# Analysis of scenarios of the transition to chaos in a discrete two-mode model of a free-electron laser

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A discrete two-parameter two-dimensional mapping, which describes the dynamics of two modes in a free-electron laser (FEL), is investigated by nonlinear dynamics methods. © *1999 American Institute of Physics.* [S1063-7850(99)01906-0]

When a free-electron laser (FEL) is employed as a generator of coherent electromagnetic radiation, great importance is attached to determining the character of the interaction of the longitudinal modes, as well as of the various routes to the onset of a multiple-frequency regime. This problem can be solved using a discrete approximation, in which the interaction of the electrons with the electromagnetic wave is taken into account only at the ends of the interaction space of the FEL.[1,2] Presuming that the gain band of the generator contains many nonequidistant longitudinal modes, we can assume that the amplification of each parasitic mode is determined only by the field of the fundamental mode and depends weakly on the other parasitic modes. The following discrete mapping, which describes the dynamics of two modes in an FEL, was obtained within this approximation in Ref. 3:

$$R^{-1}x_{n+1}-x_n=L^3|J_0(y_n)J_1(x_n)|,$$
$$R^{-1}y_{n+1}-y_n=L^3|J_0(x_n)J_1(y_n)|, \quad (1)$$

where $x_n$ and $y_n$ are the reduced amplitudes of the fundamental and parasitic modes; $n$ is the discrete time; $R$ is the transmission coefficient, which is equal to the product of the reflection coefficients of the mirrors; $L$ is the normalized length of the interaction space; and $J_0$ and $J_1$ are Bessel functions. Within the two-mode model, excitation of the fundamental mode in the absence of the competitive mode is described by $J_1$, and the influence of the competitive mode on it is described by $J_0$.

As was shown in Ref. 3, in the parameter plane of the transmission coefficient $R$ and the normalized length of the interaction space $L$ there are two lines: $L_{st}$, which corresponds to values of $L$ at which excitation of the generator occurs at a single fundamental mode, and $L_{cr}$, which corresponds to excitation of the parasitic mode. At a small supercriticality $L>L_{cr}$, a steady-state regime with the generation mainly of two longitudinal modes sets in.

The relation (1), however, merits more detailed scrutiny. From the standpoint of the theory of dynamic systems, it is a discrete two-parameter two-dimensional mapping. Here we present the results of an investigation of this mapping using nonlinear dynamics methods.

A map of the dynamic regimes in the $L$, $R$ plane and one magnified fragment of it are shown in Figs. 1 and 2. In these figures periodic regimes are indicated by different shades of gray, the numbers label the period of the cycle realized, and the letters $s$ and $n$ denote the cophasal and noncophasal regimes, respectively. Computer simulation reveals the $L_{st}$ and $L_{cr}$ lines, which, in terms of the discrete mapping, correspond to the following sequence of regimes. At $L<L_{st}$ there is a single stable fixed point at the origin of coordinates of the phase plane, i.e., at $x=0, y=0$. It becomes unstable at the $L=L_{st}$ line, and a new fixed point, for which $x>0$ and $y=0$, appears. As $L$ increases further from $L_{st}$ to $L_{cr}$, the value of $x$ increases to a definite value, and at the moment when $L=L_{cr}$ a fixed point, for which $x>y>0$, appears in a soft manner. This corresponds to the birth of a second parasitic mode.

It is not difficult, however, to see that such a picture, which was predicted in Ref. 3, is observed only on the right-hand side of the map of dynamic regimes, i.e., at fairly large values of the transmission coefficient $R$. At small values of $R$ the following occurs: the fixed point with the coordinates $x>0$, $y=0$ undergoes a period-doubling bifurcation, while the second mode remains undisturbed, and the birth of a

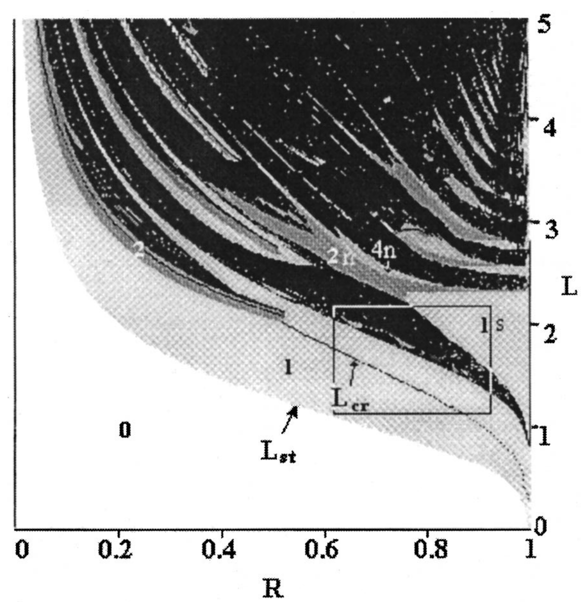


FIG. 1.

