An autonomous system with attractor of Smale – Williams type with resonance transfer of excitation in a ring array of van der Pol oscillators

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Abstract – An autonomous system we propose is a ring structure of a large number of van der Pol oscillators, which manifests cyclic propagation of the localized excitation with phase undergoing the expanding circle map transformation on each full revolution. Due to this, it is reasonable to suppose that attractor of Smale – Williams type occurs in the phase space of the system. Because of the slow spatial variation of the natural frequencies of the oscillators around the ring, it appears possible to exploit resonance mechanism for the excitation transfer; so, the system may have prospects for implementation of high-frequency chaos generators.

Key words: Attractor; van der Pol oscillator; Smale – Williams solenoid; Lyapunov exponent

1. Introduction

Uniformly hyperbolic strange attractors known in theory of dynamical systems have strong chaotic properties and allow for a far-reaching mathematical analysis. They are structurally stable, i.e. insensitive in respect to variations of functions and parameters in the governing equations [1-5]. Traditionally, in reviews and textbooks on nonlinear dynamics, the uniformly hyperbolic attractors (the Plykin attractor, the Smale – Williams solenoid) are illustrated by artificially constructed discrete-time evolution rules. An interesting problem is to find examples of such attractors in systems of physical or technical origin.

Recently, several systems were suggested, in which the Poincaré map possesses an attractor of Smale – Williams type [6-8].

Among them there are non-autonomous systems composed of two van der Pol oscillators, which become active turn by turn and pass the excitation each other. Transformation of the phase of oscillations on a whole cycle of due to the resonance excitation transfer in these systems gives rise to the expanding circle map that implies chaotic dynamics.

In addition, autonomous systems were designed based on dynamics close to a heteroclinic contour [8]. Such system may be thought as a ring set of self-oscillators activated alternately because of appropriately chosen terms in the equations responsible for nonlinear saturation. The disadvantage of the models considered in Ref. [8] is a non-resonant nature of transfer of the excitation between the oscillators. On one hand, postulated in these constructions form of equations for a single oscillator in practice corresponds to description in terms of slow complex amplitudes, which implies assumption of relatively high natural frequency. On the other hand, the exploited non-resonance mechanism of transfer of excitation between the alternately active oscillators ensures the prescribed transformation for phases only at relatively low natural frequencies. These two conditions contradict each other; so, practical implementation of the scheme is questionable.

One way to overcome the problem is to consider a large number of oscillators arranged in such way that the neighbors transferring the excitation each other possess close natural frequencies, which decrease gradually from the beginning to the end of the chain; the final stage of the excitation transfer at the closure of the chain is of a special kind been accompanied by doubling of the respective frequency and the oscillation phase to organize the expanding circle

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map. In the present paper, this idea is implemented in a ring model composed of van der Pol oscillators.

2. Equations and principle of operation

As known, a single autonomous van der Pol oscillator is governed by a differential equation

$$\ddot{x}-(A-x^2)\dot{x}+\omega_0^2 x=0\,, \tag{1}$$

where ω_0 is the natural frequency. At negative values of the parameter A the only attractor is a fixed point at the origin. At positive A this point becomes unstable, and self-oscillations occur associated with the attractive limit cycle.

Let us consider the following ring model composed of N+1 interacting van der Pol oscillators

$$\begin{aligned}
&\ddot{x}_0-(A+\tfrac{3}{2}x_N^2+x_0^2-2S)\dot{x}_0 \;+\; \omega_0^2 x_0 &&=\varepsilon x_N^2,\\
&\ddot{x}_1-(A+\tfrac{3}{2}x_0^2+x_1^2-2S)\dot{x}_1 \;+\; 2^{-2/N}\omega_0^2 x_1 &&=\varepsilon x_0,\\
&\ddot{x}_2-(A+\tfrac{3}{2}x_1^2+x_2^2-2S)\dot{x}_2 \;+\; 2^{-4/N}\omega_0^2 x_2 &&=\varepsilon x_1,\\
&\dots\\
&\ddot{x}_N-(A+\tfrac{3}{2}x_{N-1}^2+x_N^2-2S)\dot{x}_N+\;2^{-2}\omega_0^2 x_N &&=\varepsilon x_{N-1},\\
&S=\textstyle\sum_{k=0}^{N}x_k^2.
\end{aligned} \tag{2}$$

