The self-oscillating system with compensated dissipation – the dynamics of the approximate discrete map

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We study the van der Pol oscillator subject to an external forcing given by δ-pulses depending on the dynamical variable. In order to simplify the computations an approximate map is derived by solving the equations between subsequent pulses. It is shown that the period doubling cascade in this map with nonlinear dissipation converges to a critical point of Hamiltonian type with scaling properties which are known for conservative systems.

Keywords: self-oscillating system; dissipative systems; period-doubling; scaling law; conservative systems; quasiperiodicity.

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1. Introduction

Chaotic dynamics in nonlinear dynamical systems can emerge due to different scenarios among which the route via an infinite period-doubling cascade is the most prominent one. It was found to appear in many model and realistic systems (see, e.g., [1] for references). Feigenbaum proved that this route is universal for dissipative systems possessing a quadratic nonlinearity and derived the scaling behaviour for successive period-doubling bifurcations [2, 3]. Later it was demonstrated that conservative systems can also exhibit universal period-doubling behavior, but with a different manifestation of self-similarity in parameter space in the vicinity of the critical point as well as in phase space at the critical point (see [4, 5] for detailed description and references). This point is also called Hamiltonian critical point (H-point, or H-type critical point). From the point of view of renormalization group analysis, the H-point corresponds to a codimension 2 critical point in dissipative systems and to a codimension 1 critical point in conservative systems [6–8]. This has been shown for the Hénon map [8], where this codimension 2 point appears at the intersection of the line of zero dissipation and the Feigenbaum critical line of the transition to chaos. However, it was also demonstrated that in systems with constant dissipation even a small damping destroys the corresponding Hamiltonian scaling behaviour, and the period doublings begin to obey the Feigenbaum scaling law for the dissipative case [9–11]. Hence, the study of systems with constant dissipation revealed that the observation of an H-type critical point is only possible if the dissipation is exactly zero, which can not be realized in a physical experiment. Therefore, the only way to obtain such a type of critical behaviour is to construct more general systems with non-constant dissipation. It will be also interesting from the theoretical point of view to find the H-type critical point in such systems, because to the best of our knowledge it has been so far only observed in conservative systems.

The idea to obtain a realistic physical situation possessing H-type critical scaling behaviour is to take a self-oscillating system and try to compensate the energy loss in average over the oscillation period by means of an external forcing. The aim of this paper is to propose such a system which can be physically realized. To this end we employ the van der Pol oscillator, which is a classical model in nonlinear dynamics. Its analysis allows us to obtain results which can be con-

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sidered to be rather general. For simplicity we apply δ-pulses as the external forcing on the system and expect that at a certain combination of damping and nonlinearity parameters the Hamiltonian critical behavior could be observed.

The paper is organized as follows: Section 2 introduces the model of investigation and demonstrates how to derive the approximate map from the differential equations. Section 3 and 4 discuss the dynamics of this map for negative and positive nonlinear dissipation respectively. Moreover, they highlight the commonalities and the differences between the two parameter regions. Finally, Section 5 is devoted to the search of the critical point of Hamiltonian type.

We would like to note that all bifurcation curves shown in this paper were obtained by means of the software CONTENT [12].

2. System under investigation

Let us consider the van der Pol oscillator driven by external δ-pulses:

$$\ddot{x} - (\varepsilon - \mu x^2)\dot{x} + x = \sum_m F(x)\delta(t - mT) \tag{1}$$

Here x is the dynamic variable, while ε and μ are the parameters related to dissipation. The parameter ε characterizes the strength of the linear dissipation whereas μ is responsible for the nonlinear dissipation. For positive values of these parameters the system demonstrates a stable limit cycle of size $r \approx \sqrt{\varepsilon/\mu}$ in the absence of the external driving, while for negative parameter values the limit cycle is unstable. The sum on the right hand side of equation (1) corresponds to a forcing in form of δ-like pulses applied at time intervals T. The amplitude of these pulses is modulated by the function $F(x)$ which depends on the dynamic variable itself.

