

Collective phase chaos in the dynamics of interacting oscillator ensembles

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We study the chaotic behavior of order parameters in two coupled ensembles of self-sustained oscillators. Coupling within each of these ensembles is switched on and off alternately, while the mutual interaction between these two subsystems is arranged through quadratic nonlinear coupling. We show numerically that in the course of alternating Kuramoto transitions to synchrony and back to asynchrony, the exchange of excitations between two subpopulations proceeds in such a way that their collective phases are governed by an expanding circle map similar to the Bernoulli map. We perform the Lyapunov analysis of the dynamics and discuss finite-size effects. © 2010 American Institute of Physics. [doi:[10.1063/1.3527064](https://doi.org/10.1063/1.3527064)]

The behavior of high-dimensional nonlinear systems of different nature (e.g., in hydrodynamics, electronics, neurodynamics, laser physics, and nonlinear optics) in many cases can be treated in terms of a cooperative action of a large number of relatively simple elements, such as interacting oscillators. Quite often, the essential dynamics is low dimensional: even for a very large number of elements (and also in the thermodynamic limit where this number tends to infinity), one can find a few degrees of freedom that describe the macroscopic evolution. A famous example is the Kuramoto model where the global variable—complex order parameter—undergoes a Hopf bifurcation corresponding to self-synchronization in the ensemble of oscillators. Here, we report on a modification of the Kuramoto model where the behavior of the global variables becomes chaotic. Our model consists of two populations that undergo Kuramoto-type transitions, where due to an external modulation of the coupling strength the oscillators synchronize and desynchronize alternately in both subpopulations. The operation of the whole system consists of alternating patches of activity of the subensembles in the sense that their order parameters (complex mean fields) vary between small and large magnitudes. Due to a specially constructed additional nonlinear coupling between the subpopulations, the dynamics of the phases of the complex order parameters is described stroboscopically by an expanding circle map (the Bernoulli map). Apparently, such systems may be designed on a base of ensembles of electronic devices, such as arrays of Josephson junctions, or with nonlinear optical systems, such as arrays of semiconductor lasers.

I. INTRODUCTION

A long-time challenging problem motivating the development of nonlinear science concerns complex behavior of systems characterized by a large number of degrees of freedom. Such systems are of interest in various fields, e.g., in hydrodynamics (the problem of turbulence), laser physics, nonlinear optics, electronics, and neurodynamics.¹ The func-

tioning of such systems often can be thought of as cooperative action of a large number of relatively simple elements, such as oscillators, with different types of interaction between them.^{2–5} In many cases, nevertheless, the dynamics can be effectively treated as low dimensional in terms of appropriate collective modes.^{2–4,6–8} In this context, an interesting question arises about a possibility of implementing various known types of low-dimensional complex dynamic behavior on the level of the collective modes.

One basic textbook example of strong chaos is the Bernoulli map, or expanding circle map.⁹ In general form, the map is $\varphi_{n+1} = M\varphi_n \pmod{2\pi}$, where M is an integer larger than 1 and φ is an angular variable. With initial condition for $\varphi/2\pi$ represented in the numeral system of base M by a random sequence of digits, the dynamics will correspond to a shift of this sequence by one position to the left on each next step of the iterations. It means that the representative point φ_n will visit in random manner M equal segments partitioning the circle, in accordance with the mentioned sequence of the digits. This is just what we call chaos. The sensitive dependence of the dynamics on the initial conditions is characterized quantitatively by a positive Lyapunov exponent $\Lambda = \ln M$.

In a recent series of papers,^{10–12} it was shown how to implement the dynamics described by the Bernoulli map in low-dimensional systems of alternately excited self-sustained oscillators. In these systems, the existence of uniformly hyperbolic chaotic attractors of Smale–Williams type (see, e.g., Ref. 13) was established.^{14,15}

Chaotic nature of the dynamics reveals itself in the chaotic evolution of the phases of oscillations generated at successive stages of excitation of the subsystems. The purpose of the present article is to demonstrate a possibility of chaotic behavior of similar nature at the level of collective variables for multidimensional systems represented by ensembles of oscillators.

We do not start here with some arrangement motivated by a particular application, but construct an idealized ex-

ample that is simple and convenient for the theoretical description. The model is composed of two similar ensembles of self-sustained oscillators with their natural frequencies distributed in some ranges; within each of these ensembles the global coupling is switched on and off alternately. The interaction between these two subensembles is arranged with a special type of additional coupling through mean fields characterized by quadratic nonlinearity.

Due to the presence of the global coupling, each ensemble can undergo the Kuramoto transition: as oscillators synchronize, the collective field emerges with notable amplitude and definite phase of its oscillations. As the coupling is turned off, the oscillators desynchronize because of the frequency detuning of individual oscillators, and the collective field disappears. The operation of the whole system consists of alternating activities of two ensembles with the corresponding alternating meandering of their order parameters. Due to an additional coupling, the excitation is transmitted from one ensemble to another and back, so the phase of this collective excitation in the course of the process evolves in accordance with the expanding circle map mentioned above.

