Control of chaos in nonautonomous systems with quasiperiodic excitation

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(Submitted January 26, 1999)
Pis'ma Zh. Tekh. Fiz. **25**, 63–70 (August 26, 1999)

A procedure is suggested for controlling chaos in quasiperiodically excited systems by stabilizing an unstable torus, or creating a new one by means of a small action. For this purpose the controlled system is synchronized with one which is similar but in a state of stable quasiperiodic motion. The method is illustrated for a quasiperiodically perturbed logistic mapping and a Duffing oscillator. © *1999 American Institute of Physics.* [S1063-7850(99)02608-7]

1. INTRODUCTION

Controlling chaos is one of the most practically relevant directions in nonlinear physics. The most common method is the Ott–Grebogi–Yorke method of stabilizing unstable periodic orbits[1] and various modifications of this.[2,3] Its efficiency has been demonstrated for many autonomous and periodically excited systems. However, the problem of control in systems exposed to more complex multifrequency action has been left to one side. The present paper is devoted to searching for methods of solving this problem in the simplest case of a biharmonic action with two nonmultiple frequencies.

We shall analyze some characteristic features which are imposed on the control problem by the quasiperiodicity of the action. Primarily, this results in the absence of unstable periodic orbits which could be stabilized. The way out here is to use an unstable torus which can exist in the phase space of these systems. This is created as a result of bifurcations such as ''doubling'' and loss of torus symmetry and after a transition to chaos, is embedded in a chaotic attractor or exists outside it. However, the standard control procedure either requires a knowledge of the orbit being stabilized or the presence of an oscillator which generates a suitable reference signal.[1,2] A torus is a topologically more complex object than a cycle and a model of the global dynamics of the system is required to reconstruct it. The construction of such a model and the search for an unstable torus is in itself a fairly complex problem. Hence it is more productive to search for a suitable reference oscillator. As a reference orbit to stabilize an unstable torus, it seems reasonable *a priori* to use a torus evolutionally coupled with it, which existed in the system for other parameters (before loss of stability). The aim of the present study is to demonstrate the possibility of implementing this idea. A procedure is describing for stabilizing an unstable torus or creating a new orbit in its vicinity by locking the controlled system to one which is similar but in a state of stable quasiperiodic motion. This is illustrated for a quasiperiodically perturbed logistic mapping and a Duffing oscillator.

2. STABILIZATION OF AN UNSTABLE INVARIANT MAPPING CURVE

We shall analyze the two-dimensional mapping

$$x_{n+1}=f(x_n,\theta_n)=\lambda-x_n^2-\varepsilon\cos 2\pi\theta_n,$$
$$\theta_{n+1}=\theta_n+\omega,\ \mathrm{mod}\,1, \tag{1}$$

where $\omega=(\sqrt{5}-1)/2$ is the ''golden cross section.'' This mapping demonstrates a torus doubling bifurcation and a transition to a strange nonchaotic attractor and chaos. The unstable invariant curve generated by the torus doubling $x=\varphi(\theta)$, $0\leqslant\theta\leqslant 1$ is embedded in a chaotic attractor (Fig. 1a). In order to stabilize the invariant curve, we can model the mapping (1) as follows:

$$x_{n+1}=f(x_n,\theta_n)+g(x_n,\theta_n),$$
$$\theta_{n+1}=\theta_n+\omega,\ \mathrm{mod}\,1, \tag{2}$$

where the function $g(x,\theta)$ is the control action which must satisfy the following two constraints:

$$g(\varphi(\theta),\theta)=0,$$
$$\int_0^1 \ln|f'_x(\varphi(\theta),\theta)+g'_x(\varphi(\theta),\theta)|\,d\theta<0.$$

The first constraint implies that as the trajectory approaches the invariant curve, the control action tends to zero, while the second implies that the average Lyapunov exponent along the invariant curve is negative and the invariant curve becomes stable. The function $g(x,\theta)$ may be taken in the standard form of a proportional control action

$$g(x,\theta)=C(x-\varphi(\theta)),$$

where C is the coupling parameter. Then, in accordance with the fundamental idea of the method we note that the invariant mapping curves (1) in different regions of parameter space are fairly similar, as we can see from Fig. 1b where the thin line $x(\theta)$ is the unstable torus of the mapping (1) from the chaos region, and $y(\theta)$ is a stable torus from some other region. It is easily seen that the following relation holds:

$$x(\theta)\approx Sy(\theta+\tau)+B,$$

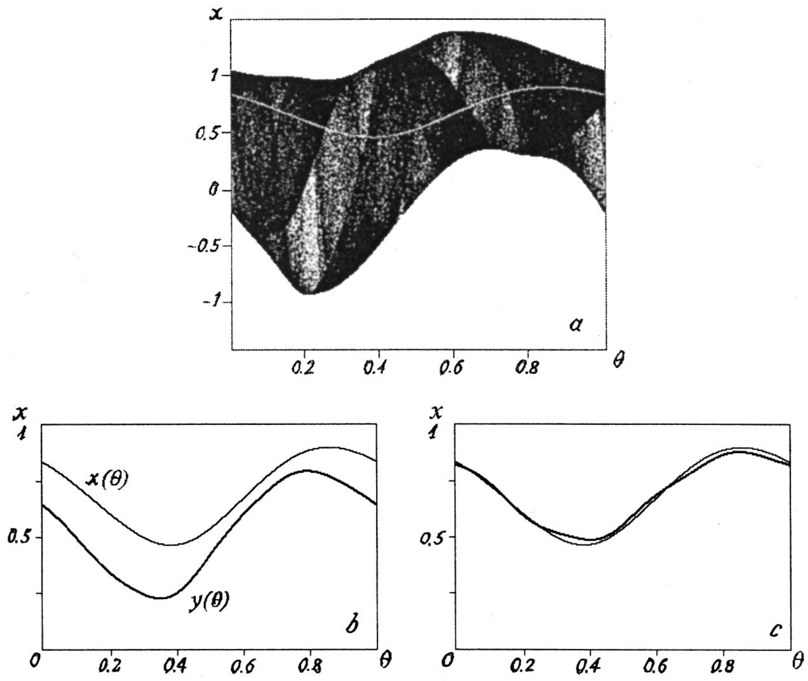

FIG. 1. a — Chaotic attractor and unstable torus of mapping (1) embedded in it (*1*) ($\lambda=1.2$, $\varepsilon=0.2$); b — thin line $x(\theta)$ — unstable torus from region of chaotic dynamics of parameter space being stabilized ($\lambda=1.2$, $\varepsilon=0.2$), thick line $y(\theta)$ — stable torus of mapping (1) used as reference trajectory for control ($\lambda=0.85$, $\varepsilon=0.2$); c — thin line — unstable torus of mapping (1), thick line — torus of mapping (3.1, 3.3) obtained from (1) by using the control procedure (for the same values of the parameters).

and the curve from its region of stability can be reduced to the form of an unstable curve from the chaos region with, a fairly high degree of accuracy, by means of a proportional contraction/expansion along the x axis and a shift parallel to the coordinate axes. The function $g(x,\theta)$ thus has the form

$$g(x_n,\theta_n)=C(x_n-Sy_n(\theta_n+\tau)-B),$$

where the parameters S and B determine the contraction/expansion and the parallel shift of the invariant curve, and τ is the phase shift. Finally, the mapping can be written in the form

$$x_{n+1}=\lambda_1-x_n^2+\varepsilon\cos 2\pi\theta_n+C(x_n-Sy_n-B), \qquad (3.1)$$

$$y_{n+1}=\lambda_2-y_n^2+\varepsilon\cos 2\pi(\theta_n+\tau), \qquad (3.2)$$

$$\theta_{n+1}=\theta_n+\omega,\ \mathrm{mod}\,1, \qquad (3.3)$$

where λ_1 is the value of the parameter for which chaos and an unstable torus exist in the mapping (1), and λ_2 is the value for which a stable torus exists, which is used as the reference trajectory.

The results of applying the stabilization algorithm are plotted in Fig. 1c. The thin line gives the unstable invariant mapping curve (1) and the thick line gives the stable torus obtained as a result of applying the control procedure. These last two lines are almost matched. The accuracy of their matching depends on the successful choice of parameters of the reference mapping (3.2–3.3) and the coupling parameters. The dependence of the controlling action decreases to a level which is no more than 5% of the variation in the parameters needed to transfer the controlled mapping to the region of regular dynamics, which suggests that the control problem has been solved.

3. FLUX MODEL

Since a one-dimensional irreversible logistic mapping is not a Poincaré cross section of any flux, we need to demonstrate separately that the proposed control method can be applied to models in the form of a system of differential equations. As such we take a biharmonically excited Duffing model, which has many physical analogs:

$$\ddot{x}+2\lambda\dot{x}+x+x^3=f_1\cos\omega_1 t+f_2\cos\omega_2 t, \qquad (4)$$

where $\omega_1/\omega_2=(\sqrt{5}-1)/2$. Since a Duffing oscillator possess a symmetric potential, the transition to chaos in this system should be preceded by symmetry-loss bifurcations of the torus. The resulting unstable torus is embedded in a chaotic attractor (Fig. 2a). To stabilize this, we modify Eq. (4) by adding a control action in a form known as continuous proportional feedback:[2]

$$\ddot{x}+2\lambda\dot{x}+x+x^3=f_1\cos\omega_1 t+f_2\cos\omega_2 t+g(x,t), \qquad (5)$$

$$g(x,t)=C(x-Sy(t)),$$

where $y(t)$ is a reference signal obtained by integrating Eq. (4) numerically for values of the parameters corresponding to the stable symmetric torus regime. These parameters are determined by trial and error: we decrease the amplitude of the fundamental-frequency signal f_1 until we enter the symmetric torus regime, after which finer tuning is achieved by varying the amplitude of f_2. As a result of the symmetry of the system, the parameters B and τ [see Eq. (3)] are zero and because of the low sensitivity of the torus to variations in the parameters, S is close to unity.