To simplify the analysis let us construct an approximate discrete map for system (1). Let us denote the values of the dynamical variables x and $y = \dot{x}$ before the n-th pulse to be x_n and y_n. Hence just after the pulse these variables will have the values

$$x_{+0} = x_n, y_{+0} = y_n + F(x_n) \tag{2}$$

Now let us consider the time instant of this pulse as t=0 and construct an approximate solution for x_{n+1} and y_{n+1} using the van der Pol averaging method which implies the representation of the dynamical variables as a combination of a high frequency oscillating part and a slow amplitude a,

$$x = ae^{it} + a^* e^{-it}, \ \dot{x} = y = \mathrm{i}ae^{it} - \mathrm{i}a^* e^{-it}. \tag{3}$$

Here the additional condition $\dot{a}e^{it} + \dot{a}^* e^{-it} = 0$ is used. Further time averaging of the system equations over the period of the high frequency oscillations leads to a differential equation describing the time evolution of the slow amplitude a between two subsequent pulses [13]:

$$\dot{a} = \tfrac{1}{2}ea - \tfrac{1}{2}\mu|a|^2 a \tag{4}$$

Using the initial condition (2), we obtain

$$a(t) = \frac{a_{+0}\exp(et/2)}{\sqrt{1 + (\mu/e)(\exp et - 1)|a_{+0}|^2}} = \tfrac{1}{2}(x - iy)\exp(-it). \tag{5}$$

Equation (5) at t=T will give us the amplitude just before the (n+1)-th pulse, and it will allow us to find the coordinate and velocity:

$$x_{n+1} = B\frac{x_n\cos T + (y_n + F(x_n))\sin T}{\sqrt{1 + C\left[x_n^2 + (y_n + F(x_n))^2\right]}},$$
$$y_{n+1} = B\frac{-x_n\sin T + (y_n + F(x_n))\cos T}{\sqrt{1 + C\left[x_n^2 + (y_n + F(x_n))^2\right]}}, \tag{6}$$

where the new parameter $B = \exp\frac{1}{2}\varepsilon T$ depends nonlinearly on the value of linear dissipation, and $C = \mu T(\exp\varepsilon T - 1)/4\varepsilon T$ is proportional to the strength of nonlinear dissipation.

Let us choose $T = \frac{\pi}{2}(4k + 1)$ with k=0,1,2,… for simplicity. In this case the map (6) transforms into

$$x_{n+1} = \frac{B(y_n + F(x_n))}{\sqrt{1 + C\left[x_n^2 + (y_n + F(x_n))^2\right]}},$$
$$y_{n+1} = -\frac{x_n B}{\sqrt{1 + C\left[x_n^2 + (y_n + F(x_n))^2\right]}}. \quad (7)$$

Now we have to choose a specific form of the function *F(x)* modulating the pulse amplitude. Note that in the case of zero nonlinear dissipation *C*=0 the denominator in both equations (7) is equal to 1 so the form of the function *F(x)* determines the nonlinear properties of the system in this case. We now choose the most simple quadratic function for the amplitude modulation $F(x) = 1 - \lambda x^2$. In this case the map (7) turns into the Hénon map with renormalized parameters. Finally, the map will take the form

$$x_{n+1} = B\frac{1 - \lambda x_n^2 + y_n}{\sqrt{1 + C\left[x_n^2 + (1 - \lambda x_n^2 + y_n)^2\right]}},$$
$$y_{n+1} = -B\frac{x_n}{\sqrt{1 + C\left[x_n^2 + (1 - \lambda x_n^2 + y_n)^2\right]}}. \quad (8)$$

and can be considered in some sense as the parametrized version of the Hénon map.

The structure of the parameter plane of system (7) at *C*=0 is shown in Figure 1. The line *B*=1 corresponds to the conservative Hénon map. Along this line we find a period doubling cascade possessing the well-known Hamiltonian scaling behaviour. The terminal point of this period doubling cascade is therefore an H-type critical point. This point is exactly the intersection of the line of zero dissipation (*B*=1) and the line of Feigenbaum points – the accumulation points of the period doubling cascade for *B*<1. For *B*>1 the trajectories diverge.