One of the goals of this paper is to compare the global realization of hyperbolic chaos, as outlined above, with other types of collective chaos in ensembles of dynamical systems. Postponing a detailed discussion to the conclusion, we mention here that collective chaos has been mainly studied in two different contexts: in ensembles of maps,^{16–18} where individual elements are chaotic, and in populations of oscillators (which as individual elements are, contrary to the case of maps, nonchaotic), either with a distribution of natural frequencies^{19,20} or identical.^{6,21–24} In this context, our study is closely related to that in Refs. 19 and 20 because we also consider nonidentical oscillators; the difference is that we organize the coupling in a special way to ensure desired properties of collective chaos.

II. BASIC MODELS

In this section, we formulate models of interacting oscillator ensembles using three different levels of reduction. First, we consider the equations in the “original” form, where each oscillator is described by the van der Pol equation. Next, we exploit the method of averaging and formulate the model in terms of slowly varying complex amplitudes of oscillations. Finally, we proceed with neglecting amplitude variations for single oscillators and account only for variations of the phases on their limit cycles; that is the model of ensemble of phase oscillators.

As outlined in Sec. I, we do not study a general system of coupled alternately synchronized ensembles, but construct a model that should produce chaos, having specific properties (hyperbolicity). Moreover, we want to check how reductions to amplitude and phase equations influence the dynamical properties. Therefore, the model in the original variables [Eq. (3), below] looks rather cumbersome, while its reduction to complex amplitude [Eq. (5)] and phase [Eq. (8)] variables is rather close to the standard Kuramoto model.

A. Coupled van der Pol oscillators

Let us consider two interacting ensembles of self-sustained oscillators. We assume that the sizes K of these ensembles are the same, and the oscillators are described by variables x_k and y_k , respectively, where $k=1, \dots, K$. Next, we assume that the distributions of natural frequencies ω_k of the oscillators are identical for both ensembles, with some mean frequency ω_0 . Within each ensemble, oscillators are coupled via their mean fields X and Y , defined according to

$$X = \frac{1}{K} \sum_k x_k, \quad Y = \frac{1}{K} \sum_k y_k. \quad (1)$$

To account for the dissipative nature of the coupling (i.e., a tendency to equalize the instant states of the interacting subsystems), we assume that it is introduced by terms in the equations containing time derivatives of the fields X and Y . We suppose that the coupling strength varies in time periodically, with a slow period $T \gg 2\pi/\omega_0$, between zero and some maximum value, which exceeds the synchronization threshold of the Kuramoto transition. Thus, each ensemble goes periodically through the stages of synchrony (when the coupling is large) and asynchrony (when the coupling is small). The coupling is organized in such way that these stages in the two ensembles occur alternately. Finally, there is a *non-linear* interaction between ensembles via the second-order mean fields,

$$X_2 = \frac{1}{K} \sum_k x_k^2, \quad Y_2 = \frac{1}{K} \sum_k y_k^2. \quad (2)$$

Being represented as sums of the squares of the original variables, these fields X_2 and Y_2 contain components with the double frequency in comparison to the frequency of the variables X and Y . To ensure an efficient *resonant interaction*, we need the components with the main oscillator frequency; to this end, the coupling terms are chosen as products of time derivatives of the fields X_2 and Y_2 and of an auxiliary signal $\sin \omega_0 t$. These products then contain the components with the basic frequency ω_0 . The set of governing equations for the model reads as

$$\begin{aligned} \frac{d^2 x_k}{dt^2} - (Q - x_k^2) \frac{dx_k}{dt} + \omega_k^2 x_k &= \kappa f_1(t) \frac{dX}{dt} + \varepsilon \frac{dY_2}{dt} \sin \omega_0 t, \\ \frac{d^2 y_k}{dt^2} - (Q - y_k^2) \frac{dy_k}{dt} + \omega_k^2 y_k &= \kappa f_2(t) \frac{dY}{dt} + \varepsilon \frac{dX_2}{dt} \sin \omega_0 t, \end{aligned} \quad (3)$$

where $k=1, 2, \dots, K$. The functions $f_1(t) = \cos^2(\pi t/T)$ and $f_2(t) = \sin^2(\pi t/T)$ describe the alternate on/off switching of the couplings inside the ensembles. The parameter Q determines the amplitude of each single van der Pol oscillator; parameters κ and ε characterize the internal and mutual couplings for the ensembles, respectively. It is assumed that the period of coupling modulation contains a large integer number of periods of the auxiliary signal, i.e., $\omega_0 T / 2\pi = N \gg 1$.