The factors at the first derivatives may be rewritten in a form $A-\frac{1}{2}x_{j-1}^2-x_j^2-2\sum\limits_{k\neq j-1,j}x_k^2$. Its structure ensures a situation that each given oscillator contributes to a strong suppression of all other partners, except one numbered with the next index. Due to the presence of the additional coupling characterized by parameter ε the excitation of each j-th oscillator is stimulated by oscillations from the (j-1)-th partner. Because of such structure, the system manifests cyclic propagation of the localized excitation around the ring in direction of increasing indices.

Indeed, let us suppose that initially all N variables are relatively small, but, say, x_1 is the largest. Then, it grows in accordance with the equations (2), reaches a level of order 1 and nearly saturates. Now, it suppresses strongly the variable x_3, because x_1 enters in the third equation with factor 2, but it does not restrain the growth of the variable x_2, as x_1 is present in the second equation with factor ½. On the next stage, the variable x_2 grows, saturates, and suppresses the variable x_1, but allows growth of x_3, and so on. So, the elements become active turn by turn in cyclic order $x_1\to x_2\to x_3\to x_4\to\dots$.

Excitation of each next oscillator occurs in presence of stimulating force from the previous one, which corresponds to the right-hand part terms proportional to the coupling constant ε in Eqs. (2). As N is large enough, the natural frequencies of the neighbor oscillators are close, and the stimulation is practically resonance. Only at the edge of the chain, the transfer from N-th oscillator to 0-th one is arranged in other way: the first one has the natural frequency twice less that the last one, and the resonance stimulation is produced by the second harmonic presenting in the coupling term proportional x_N^2 in the first equation of the set (2). It ensures doubling of the phase shift. Indeed, if $x_N\sim\cos(\frac{1}{2}\omega_0 t+\varphi)$, then, $x_N^2\sim\cos^2(\frac{1}{2}\omega_0 t+\varphi)=\frac{1}{2}\cos(\omega_0 t+2\varphi)+\dots$, where three dots mean a non-resonant term. Hence, the 0-th oscillator will accept the phase shift $2\varphi+const$ at the next stage of activity. So, on successive bypasses of the ring by the excitation, the phase shifts will evolve in accordance with the expanding circle map, or the Bernoulli map

$$\varphi_{n+1}=2\varphi_n+\text{const}\,. \tag{3}$$

This is chaotic map with the Lyapunov exponent $\Lambda=\ln 2\approx 0.693$; so the described mode of operation corresponds in fact to chaotic attractor in the ring system.

3. Results of numerical simulation

Figure 1 shows in panel (a) the time dependences for generalized coordinates of oscillators in the ring structure for N=14 in accordance with results of numerical computer solution of equations (2) after exclusion of the transients.

In accordance with the idea of the design, the oscillatory excitation has to undergo doubling of phase with each bypass of the ring. This property is crucial for the realization of the attractor of Smale - Williams type, and it is highly desirable to illustrate it in the numerical computations.

It is not so trivial to construct an appropriate Poincaré map: simple definitions appear to be unsuccessful for illustrating the desirable phase transformation. Indeed, a single van der Pol system is not so close to perfect symmetry intrinsic to the shortened amplitude equations [8], and it is problematic to locate the instant of the cross-section relatively to the envelope of the oscillations with inaccuracy essentially less then the period associated with the natural frequency and obtain good definition of the phases. To overcome the problem let us introduce an auxiliary variable z governed by the differential equation

$$\dot{z} + \gamma z = +x_0^2 + \omega_0^{-2}\dot{x}_0^2 - x_1^2 - 2^{2/N}\omega_0^{-2}\dot{x}_1^2 . \qquad (4)$$

Here the right-hand part is the difference of squared amplitudes for the oscillatory elements numbered 0 and 1, and γ is positive constant. The behavior of the variable z is illustrated by panel (b) of Fig.1 at particular γ=0.1. The value z grows during the epoch of activity of the oscillator 0, and then decreases below zero during the epoch of activity for the oscillator 1. Then, it decays nearly to zero until the next activation of the oscillator 0 occurs.