Depending on the sign of *C* our system can demonstrate two different regimes: (i) For *C*<0 the

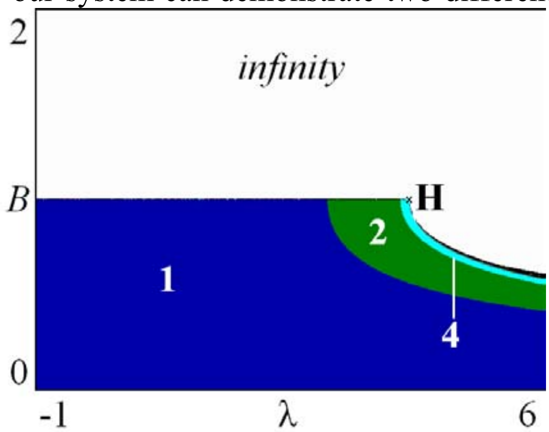

Figure 1 (colour online). The structure of the parameter plane of map (7) with the function $F(x) = 1 - \lambda x^2$ at *C*=0. Areas of different colour correspond to regions of existence of cycle of certain periods (see numbers in the figure). The black area corresponds to non-periodic motion. The black cross marks the location of the critical point of H-type [8]. "Infinity" here and further denotes the area where trajectories diverge.

nonlinear dissipation is negative (μ<0) and acts therefore like an additional forcing. In this case the system possesses the subcritical Hopf bifurcation leading to an unstable limit cycle. For ε<0 there exists a stable equilibrium at the origin and an unstable limit cycle, while for ε>0 the equilibrium in the origin becomes unstable. (ii) C>0 corresponds to a positive nonlinear dissipation. Hence system (1) turns into the usual van der Pol system with a supercritical Hopf bifurcation at ε=0. Let us consider these two cases separately.

3. System with negative nonlinear dissipation

Let us analyze the structure of the parameter plane spanned by the parameter related to the linear dissipation *B* and the nonlinearity in the pulse amplitude λ. Figure 2 represents areas of the exis-

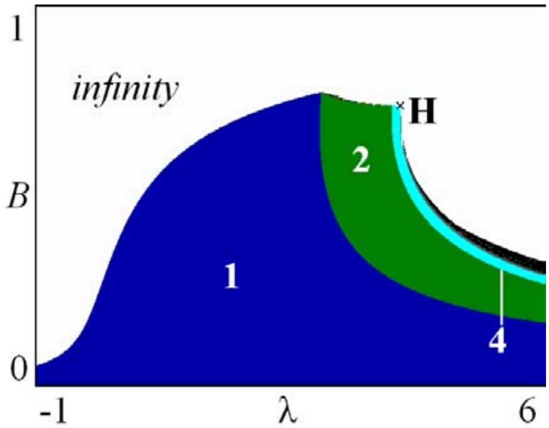

Figure 2 (colour online). The structure of the parameter plane of map (8) at *C*=-0.9. Areas of different colour correspond to regions of existence of cycle of certain periods (see numbers in the figure). The black area corresponds to non-periodic motion. The black cross marks the location of critical point of H-type (see discussion in Section 5).

tence of different dynamical regimes on the plane (λ, *B*). As one can see, in this case the area, in which trajectories escape to infinity grows dramatically compared with the linear dissipation case (cf. Figure 1). To emphasize the important bifurcations we show additionally the structure of the bifurcation lines in Figure 3. Areas of periodic regimes are bounded by lines of Neimark-Sacker (NS) bifurcations indicating the transition to quasiperiodic motion. The Neimark-Sacker bifurcations can be either super- or subcritical leading to stable or unstable quasiperiodic motion respectively. The type of the bifurcation (super- or subcritical) changes with the transition to the next period, when the line of the corresponding period doubling is crossed (see Figure 2b). Lines of Neimark-Sacker bifurcations and period-doubling lines intersect in the resonance 1:2 – point with double multiplier -1 (R2) (in the terminology of [14]). Such structure of the bifurcation lines reflect the results of the mathematical analysis in the vicinity of the R2 point (see again [14]) and it repeats for all periods of the period-doubling cascade. However it is important to note that at each subsequent period doubling bifurcation the type of the NS bifurcation changes from super- to subcritical and vice versa. This sequence of codimension 2 bifurcation points where period doubling and Neimark-Sacker bifurcations meet converges to a certain limit whose properties will be discussed in detail below.