B. Description in terms of complex amplitudes

The method of averaging allows us to describe weakly nonlinear oscillators in a simplified form, via the slow dynamics of the complex amplitudes; this description is appropriate under the assumptions that $|\omega_k - \omega_0| \ll \omega_0$, $2\pi/T \ll \omega_0$. For the van der Pol oscillators (3), the averaging can be accomplished by introducing complex amplitudes a_k and b_k according to

$$a_k(t) = e^{-i\omega_0 t} \frac{\dot{x}_k + i\omega_0 x_k}{i\omega_0}, \quad b_k(t) = e^{-i\omega_0 t} \frac{\dot{y}_k + i\omega_0 y_k}{i\omega_0}. \quad (4)$$

Substituting this into Eq. (3) and averaging over the period of fast oscillations $2\pi/\omega_0$ (practically, this may be done simply by dropping all terms in the right hand side, which contain fast time dependencies such as $\sim e^{\pm i\omega_0 t}$, $e^{\pm 2i\omega_0 t}$, etc.), we obtain

$$\begin{aligned} \dot{a}_k &= i\Omega_k a_k + \frac{1}{2}(Q - |a_k|^2)a_k + \frac{1}{2}\kappa f_1(t)A + \frac{1}{2}i\varepsilon B_2, \\ \dot{b}_k &= i\Omega_k b_k + \frac{1}{2}(Q - |b_k|^2)b_k + \frac{1}{2}\kappa f_2(t)B + \frac{1}{2}i\varepsilon A_2. \end{aligned} \quad (5)$$

Here, $\Omega_k = \omega_k - \omega_0$ are the frequency differences with respect to the average one. The mean fields A , B , A_2 , and B_2 are defined similar to Eqs. (1) and (2) as

$$\begin{aligned} A &= \frac{1}{K} \sum_k a_k, \quad B = \frac{1}{K} \sum_k b_k, \quad A_2 = \frac{1}{K} \sum_k a_k^2, \\ B_2 &= \frac{1}{K} \sum_k b_k^2. \end{aligned} \quad (6)$$

C. Phase approximation

One more step in the simplification of the model is to completely neglect the amplitude variations for single elements and to reduce the description to that in terms of ensembles of phase oscillators. In this approximation, the individual oscillators are assumed to have constant amplitude (that of the limit cycle of a single oscillator) throughout the process, while the dynamics manifests itself only in the evolution of their phases. This is a widely used approximation in the theory of synchronization (see Refs. 2 and 4).

To derive the phase equations, we substitute

$$a_k(t) \approx Re^{i\theta_k(t)}, \quad b_k(t) \approx Re^{i\phi_k(t)} \quad (7)$$

into Eq. (5), where R is a constant, equal for all oscillators. We specify $R = \sqrt{Q + \kappa}/2$ according to the following reasons. First, if one switches off the coupling between the ensembles setting $\varepsilon = 0$, then each single oscillator is described by the equation $\dot{a} = i\Omega a + \frac{1}{2}(Q - |a|^2)a + \frac{1}{2}\kappa A$. In the absence of the synchronization, $A = 0$, and the amplitude of the stationary self-sustained oscillations is determined from $|a|^2 = Q$. On the other hand, in the case of maximal synchronization, one has $f_1 = 1$, $A = a$, and hence, $|a|^2 = Q + \kappa$. Since the dynamics consists of the alternating epochs of synchronization and desynchronization, it is reasonable to take the average value.²⁵

Then, from Eq. (5), we obtain the following equations for the phase dynamics:

$$\dot{\theta}_k = \Omega_k + \frac{1}{2}\text{Im}[(\kappa f_1(t)U + i\varepsilon RV_2)e^{-i\theta_k}],$$

$$\dot{\phi}_k = \Omega_k + \frac{1}{2}\text{Im}[(\kappa f_2(t)V + i\varepsilon RU_2)e^{-i\phi_k}], \quad (8)$$

where the complex mean fields U , V , U_2 , and V_2 are defined according to

$$U = \frac{1}{K} \sum_k e^{i\theta_k}, \quad V = \frac{1}{K} \sum_k e^{i\phi_k}, \quad U_2 = \frac{1}{K} \sum_k e^{2i\theta_k},$$

$$V_2 = \frac{1}{K} \sum_k e^{2i\phi_k}. \quad (9)$$

D. Thermodynamic limit

Above, we have assumed the number of oscillators in the ensembles to be finite. Now let us write the equations in the “thermodynamic limit” $K \rightarrow \infty$. In the case we deal with, the oscillators differ only by their natural frequencies, then it is convenient to parametrize the oscillators by this continuous variable, i.e., by the frequency ω , and introduce a distribution over the frequencies characterized by a function $g(\omega)$. After transformation of the equations to slow amplitudes and in the phase approximation, we will use the index $\Omega = \omega - \omega_0$, designating the frequency difference. Then, we characterize the distribution by the density $\tilde{g}(\Omega) = g(\omega_0 + \Omega)$.