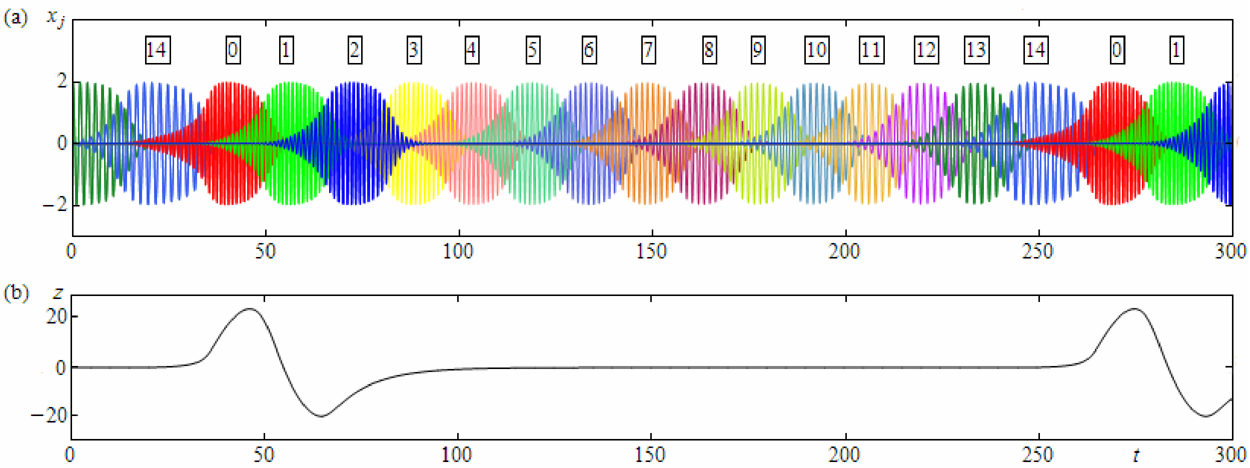

Fig. 1. (Color on-line.) Time dependences for x_j, j=0,…, N (a) and auxiliary variable z (b) used for construction of the return map obtained from the numerical computer solution of Eqs. (2) at $N = 14,\ A = 1,\ \omega_0 = 2\pi,\ \varepsilon = 0.2$.

For the record of the system states and evaluation of the phases $\varphi = \arg(x_0 + i\dot{x}_0 / \omega_0^2)$ we take sequence of the instants of the transitions from positive to negative values of the variable z. Fig.2 shows the diagram for φ_{n+1} versus φ_n. Observe that it looks topologically corresponding to the Bernoulli map: one complete revolution for pre-image gives rise to two revolutions for image. Accounting strong compression in other directions in the phase space (excluding the cyclic phase variable and the neutral direction along the phase trajectory), it gives a reason for the conjecture that hyperbolic attractor does exist, which corresponds to suspension [1-5] for the Smale – Williams solenoid.

Figure 3 presents portraits of the attractor in projection onto the phase plane of oscillator 0. Panel (a) shows the attractor for the flow system in projection from the phase space of dimension 2(N+1). In panel (b) attractor in the cross-section defined with the help of the auxiliary variable is plotted. The last one is in fact a picture of solenoid; the intrinsic fine transversal Cantor-like structure is not evident due to strong compression of the phase space.