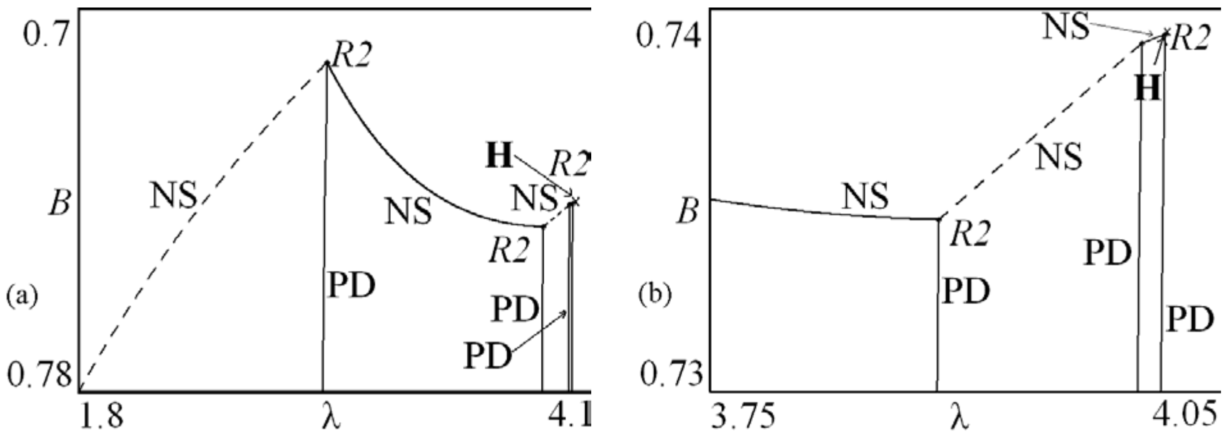

Figure 3. Structure of the bifurcation lines for map (8) at *C*=-0.9 and its magnification. NS – line of Neimark-Sacker bifurcation, PD – period doubling line, R2 – resonance 1:2 point, H – critical point. The dashed line corresponds to the subcritical NS bifurcation, the solid line denotes the supercritical one.

Let us now turn to the analysis of the quasiperiodical area and the synchronization tongues which bifurcate from the regime of period 2 (see magnification of the corresponding part of the parameter plane in Figure 4). The form of these synchronization tongues differs from the “classical” Arnol'd tongues. A detailed investigation by means of bifurcation analysis shows that their struc-

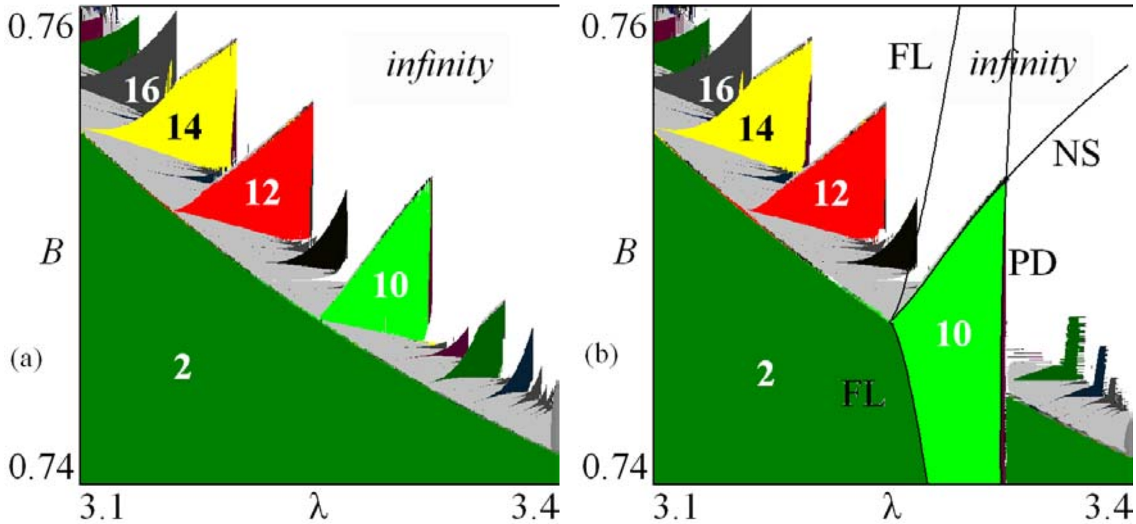

Figure 4 (colour online). Magnification of the parameter plane (a) and parameter plane with bifurcation lines overlayed (b) for map (8) at C=-0.9. Areas of different colour correspond to regions of existence of cycle of certain periods (see numbers in the figure). The light gray area corresponds to non-periodic motion. NS – line of Neimark-Sacker bifurcation, PD – period doubling line, FL – fold line.