With these notations, the set of the phase equation (8) can be rewritten as

$$\dot{\theta}_\Omega = \Omega + \frac{1}{2}\text{Im}[(\kappa f_1(t)U + i\varepsilon RV_2)e^{-i\theta_\Omega}],$$

$$\dot{\phi}_\Omega = \Omega + \frac{1}{2}\text{Im}[(\kappa f_2(t)V + i\varepsilon RU_2)e^{-i\phi_\Omega}], \quad (10)$$

where the mean fields are defined now via the integrals over the density

$$U = \int d\Omega \tilde{g}(\Omega) e^{i\theta_\Omega}, \quad V = \int d\Omega \tilde{g}(\Omega) e^{i\phi_\Omega},$$

$$U_2 = \int d\Omega \tilde{g}(\Omega) e^{2i\theta_\Omega}, \quad V_2 = \int d\Omega \tilde{g}(\Omega) e^{2i\phi_\Omega}. \quad (11)$$

In a similar way, the equations for the complex amplitudes and for the original van der Pol oscillators may be easily reformulated in the thermodynamic limit.

III. NUMERICAL EVIDENCE FOR COLLECTIVE CHAOS

In this section, we present numerical studies of models (3), (5), and (8). As is known from the theory of synchronization in populations of oscillators developed by Kuramoto, the properties of the synchronization transition are qualitatively the same for all unimodal smooth distributions of oscillators over their frequencies. In computations below, we specify the distribution for model (3) as

$$f(\omega) = \begin{cases} \frac{\pi}{2} \cos \frac{\pi(\omega - \omega_0 - \Delta\omega)}{2\Delta\omega}, & \omega \in [\omega_0 - \Delta\omega, \omega_0 + \Delta\omega] \\ 0 & \text{otherwise,} \end{cases} \quad (12)$$

and set $\omega_0 = 2\pi$ and $\Delta\omega = \pi/8$.

We select the form (12) because it allows us to choose the discrete set of frequencies for a finite ensemble according to a simple relation

$$\omega_k = \omega_0 - \Delta\omega + \frac{2\Delta\omega}{\pi} \arccos \left[\frac{2k-1}{K} - 1 \right],$$

$$k = 1, 2, \dots, K. \quad (13)$$

Furthermore, the absence of significant tails of the distribution (compared, e.g., to a Lorentzian one) simplifies numerical studies, as we do not have to bother about nonresonant oscillators with too large or too small frequencies. For models (5) and (8), analogous distributions were used obtained from Eqs. (12) and (13) with the substitution $\Omega = \omega - \omega_0$.

With this setup, we simulated the dynamics of models (3), (5), and (8) in computations at parameters $T=100$, $\kappa=1$, $\varepsilon=0.1$, and $Q=3$ for system sizes $K=1000$ and $K=10\,000$. The main quantities of interest are the phases of the mean fields. For the ensembles of van der Pol oscillators, we define them according to

$$\Phi_X = -\arctan \frac{\dot{X}}{\omega_0 X}, \quad \Phi_Y = -\arctan \frac{\dot{Y}}{\omega_0 Y}. \quad (14)$$

For ensembles (5) and (8), we define the phases simply as the arguments of the complex mean fields

$$A = |A|e^{i\Phi_A}, \quad B = |B|e^{i\Phi_B}, \quad U = |U|e^{i\Phi_U}, \quad V = |V|e^{i\Phi_V}. \quad (15)$$

Figure 1 illustrates the dynamics of the mean fields $X(t)$ and $Y(t)$ in the ensemble of van der Pol oscillators (3). First, we mention that because of the modulation $\sim \kappa f_{1,2}(t)$, the mean fields of two ensembles vary significantly: they drop nearly to zero in the epochs where the corresponding values of $f_{1,2}$ are small and attain large values in the epochs where the coupling inside a population becomes large. In panel (c), we show the time dependences of the phases of the mean fields $\Phi_{X,Y}$. One can see slight regular variations of the phases correlated with the amplitudes of the mean fields and additional phase shifts close to the moments of time when the corresponding mean fields nearly vanish [at times $t \approx 40, 140, 240$ for $X(t)$ and at times $t \approx 90, 190, 290$ for $Y(t)$]. These phase shifts correspond to *transfer of the phase* from one ensemble to another through their mutual coupling terms proportional to ε .