Figure 4 shows portraits of the attractor for the flow system in projections on coordinate axes corresponding to neighboring oscillators in the ring. The first picture on the plane (x_N, x_0) is of a specific kind; all the rest ones are qualitatively similar to each other. It is so because just transfer of the excitation from the N-th oscillator to 0-th is arranged through the quadratic coupling term and the second harmonic; all others are stimulated directly by linear terms in the equations (2).

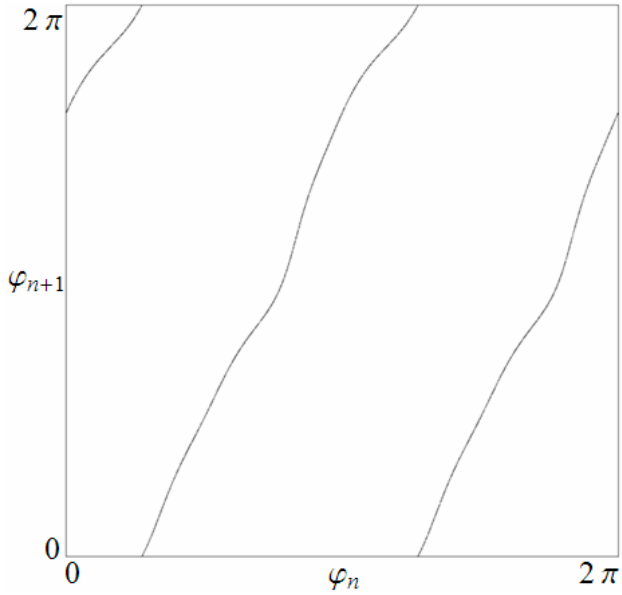

Fig. 2. Iteration diagram for phases obtained in computations for the model (2) at $N=14,\ A=1,\ \omega_0=2\pi,\ \varepsilon=0.2$. The phases are determined at instants of each successive passage of the auxiliary variable z evolving in accordance with the equation (4), from positive to negative values.

By means of the Benettin algorithm [9] applied to the set of equations (2) and respective variation equations, the larger Lyapunov exponents were evaluated:

$$\lambda_1=0.003044,\ \lambda_2=0.000002,\ \lambda_3=-0.3353,\ \lambda_4=-0.6318\ \ldots \tag{5}$$

The largest Lyapunov exponent is positive that means that dynamics is chaotic. Accounting that intersections of the axis of the auxiliary variable z=0 (see (4)) used in the analysis of the phase map occur in the present case with average time period T_{av}=227.85, it is worth considering normalization of the oldest Lyapunov exponent to this time interval that yields $\Lambda_1=\lambda_1 T_{av}\approx 0.6934$. This estimation corresponds well with the Lyapunov exponent of the Bernoulli map, that transforms the phases, $\ln 2=0.6931$. Therefore, we interpret the largest exponent as associated with cyclic coordinate in the phase space of dimension 2(N+1) along which the expansion takes place. The second exponent is close to zero, up to the numeric error, and should be related to the perturbations of a shift along the reference phase trajectory. The third exponent is negative, and its absolute value is large enough to ensure overall compression of the phase volume and sufficient to evaluate the Kaplan – Yorke dimension [10] of the attractor: $D_{KY}\approx 2.009$ (in application to the flow system). Although the full spectrum of Lyapunov exponents contains 2(N+1) numbers, it seems not necessary to compute all of them because the rest ones correspond to strong compression in the phase space and do not influence the estimate of the dimension.