The results of applying the control procedure are plotted in Fig. 2b. It can be seen that the torus and reference orbit do not agree at all, However, the control action does not exceed 8% of the variation in the principal control parameter f_1

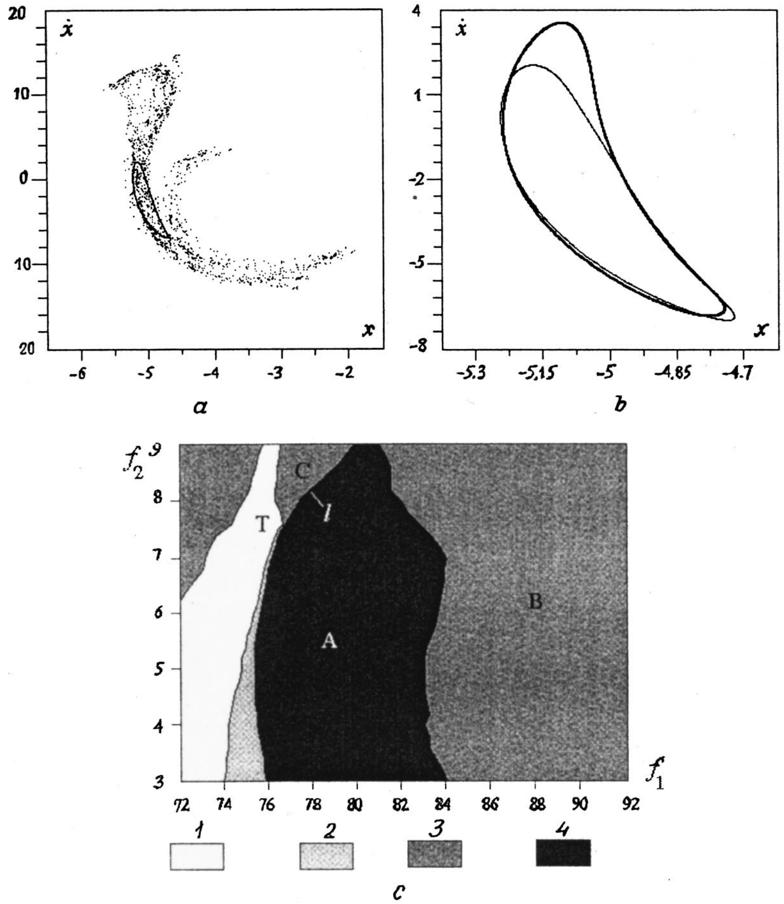

FIG. 2. Cross section of chaotic attractor and unstable torus of biharmonically excited Duffing oscillator($f_1=78$, $f_2=5$, $\lambda=0.1$, $\omega_1=1.5$) (a). Results of applying the control procedure: thin line — cross section of reference torus, thick line — stabilized (for the same parameters as a and b). Fragment of the cross section of the parameter space of a biharmonically excited Duffing oscillator ($\lambda=0.1$, $\omega_1=1.5$) (c). *1* — symmetric torus, *2* — asymmetric torus, *3* — chaos, and *4* — region in which control is possible.

needed to transfer the system to a state with regular dynamics and thus, in this case we can talk of chaos control.

The extent of the region of applicability of the proposed control method can be estimated from Fig. 2c. In region *A* this control procedure can be applied. In region *B* this is impeded by the increasing deformation of the unstable torus, since this region is extremely far from the region of existence of a stable symmetric torus. In region *C*, control cannot be applied because of the device characteristics of the parameter space imposed by the quasiperiodicity of the action. The bifurcation line of the torus symmetry loss truncates at point *T*, which is the critical point of codimensionality 2. Thus, a transition to chaos can take place in the system without loss of symmetry and the creation of an unstable torus (above point *T*). Thus, the region of chaos is divided into two sections: *A*–*B* (with an unstable torus) and *C* (no torus). These sections are divided by the line *l* on which the unstable torus is disrupted by (preferably) internal contact between the chaotic attractor boundaries. Thus, in region *C* there is no object to which the stabilization procedure can be applied.

CONCLUSIONS

These results indicate that the proposed chaos control procedure of stabilizing an unstable torus is effective in systems exposed to a quasiperiodic action.

The proposed method can be used to control chaos in real physical systems because of the universal nature of the structures of bifurcation sets of quasiperiodically excited systems of various types, which was demonstrated in Ref. 4 for a quasiperiodically excited logistic mapping and a diode resonator in the vicinity of the critical end point of the torus doubling bifurcation line. Universality implies the presence of identical regions of existence of regular and chaotic dynamics, unstable orbits, and bifurcation transitions. This means that the method described can be applied to a system of two unidirectionally coupled *LR*-diode circuits.

This work was supported by the Russian Foundation for Basic Research Grant No. 96-02-16755 and also by the ‘‘Integration’’ Federal Program, Grant No. 696.3.

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Translated by R. M. Durham