Usually, as an ensemble of oscillators passes through the Kuramoto transition from a nonsynchronous to a synchronous state while the coupling strength increases, the phase (potentially, of arbitrary value) of the arising collective mode is determined by fluctuations stimulating the excitation of this mode. In our setup, however, the excitation occurs in the presence of a small driving force $\sim \varepsilon$ because of the action of

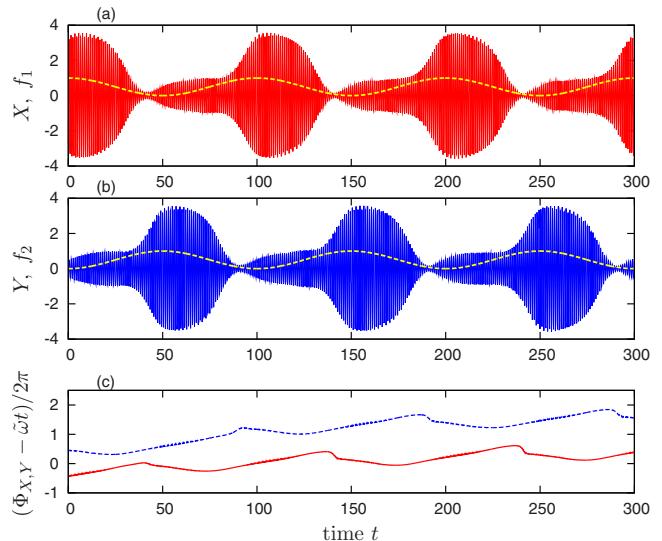


FIG. 1. (Color online) The mean fields [panels (a) and (b)] and their phases [panel (c)] for two interacting ensembles of van der Pol oscillators (12) with $K=1000$. The modulation function of the coupling $f_{1,2}$ are shown in (a) and (b) with dashed lines. To make picture (c) clearer, we subtract a linear phase growth and plot the value $\Phi_{X,Y} - \tilde{\omega}t$, where $\tilde{\omega} = \omega_0 - 0.2$ is chosen close to the empirical mean frequency $\tilde{\omega}$, which slightly differs from ω_0 due to nonlinearity. $\Phi_X - \tilde{\omega}t$ is shown with solid line and $\Phi_Y - \tilde{\omega}t$ with dashed one.

another ensemble, which is synchronous at that moment and generates notable mean field. This stimulation determines the phase on the appearing collective mode, which accepts this externally designated phase. This is the mechanism of the phase transfer.

Because the mutual couplings are proportional to the *second-order* mean fields characterized by a doubled frequency, the phase transfer is accompanied with doubling of the phase. To explain this, let us assume that at the transfer of excitation from the first to the second subensemble, we have $X \sim \cos(\omega_0 t + \Phi)$ and $\dot{X}_2 \sim \sin[2(\omega_0 t + \Phi) + \text{const}]$, respectively. Then, the driving force that affects the second subensemble contains the resonance component of $\dot{X}_2 \sin \omega_0 t \sim \cos(\omega_0 t + 2\Phi + \text{const})$. The arising mean field will be of the form $Y \sim \cos(\omega_0 t + 2\Phi + \text{const})$.

As doubling of the phase occurs at each transfer from one subsystem to another and back through the full cycle (i.e., over the period T), as a result, we expect that the phase is multiplied by a factor of 4 (up to an additive constant). To check the supposed mechanism of the phase transfer, we constructed numerically a stroboscopic iteration phase diagram $\Phi_X(nT) \rightarrow \Phi_X((n+1)T)$, relating the phases at the successive moments of maximal amplitude of the mean field X . This map, shown in Fig. 3(a), clearly indicates that the transformation is indeed close to the Bernoulli-type map,

$$\Phi_X((n+1)T) = [4\Phi_X(nT) + \text{const}] \pmod{2\pi}. \quad (16)$$

For the same values of the parameters, we also simulated the dynamics of the coupled oscillator ensembles in the slow complex amplitude version of Eq. (5) [see Figs. 2(a) and 2(b)] and in the phase approximation of Eq. (8) [see Figs. 2(c) and 2(d)]. In these situations, the second-order effects of amplitude-dependent frequency shifts, like those seen in Fig. 1(c), are not observed. As a result, the phases between the