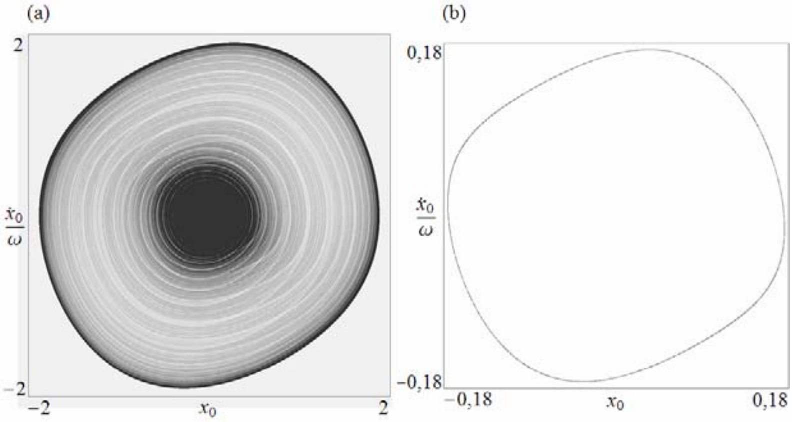

Fig. 3. Attractor of the model (2) at $N = 14$, $A = 1$, $\omega_0 = 2\pi$, $\varepsilon = 0.2$ in projection onto the phase plane of the oscillator 0 from the multidimensional phase space of the flow system (a) and in the cross-section constructed with using the auxiliary variable z as explained in the text (b).

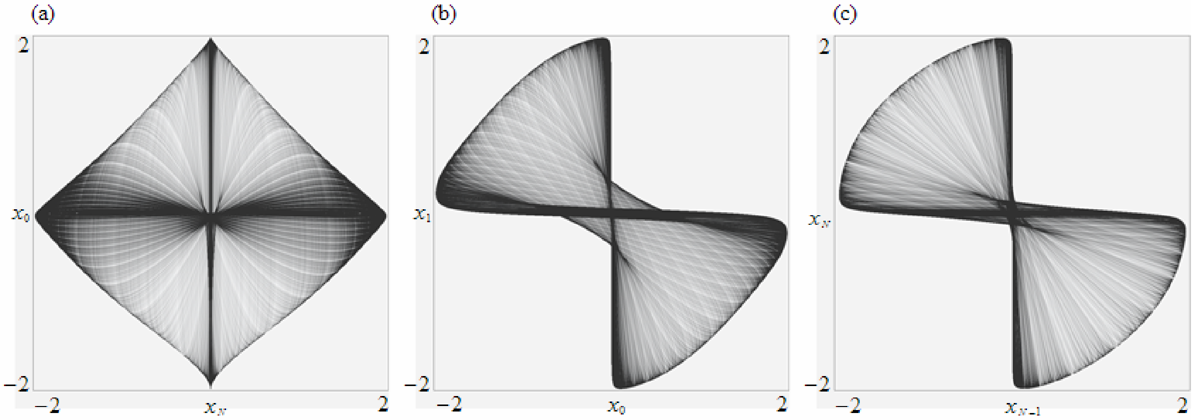

Fig. 4. Portraits of the attractor of the model (2) at $N = 14$, $A = 1$, $\omega_0 = 2\pi$, $\varepsilon = 0.2$ in projections from the multidimensional phase space of the flow system onto the planes of generalized coordinates of two neighboring oscillators in the ring structure: N and 0 (a), 0 and 1 (b), N–1 and N (c). For other neighboring pairs the diagrams look qualitatively similar to (b) and (c).

In fact, the attractor is a kind of the Smale – Williams solenoid embedded in the multi-dimensional phase space of the Poincaré map. Indeed, the cyclic coordinate φ undergoes the expanding transformation on each step of the dynamical evolution associated with bypass of the excitation around the ring, while in other directions in the phase space the phase volume undergoes compression. (As the compression is too small, and dimension is very close to integer number, the transverse fractal structure in of Fig.3b is not resolvable.)

4. Conclusions

The obtained results indicate occurrence of Smale-Williams solenoid in the phase space of the proposed system. The real device, based on this model, may be implemented e.g. in electronics or optics. In conclusion, we can assume that similar ring structures with other methods of introducing of coupling, may be implemented to organize the dynamics associated with the Arnold cat map and with hyper-chaotic attractors as well in similar way like it is done in Refs.[8] and [11].

Acknowledgements

V.P.K. acknowledges support from Grant of President of Russian Federation MK-905.2010.2. S.P.K acknowledges support from RFBR grant No 09-02-00426.

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