an understanding of *universality* (the evolution operator g_k depends on the three parameters C_i only) and *scaling* (the substitution $C_i\to C_i/\delta_i$ leaves the evolution operator unchanged under time scale doubling $k\to k+1$). The parameters C_i are called *scaling coordinates.*

Let us consider the model one-dimensional map

$$x\to 1-Ax^2-Bx^4-Cx \quad (4)$$

to illustrate universality and scaling properties of the three-parameter critically. Table 3 presents the coordinates (A_c, B_c, C_c) of the T, S, S′, and E type critical points found as the accumulation points of period doubling cascades along curves (i)–(iv) in the parameter space (A, B, C).

To observe three-parameter scaling in the parameter space it is necessary to obtain explicit relations between parameters A, B, C and scaling coordinates C_1, C_2, C_3. For this we define a tangent coordinate system (ξ_1, ξ_2, ξ_3) which coincides with (C_1, C_2, C_3) within the linear approximation and is related to the parameters A, B, C via the affine transformation

$$(\Delta A, \Delta B, \Delta C)=\xi_1\boldsymbol{r}_1+\xi_2\boldsymbol{r}_2+\xi_3\boldsymbol{r}_3 .$$

Here ΔA, ΔB, ΔC denote deflections from the critical point. The vectors $\boldsymbol{r}_i$ should be calculated specifically for each critical point and for each concrete map.

However, the assumption of a linear relation between old and new coordinates appears to be insufficient for verification of the scaling properties for the S, S′, and E types of criticality. In general, we suppose that scaling coordinates C_i are connected with ξ_i via nonlinear relations

$$C_i=\xi_i+\sum a_{jkl}^{(i)}\xi_1^j\xi_2^k\xi_3^l, \quad i=1, 2, 3 , \quad (5)$$

where $a_{jkl}^{(i)}$ are some constants, and summation is carried out over all non-negative indices j, k, l, provided $i+j+k>0$. However, by taking into account the concrete values of δ_i we may simplify Eq. (5).

If in (5) we make the substitution $C_i\to C_i/\delta_i^n$ and $\xi_i\to\xi_i/\delta_i^n$, we obtain

$$C_i=\xi_i+\sum [(\delta_i/\delta_1^j\delta_2^k\delta_3^l)^n]a_{jkl}^{(i)}\xi_1^j\xi_2^k\xi_3^l .$$

Let us neglect the terms which do not increase with $n\to\infty$. In other words, we retain only the terms with the factor $\delta_i/\delta_1^j\delta_2^k\delta_3^l$ greater than unity.

So, we introduce the ersatz-scaling coordinates $\tilde{C}_i$:

– for tricriticality we simply suppose

$$\tilde{C}_1=\xi_1, \quad \tilde{C}_2=\xi_2, \quad \tilde{C}_3=\xi_3 ;$$

– for S type:

$$\tilde{C}_1=\xi_1+a_{002}^{(1)}\xi_3^2, \quad \tilde{C}_2=\xi_2, \quad \tilde{C}_3=\xi_3 ;$$

– for S′ type:

$$\tilde{C}_1=\xi_1+a_{002}^{(1)}\xi_3^2+a_{011}^{(1)}\xi_2\xi_3, \quad \tilde{C}_2=\xi_2, \quad \tilde{C}_3=\xi_3 ;$$

– for E type:

$$\tilde{C}_1=\xi_1+a_{002}^{(1)}\xi_3^2+a_{011}^{(1)}\xi_2\xi_3+a_{003}^{(1)}\xi_3^3,$$

Table 3
Codimension-three critical points for the map (4)

Type	A_c	B_c	C_c
T	0	1.594901356229	0
S	1.872448192264	−1.625205284712	1.094016101529
S′	1.379909480783	−0.557409701182	1.181821122326
E	2.449366934076	−1.260415730596	0.700954625016