ture is much more complicated. Here we will show the results concerning the tongue of period 10, but the described structure is rather general and applies to the other tongues as well. The lower border of this synchronization area is formed by a saddle-node (fold) bifurcation, which ends up in a cusp point on the line of NS bifurcation of the period 2 orbit. The upper part of the synchronization tongue consists of two parts. The first one close to the cusp point is formed by a second fold line. The second part is build by a Neimark-Sacker bifurcation for the period 10 cycle. The NS bifurcation turns out to be subcritical, and, hence, plays the role of the border of the synchronization area. There exists also a region where the period 2 cycle and the invariant curve bifurcated from it coexists with the period 10 cycle. On a certain line in parameter space the invariant curve undergoes a homoclinic bifurcation colliding with the saddle period 10 cycle. Phase portraits illustrating this process are shown in Figure 5. Here we mention that this synchronization tongue corresponds exactly to the one described in [15] for another 2D parametrization of the Hénon map – the generalized Hénon map [16].

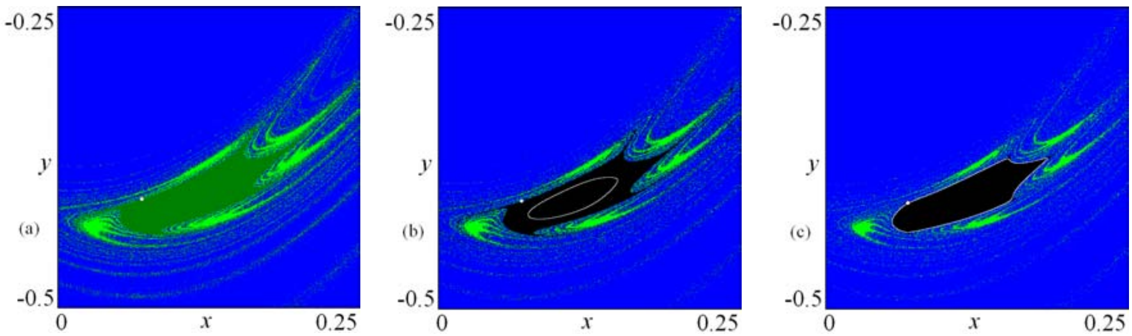

Figure 5 (colour online). Basins of attraction in different dynamical regimes. C=-0.9, λ=3.3, values of parameter B: (a) 0.7439, (b) 0.7452, (c) 0.74615. Dark gray (indigo online) corresponds to the divergence to infinity, white point marks the location of one of the points of the period-10 saddle cycle. The white curve in (b) and (c) denotes the invariant curve.

4. System with positive nonlinear dissipation

Now let us turn to the analysis of system (8) when the autonomous system in (1) is the van der Pol oscillator with the stable limit cycle. The chart of dynamical regimes of the map (7) in this case is shown in Figure 6 together with the one for the initial system (1). In contrast to the previously examined case of negative nonlinear dissipation, the area of existence of periodic regimes