 

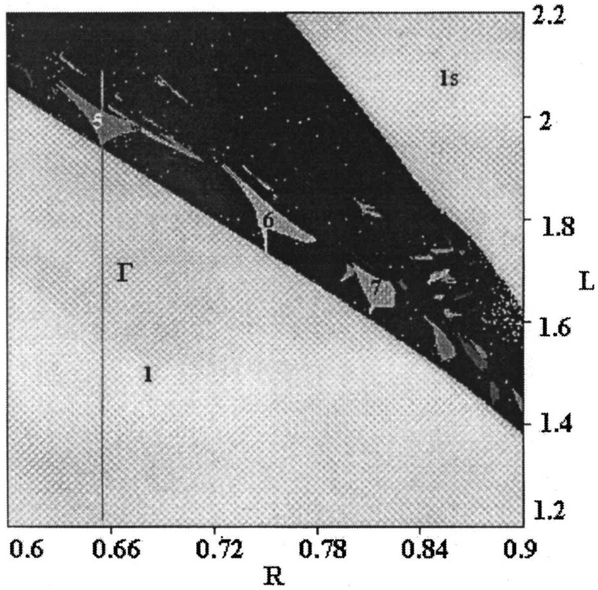


FIG. 2.

parasitic mode occurs only when $L$ increases further. Thus, in the range $R<0.5$, of the two possible mechanisms for a loss of stability of the single-frequency regime, viz., the amplitude regime (which is caused by modulation of one longitudinal mode) and the frequency regime (which is caused by competition between different modes), only the first is realized in the two-mode model of an FEL investigated.

Let us next examine the behavior of the mapping (1) when $L$ exceeds the critical value $L_{cr}$.

In the region $R<0.5$ the birth of two invariant curves from two points which previously belonged to a cycle of period 2 takes place on the $x$, $y$ phase plane as $L$ increases. (In terms of the original generator, self-modulation of the modes with incommensurate frequencies under consideration takes place). Then this ''2-torus'' breaks up through a lose of smoothness to form a chaotic attractor.

If the dimensionless length of the interaction space is increased at $R>0.5$, we discover a ''neutrality line,'' toward which numerous synchronization tongues extend. The magnified fragment of the map in Fig. 2 shows tongues, which correspond to rotation numbers equal to 1/5, 1/6, and 1/7. As we move along line $\Gamma$ on the parameter plane with increasing $L$, computer simulation of the phase portraits reveals synchronization with the formation of a cycle of period 5. Then an invariant curve arises from each of the five points of the cycle. As $L$ is increased further, ''corners'' appear on these curves, and an intricate attractor, which has a positive Lyapunov exponent, forms. Finally, after an even larger increase in $L$, two Lyapunov exponents become positive, attesting to the presence of hyperchaos in the system. As we move over the parameter plane, an invariant curve is observed near the left-hand boundary of the synchronization tongue with a rotation number of 1/5. This curve loses its smoothness and breaks up in accordance with the Afraĭmovich–Shil'nikov theorem to form a chaotic attractor.

Another region for the existence of a stable fixed point can be detected on the right-hand side of the map of dynamic regimes. In this case a regime of cophasal oscillations, in which $x=y$, is realized (i.e., in terms of the original system, the generation of a fundamental mode and a parasitic mode with equal amplitudes is observed). In Fig. 1 this region is labeled 1s. This regime of parameter variation also becomes unstable, and period doubling occurs. It is significant that the 2-cycle generator is asymmetric. Then a noncophasal 4-cycle and an 8-cycle appears, the boundaries of their regions of stability having a complex form. Thereafter, quasiperiodic, chaotic, and hyperchaotic regimes appear again.

We note that the regions of hyperchaos occupy a considerable fraction of the parameter plane, so that this regime is quite typical for the mapping (1).

Our treatment revealed fine details and interesting features of the complex regimes of a model mapping, which describes the dynamics of an FEL with a low-$Q$ electrodynamic system. The results presented here, especially the map of dynamic regimes, can serve as a useful ''guide'' for computer investigations of the original equations in partial derivatives.

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