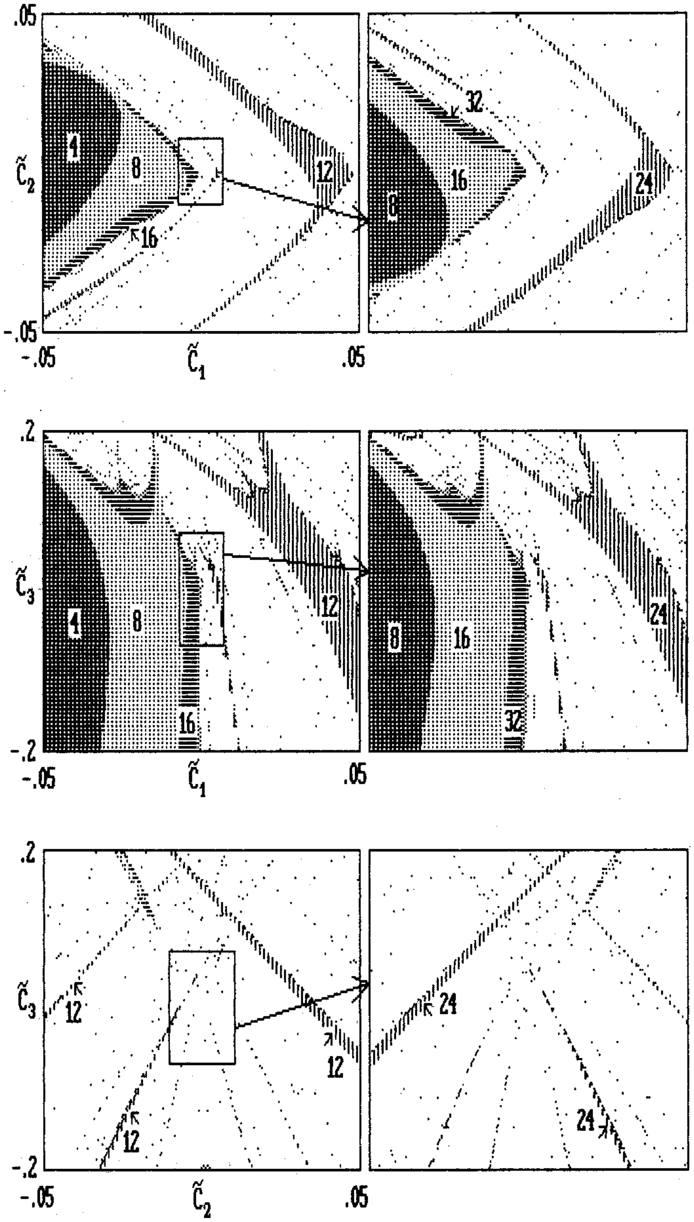

Fig. 2. Topography of the parameter space cross-sections near the codimension-three tricritical point for the map (4). The scaling coordinates (6) are used. Domains of different period attractive cycles are shown by different shading. The tricritical point is located exactly at the middle of the pictures.

$$\tilde{C}_2=\xi_2, \quad \tilde{C}_3=\xi_3 .$$

Taking the critical points of Table 3 we have calculated all the necessary coefficients and find the following relations between old and new coordinates:

the T point:

$$\Delta A=-0.80991808\tilde{C}_3,$$

$$\Delta B=\tilde{C}_1-0.60297275\tilde{C}_2+\tilde{C}_3, \quad \Delta C=\tilde{C}_2, \tag{6}$$

the S point:

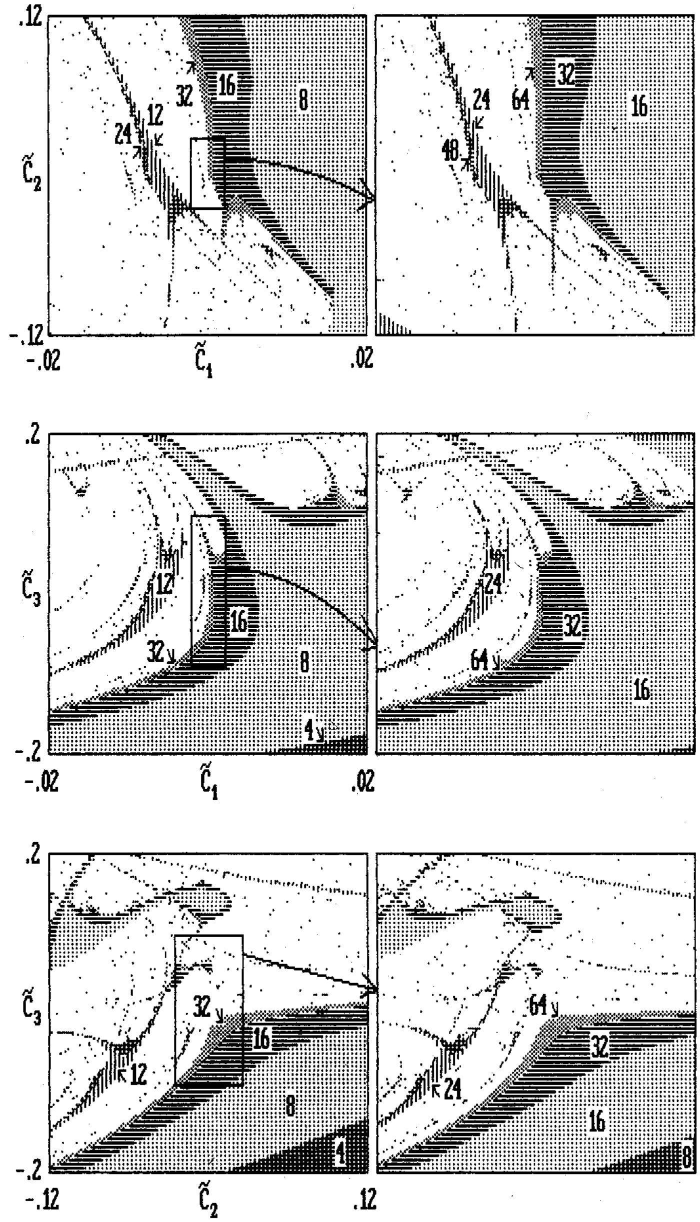

Fig. 3. Topography of the parameter space cross-sections near the S type critical point for the map (4). The scaling coordinates (7) are used. Domains of different period attractive cycles are shown by different shading. The critical point is located exactly at the middle of the pictures.

$$\Delta A=-0.52856711(\tilde{C}_1+0.780253\tilde{C}_3^2)$$
$$-0.80973728\tilde{C}_2+\tilde{C}_3,$$

$$\Delta B=(\tilde{C}_1+0.780253\tilde{C}_3^2)+\tilde{C}_2-0.848712657\tilde{C}_3,$$

$$\Delta C=-0.12666138(\tilde{C}_1+0.780253\tilde{C}_3^2)$$
$$+0.08221729\tilde{C}_2+0.59074778\tilde{C}_3, \tag{7}$$

the S′ point:

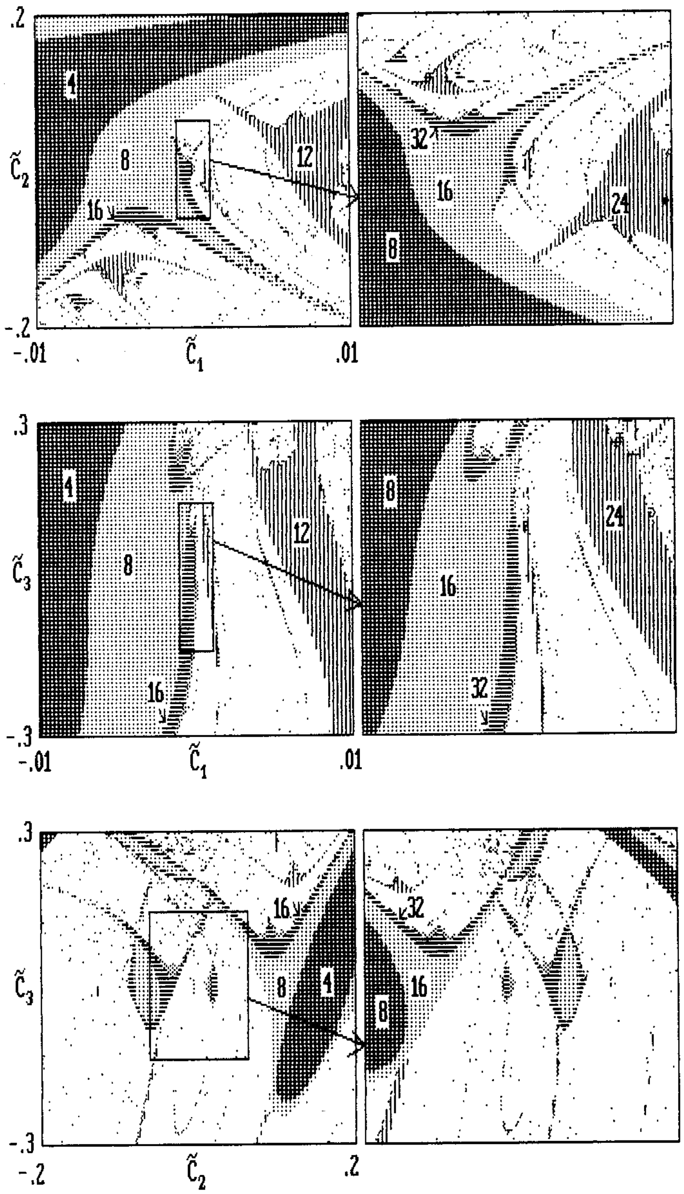

Fig. 4. Topography of the parameter space cross-sections near the S′ type critical point for the map (4). The scaling coordinates (8) are used. Domains of different period attractive cycles are shown by different shading. The critical is located exactly at the middle of the pictures.