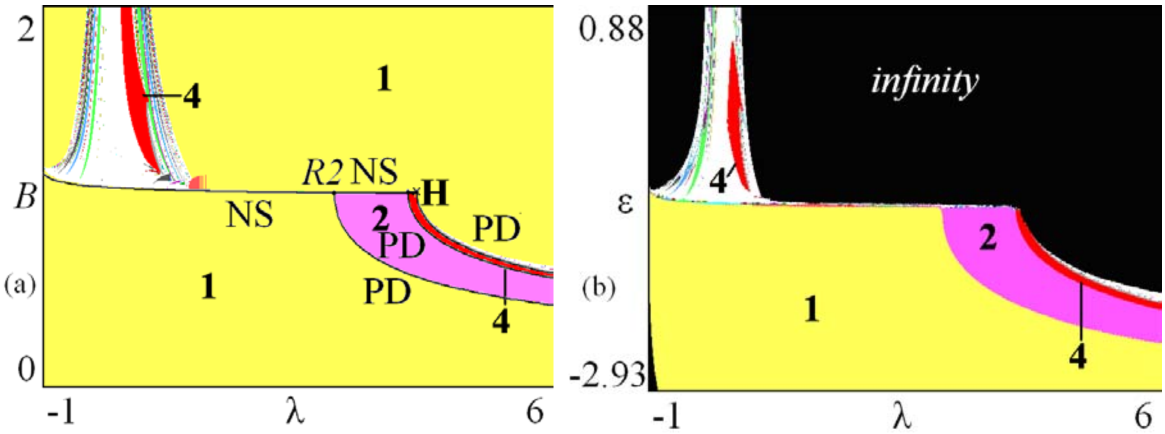

Figure 6 (colour online). Structure of the parameter plane of map (8) at C=0.1 (a) and of the original system (1) at μ=0.5. Areas of different colour correspond to regions of existence of cycle of certain periods (see numbers in the figure). The white area corresponds to non-periodic motion. The black cross marks the location of the critical point of H-type (see discussion in Section 5). For system (1) the scale of the ε axis is nonlinear and corresponds to the linear scale for parameter B in map (8).

grows in size. Additionally we observe the appearance of the area of quasiperiodic motion with embedded synchronization tongues near the axis λ=0 which corresponds to the van der Pol oscillator subject to external δ-pulses with fixed amplitude. These findings fit well to known results for such systems. Another difference which one can easily detect is disappearance of the area of unstable dynamics (diverging trajectories). In place of it the region of period 1 appears. However, comparing Figures 6(a) and 6(b), it becomes obvious that this regime is nonphysical and appears just due to the approximations made when deriving the map.

In the case of positive nonlinear dissipation (C>0) the lines of NS bifurcation again change their type (super- to subcritical and vice versa) at the transition to the next period in the period-doubling cascade, but their type is now opposite to the one they had in the previous case (C<0). In contrast to the case of negative C we now obtain a supercritical bifurcation for the stable fixed point, while for the cycle of period 2 a subcritical bifurcation is observed (see enlarged fragments of the parameter plane with the superimposed bifurcation lines in Figure 7). As a consequence, the lines of NS bifurcation change their type when crossing the line of C=0. This resembles the situation at C=0 in the autonomous system in (1), where the Hopf bifurcation also changes its type.

Here we also have to mention that in case of positive nonlinear dissipation points of R2 type also form a converging sequence.

Now let us turn to the more detailed analysis of the properties of these sequences.

5. Search of the critical point of Hamiltonian type

As it has been mentioned above, points with double -1 multiplier (R2-points) form a sequence in both cases of negative and positive nonlinear dissipation respectively. These sequences are given in Tables 1 and 2 correspondingly. From these tables one can see the convergence of these sequences to limit points. These points have the same properties as the terminal points of the period-doubling lines in Hénon map – they also have double -1 multipliers. Moreover, at these points the NS bifurcation changes its type. Normally, the change of type of NS bifurcation, which occurs at the Chenciner bifurcation point, corresponds to the neutral nonlinear stability of the fixed points [14], which, in turn, is also a feature of conservative systems. Hence, we conclude, that our system obeys some properties of conservative systems when moving along this route. As a consequence this sequence of R2 points has to end up in a critical point of Hamiltonian type. To illustrate this result we have calculated the scaling constants as the eigenvalues of