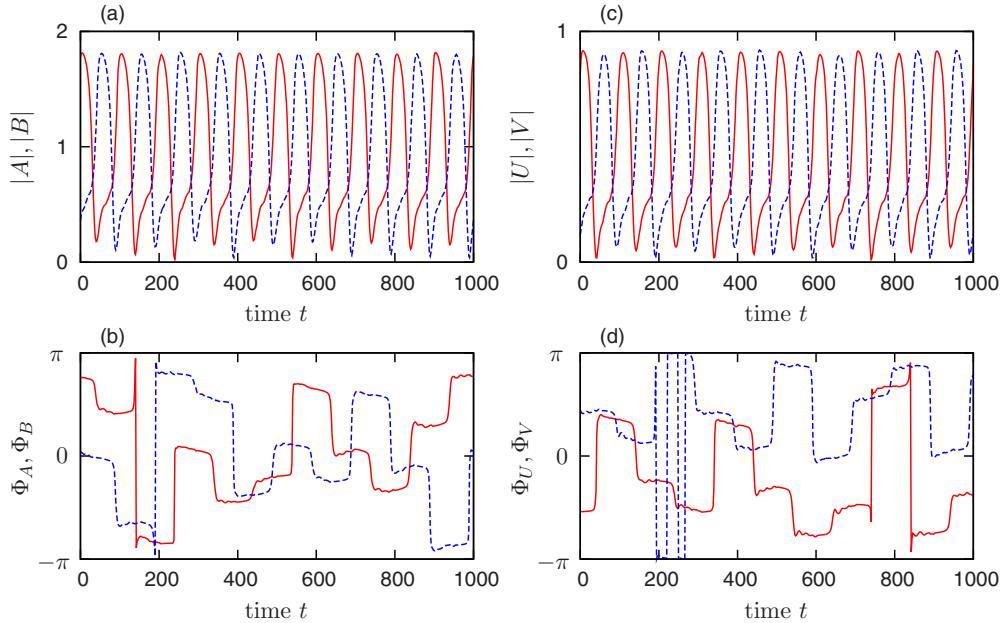


FIG. 2. (Color online) Evolution of amplitudes and phases for the mean fields of coupled ensembles described by slow complex amplitudes [panels (a) and (b)] and in the phase approximation [panels (c) and (d)] with $K=1000$. Solid lines: variables $|A|, \Phi_A, |U|, \Phi_U$; dashed lines: variables $|B|, \Phi_B, |V|, \Phi_V$.

short intervals of phase transfers are nearly constant, and the phase shifts at the transfers are clearly visible. The stroboscopic maps of the phases are shown in Figs. 3(b) and 3(c). They demonstrate Bernoulli-type maps similar to that for the ensemble of van der Pol oscillators of Eq. (16) [see Fig. 3(a)].

For further numerical characterization of the chaotic dynamics of the mean fields, we constructed the stroboscopic maps of the complex mean fields via period of modulation T . Portraits of attractors of these maps for our three levels of description of the ensembles are depicted in Fig. 4. For a dynamical system where the phase (or another cyclic variable) undergoes a Bernoulli-type transformation, while in other directions in the phase space the phase volume compresses, one expects the strange attractor to be of the Smale–Williams type, i.e., represented by a solenoid. In a two-dimensional projection, this attractor looks like a circle with a fractal transversal structure. Figure 4 confirms this picture for the dynamics of the mean field.

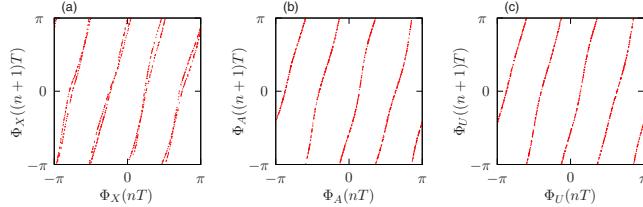


FIG. 3. (Color online) Stroboscopic maps over the period of external modulation T for (a) the ensembles of van der Pol oscillators (12), (b) for the ensembles described by slow complex amplitudes (5), and (c) for the ensembles in the phase approximation (8); in all cases, $K=1000$. In all cases, the dynamics seems to be well described by the Bernoulli map (16). The observed splitting of the “lines” [the most pronounced in panel (a)] appears because of the presence of transversal fractal structure of the attractors (see Fig. 4): distinct filaments of the attractor give rise to distinct filaments on the phase iteration diagram due to imperfection of the phase definition.

We would like to stress that the Bernoulli map describes the *collective* phase (i.e., the phase of the mean field) but not individual phases of the oscillators. Indeed, as is evident from topological considerations, the doubling of the phase is only possible if the amplitude vanishes at some stage. For the complex mean field, the phase doubling is achieved by synchronization-desynchronization, while all individual oscillators always have finite amplitude and therefore cannot be described by the doubling Bernoulli map. We illustrate this in Fig. 5, where the maps are shown similar to that in Fig. 3, but for the individual phases of two representative oscillators of the ensemble. Mostly close to the Bernoulli map is the behavior of the oscillator at the center of the