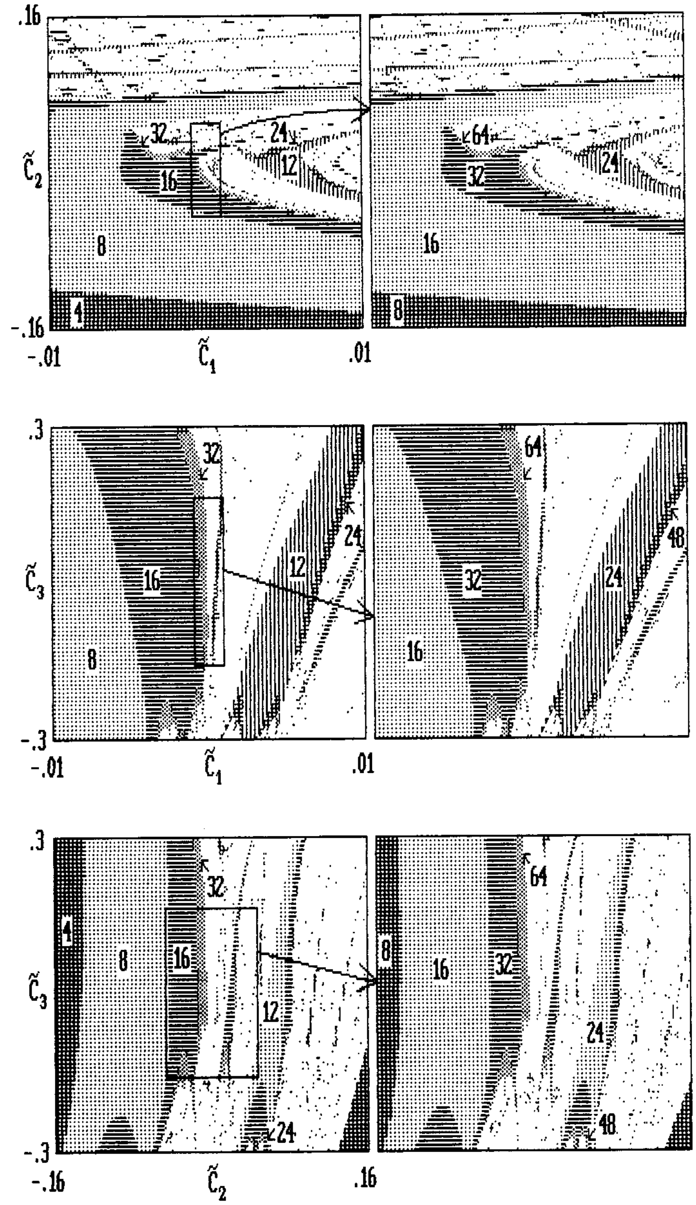

Fig. 5. Topography of the parameter space cross-sections near the E type critical point for the map (4). The scaling coordinates (9) are used. Domains of different period attractive cycles are shown by different shading. The critical point is located exactly at the middle of the pictures.

$$\Delta A = 0.83522562(\tilde{C}_1 - 0.0824\tilde{C}_3^2 + 0.1898\tilde{C}_2\tilde{C}_3) - 0.94065706\tilde{C}_2 + \tilde{C}_3\,,$$

$$\Delta B = -0.06885319(\tilde{C}_1 - 0.0824\tilde{C}_3^2 + 0.1898\tilde{C}_2\tilde{C}_3) + \tilde{C}_2 - 0.6381335\tilde{C}_3\,,$$

$$\Delta C = (\tilde{C}_1 - 0.0824\tilde{C}_3^2 + 0.1898\tilde{C}_2\tilde{C}_3) - 0.14833524\tilde{C}_2 - 0.07558422\tilde{C}_3\,, \qquad (8)$$

the E point:

$$\Delta A = (\tilde{C}_1 - 0.200585\tilde{C}_3^2 + 0.3443\tilde{C}_2\tilde{C}_3 - 0.13839\tilde{C}_3^3) + \tilde{C}_2 - 0.94564636\tilde{C}_3\,,$$

$$\Delta B = -0.55973816(\tilde{C}_1 - 0.200585\tilde{C}_3^2 + 0.3443\tilde{C}_2\tilde{C}_3 - 0.13839\tilde{C}_3^3) - 0.45710026\tilde{C}_2 + \tilde{C}_3\,,$$

$$\Delta C = -0.33884190(\tilde{C}_1 - 0.200585\tilde{C}_3^2 + 0.3443\tilde{C}_2\tilde{C}_3 - 0.13839\tilde{C}_3^3) + 0.31758086\tilde{C}_2 - 0.00651569\tilde{C}_3\,. \qquad (9)$$

In Figs. 2–5 the topography of different dynamical regimes is shown in neighborhoods of the critical points for the map (4). They correspond to cross-sections of the parameter space by coordinate surfaces ($\tilde{C}_i$, $\tilde{C}_j$). A critical point is located exactly at the center of each picture. Domains of cycles of different periods are shown with different shading. The outlined rectangles are shown separately with magnification by the factors δ_i and δ_j along $\tilde{C}_i$ and $\tilde{C}_j$ axes, respectively. It may be seen that location and shape of different domains are well reproduced in the chosen coordinates under the rescaling.

In conclusion we have to emphasize that the concept of multiparameter criticality seems to be a natural generalization of the idea of "roads to chaos" for a multiparameter case. It is a promising research field in nonlinear science because it leads to the introduction of new universal models and scaling properties at the boundary of chaos. In this paper we have surveyed only a small part of this field concerning the critical behaviour types associated with fixed points of the RG equation for period doubling one-dimensional maps.

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