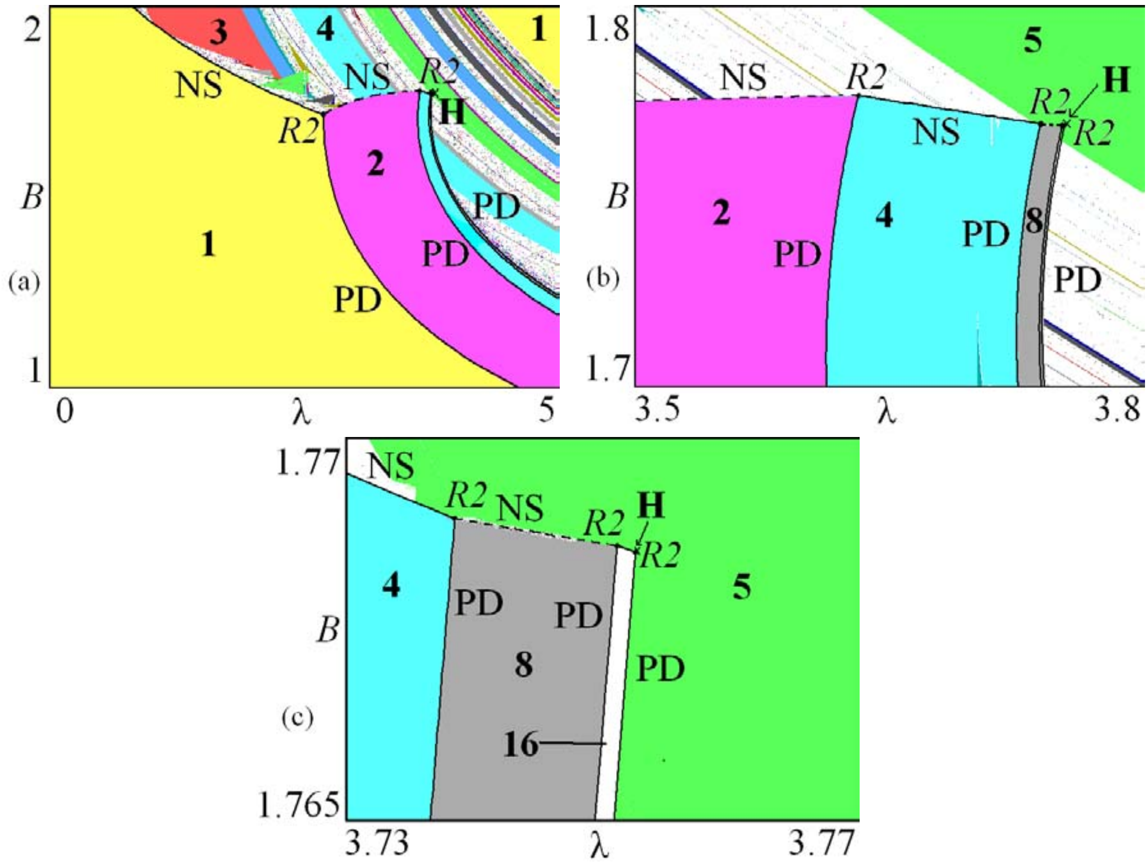

Figure 7 (colour online). Magnifications of the parameter plane for map (8) at C=4 and structure of bifurcation lines. Areas of different colour correspond to regions of existence of cycles with certain periods. The white area corresponds to non-periodic motion (in (c) also period 16, where it is marked). NS – line of Neimark-Sacker bifurcation, PD – period doubling line, R2 – resonance 1:2 point, H – critical point. The dashed line corresponds to the subcritical NS bifurcation, the solid line denotes the supercritical one.

Table 1. Sequence of resonance 1:2 points for map (8) at C=-0.9. n denotes the period.

n	λ	B
16	4.019218352	0.739364303
32	4.019398386	0.739368355
64	4.019419057	0.739368905
128	4.019421426	0.739368962
256	4.019421697	0.739368969
	4.019421…	0.73936896…

Table 2. Sequence of resonance 1:2 points for map (8) at C=4. n denotes the number of period.

n	λ	B
16	3.752589280	1.768497964
32	3.752754648	1.768491542
64	3.752773535	1.768490657
128	3.752775706	1.768490565
	3.75277…	1.768490…

the matrix Γ, which links the intervals of existence of two subsequent cycles of the period-doubling cascade in the parameter plane:

$$\begin{pmatrix} \Delta\lambda_{2n} \\ \Delta B_{2n} \end{pmatrix} = \Gamma \begin{pmatrix} \Delta\lambda_{n} \\ \Delta B_{n} \end{pmatrix} \tag{9}$$

Using this method, we have obtained for the case C=-0.9 $\delta \approx 8.7211$ and for the case C=4 $\delta_1 = 8.72109$, which is in very good correspondence with the theoretically known value $\delta = 8.7210972\ldots$.