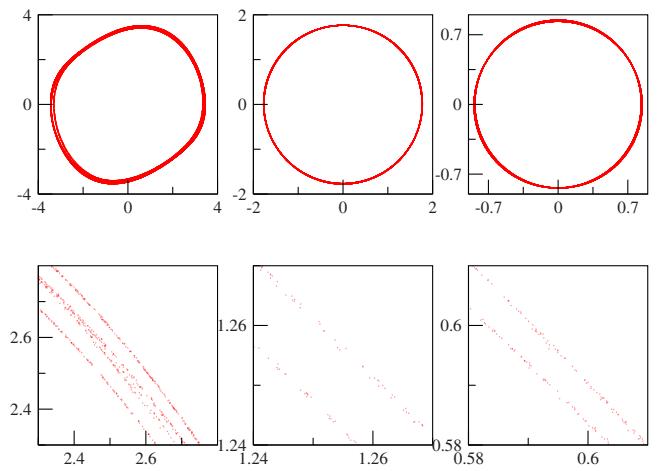


FIG. 4. (Color online) Projections of the stroboscopic maps on the plane of the order parameters: (X, \bar{X}) for the van der Pol oscillators (left column); $(\text{Re } A, \text{Im } A)$ for the ensembles described by slow complex amplitudes (center column); and $(\text{Re } U, \text{Im } U)$ for the ensembles in the phase approximation. The bottom row shows enlargements to make the fractal transversal structure evident. Here, $K=10\,000$.

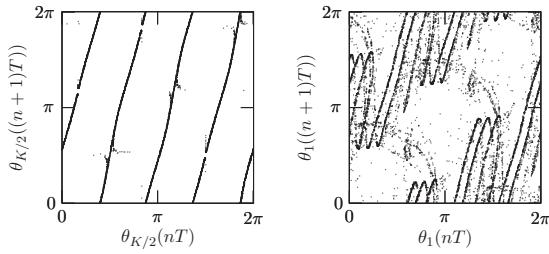


FIG. 5. Stroboscopic maps over the period of external modulation T for individual oscillators. Left panel: oscillator at the center of the band with $\omega = \omega_0$; right panel: oscillator with $\omega = \omega_0 - \Delta\omega$.

frequency band, as this oscillator nearly perfectly follows the mean field. Nevertheless, one can clearly see the “defects” of the transformation appearing because the “amplitude” of the oscillator cannot vanish. The oscillator at the edge of the band does not generally follow the phase of the mean field, and its dynamics is far from the Bernoulli map.

IV. FINITE-SIZE EFFECTS

In this section, we characterize collective chaos for different values of the ensemble size K . We focus here on the properties of the model [Eq. (5)] in terms of complex amplitudes. First, we analyzed the stroboscopically observed phases of the mean field and found that for some ensemble sizes K a periodic behavior is observed. The periods of the obtained periodic regimes are shown in Fig. 6(a) versus K . The values of K for which no period is plotted correspond to the chaotic states. At first glance, this contradicts the robustness of chaos expected because of its approximate description in terms of the expanding Bernoulli map. Apparently, changing the ensemble size we make an essential perturbation of the effective collective dynamics, so that the usual arguments of structural stability do not apply.

The analysis via the bifurcation diagrams [Fig. 6(a)] is confirmed by calculation of the Lyapunov exponents for the ensembles. We have performed this analysis for the ensembles described by complex amplitudes (5) of different sizes. First, in Fig. 7, we present the full spectrum of the Lyapunov exponents for three ensemble sizes. For these parameters, we observe chaos, and in all cases, only one Lyapunov exponent is positive. We interpret this as the existence of one collective chaotic mode, contrary to the cases when many Lyapunov exponents in populations of oscillators are positive (cf. Refs. 6, 24, and 26).

In Fig. 6(b), we present the three first Lyapunov exponents for the same data as in Fig. 6(a). Again, like in Fig. 6(a), one can see that for certain system sizes the dynamics is regular, as the largest exponent is negative. One can notice that the largest positive Lyapunov exponent decreases with K for $K < 150$, but, as panel Fig. 6(c) shows, saturates and presumably does not further decrease for larger system sizes (contrary, e.g., to the dependence $\lambda_{\max} \sim K^{-1}$ in the standard Kuramoto model reported in Ref. 27), although periodic windows can be observed for large K as well. (Preliminary calculations demonstrate that for large K these windows become extremely rare.)

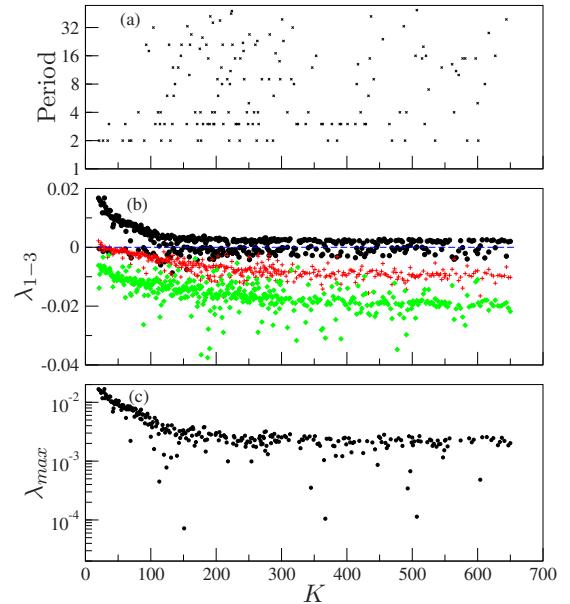


FIG. 6. (Color online) Bifurcation diagrams for the ensembles in the complex amplitude formulation (5) showing dynamical regimes in dependence on the ensemble size K . (a) Periods for periodic regimes. The values of K for which no period is plotted correspond to the chaotic states. (b) Three largest Lyapunov exponents (first: filled circles; second: pluses; and third: diamonds). Only a few regimes with small number of oscillators have two positive exponents. In panel (c), we show only positive largest Lyapunov exponents, in a logarithmic scale, to demonstrate that it does not tend to decrease for large ensemble sizes K .