But it is impossible to evaluate the second scaling constant correctly using this sequence of points, because the second scaling number is responsible for the scaling of the damping, and we are already on the route without dissipation. In order to calculate this second scaling constant we have to find another sequence of points with certain scaling properties. Such a sequence is given by points in the parameter plane where periodic orbits of period n and $2n$ have equal multipliers. This eigenvalue-matching method [17] is based on the fact that while getting close to the critical point multipliers of all unstable cycles from the period-doubling cascade tend to some fixed values, which are the universal constants for the certain type of critical point. The sequence of points in the parameter plane with equal multipliers of cycle of original and double period have to converge to the critical point rather quick, and their multipliers have to converge to the universal values. Scaling constants in this case can be found as the eigenvalues of the matrix Γ, now determined by the following relation:

$$\Gamma = \Gamma_n^{-1}\Gamma_{2n}, \quad \Gamma_n = \begin{pmatrix} \frac{\partial S_n}{\partial \lambda} & \frac{\partial S_n}{\partial C} \\ \frac{\partial J_n}{\partial \lambda} & \frac{\partial J_n}{\partial C} \end{pmatrix}, \tag{10}$$

where S_n and J_n are the trace and the determinant of the monodromy matrix (Jacobi matrix) of the period-n cycle in this point, respectively.

The coordinates of the points of the specified sequence and the values of cycle multipliers are given in the Table 3. The scaling constants estimated using these data are listed in the Table 4. One can see that these points demonstrate a rather good convergence to the limit point, and multipliers and scaling constants are in rather good correspondence with the known universal values for this type of critical behavior: $\delta_1 = 8.7210972\ldots$, $\delta_2 = 2$, $\mu_1 = -2.0574783\ldots$, $\mu_2 = -0.4860318\ldots$[8]

Table 3. Sequence of points with equal multipliers for cycles of the original and the doubled period for map (8) at B=4. n denotes the period of the original cycle while $2n$ denotes the doubled period.

n	$2n$	Λ	C	μ_1	μ_2
16	32	2.641979414	22.21231986	–2.057641674	–0.486079481
32	64	2.641972827	22.21235779	–2.057456301	–0.486025619
64	128	2.6419732641	22.21235527	–2.057481305	–0.486032664
128	256	2.641973235	22.212355441	–2.057477956	–0.486031738

Table 4. Scaling constants estimated using the data from Table 3.

n	$2n$	δ_1	δ_2
16	32	8.721435908	2.000128521
32	64	8.721042264	1.998814272
64	128	8.721096857	1.999608640
128	256	8.721095561	2.002160475

6. Conclusions

In the present work we have considered a self-oscillating system, the van der Pol oscillator, driven by a periodic train of δ-pulses whose amplitude depends on the dynamical variable in a nonlinear way. We have investigated the dynamics of an approximate discrete map derived for this system using averaging methods. This map exhibits different behaviour for negative and positive nonlinear dissipation, respectively. The most interesting feature of this map is the given by the existence of a sequence of vertex points in which the period doubling bifurcations end in a Neimark-Sacker bifurcation (Figures 3, 6, 7). These vertex points are additionally accompanied with a change in the type of Neimark-Sacker bifurcation from super- to subcritical and vice versa. This sequence of points is related to a double multiplier -1 and converges to the critical point of Hamiltonian type. The H-type critical point was found both as a limit of sequence of points with double -1 multiplier and with equal multipliers of two consecutive cycles from the period-doubling cascade. This result shows that this critical point can not only be found in conservative systems but also in systems with dissipation which effectively depends on the values of the dynamical variables†. The H-type critical point appears as a codimension 2 point in the parameter plane or as a line in the 3D parameter space. In the vicinity of such point one can obtain not only period-doublings, but also quasiperiodical regimes. Such a critical point can be found in different dynamical regimes of the system, namely when the autonomous system has a stable or an unstable limit cycle, respectively. Though in the present work we discuss only the results for just one value of the external period forcing, the general structure of the parameter space does not change with a change of the period. We believe that all results discussed above are valid to the original ODE system (1) too. Although our system (8) is not the precise Poincaré map for system (1), one can construct another ODE system corresponding to map (8) which will demonstrate all described phenomena. Since critical phenomena are universal and we studied a system of rather general type, one can expect that the picture of the critical behaviour as well as the bifurcation lines for the reconstructed and the original ODE systems will not differ dramatically. Moreover, one can expect that the qualitative features of the parameter space structure and quantitative scaling relations in the vicinity of the critical point will exist independently on the specific details of the structure of the system, e.g., the external influence on the oscillator could be modeled in form of finite length pulses or a continuous signal.

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† Here we mean that the dependence of dissipation on the values of the dynamical variables is caused by the nature of the described phenomena. Hence such systems can not be reduced to the form with only parameter-depending dissipation which would be necessary to obtain conservative dynamics.

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