We suggest that the observed peculiarities are specific just to finite-size effects, and that they will possibly disappear in the thermodynamics limit. Indeed, the computations show that they become less expressed under the increase of the ensemble size; however, as may be concluded from computations, the convergence to the large-size behavior is slow enough.

V. CONCLUSION

We have proposed and studied a model system that demonstrates chaotic dynamics of the phases of the mean fields. These collective variables are described by an expanding circle map transformation in the course of the transfer of collective excitation alternately between two synchronizing and desynchronizing groups of oscillators. We have demonstrated this on different levels of description: for the original system of coupled van der Pol oscillators, for the model based on the equations for slowly varying complex amplitude, and for the phase oscillators. While collective chaos is observed in a wide range of parameters, it is not completely robust with the respect to variations of the ensemble sizes: here, we observe regularity windows. These finite-size effects require further investigations.

An interesting property of the model studied is that in the chaotic state it has only one positive Lyapunov exponent (except for several regimes with a very low number of oscillators in ensemble), all others are negative. In this respect, our model differs from the ensemble of identical oscillators studied in Refs. 6 and 24 where the number of positive Lyapunov exponents in the regime of collective chaos was

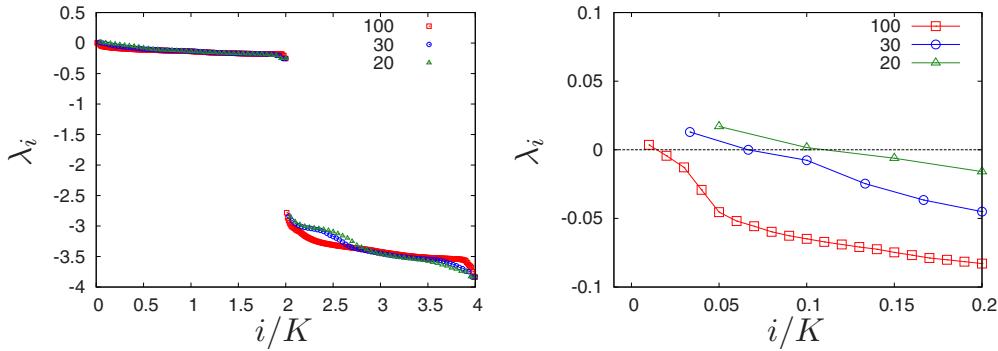


FIG. 7. (Color online) Lyapunov exponents λ_i plotted vs index i/K for the ensembles described by complex amplitudes (5) with $K=20, 30, 100$. $2K$ exponents corresponding to the phases are close to zero, while the rest $2K$ exponents corresponding to the stable amplitudes are negative. The right panel shows the region with small i ; in all cases, only the first exponent is positive.

macroscopic (proportional to the size of the ensemble). It also differs from the ensemble of identical Josephson junctions,⁷ where few Lyapunov exponents are positive but the macroscopic majority of them vanish due to partial integrability of the system. At the moment, it is not clear which physical properties of the systems are responsible for this difference. Possibly, a comparison with the Lyapunov spectrum in models with distribution of frequencies^{19,20} could shed light on this problem. A promising approach for future studies is a calculation of finite-size Lyapunov exponents like in Refs. 16 and 17.

Another peculiar property of the system studied is its sensitivity to the number of oscillators in the ensemble and appearance of periodic “windows” in dependence on this parameter. Such sensitivity has not been reported for other models demonstrating collective chaos. We speculate that this property might be related to the abovementioned existence of one positive Lyapunov exponent, which makes the collective chaos in the ensemble less robust. This issue requires special attention in the future work.

We believe that the model proposed, although rather artificial to be observed in natural oscillator ensembles, provides a useful test system for analysis. On the other hand, our research opens a possibility of constructing realistic systems with collective chaotic phase dynamics based on ensembles of such individual elements that show only regular dynamics. This might be feasible, e.g., on the basis of electronic devices, such as arrays of Josephson junctions,²⁸ or with nonlinear optical systems, such as arrays of semiconductor lasers.²⁹ Such systems are expected to generate robust chaos, providing the power level much higher than that characteristic for the individual elements. Systems of this kind may be of interest for applications requiring generation of chaotic signals, such as communication schemes,^{30,31} noise radar,³² etc.

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