Forced synchronization of quasiperiodic oscillations.

Stankevich N.V., Kurths J., Kuznetsov A.P.

April 20, 2014

Saratov State Technical University,
Politehnicheskaya 77, Saratov, 410054, Russian Federation.
Potsdam Institute for Climate Impact Research,
P.o. BOX 60 12 03, Potsdam, 14412, Germany.
Kotel'nikov's Institute of Radio-Engineering and Electronics of RAS,
Saratov Branch,
Zelenaya 38, Saratov, 410019, Russian Federation.

Abstract

A model of a generator of quasiperiodic oscillations forced by a periodic pulse sequence is studied. We analyze synchronization when the autonomous generator demonstrates periodic, quasiperiodic, respective weakly chaotic oscillations. For the forced quasiperiodic oscillations a picture of synchronization, consisting of small-scale and large-scale structures was uncovered. It even includes the existence of stable the three-frequency tori. For the regime of weak chaos a partial destruction of this features and of the regime of three-frequency tori are found.
Keywords: dynamical system; quasiperiodic oscillations; synchronization; Lyapunov exponents.

PACS: 05.45.-a

1 INTRODUCTION

Quasiperiodic oscillations are wide-spread in different areas of science and technology [1] - [8]. That is why the problem of synchronization of these oscillations with an external forcing is very important in nonlinear dynamics. The problem of synchronization of quasiperiodic oscillations has been studied less than synchronization of regular and chaotic regimes. There is a large set of problems in the theory of synchronization of quasiperiodic oscillations that have not been solved. In particular, autonomous systems forced by an external action can have different structure: firstly, it can be a system of coupled self-oscillatory elements [9, 10, 11, 12]; two new variants occur for cases of dissipative and conservative coupling [13, 14]; further, the equations of the isolated elements can be of van der Pol equation type, phase equations like the Adler equation [12, 15] or can be a map for phases (map on a torus) [16, 17].

The problem when an external force acts on an *autonomous* system with a quasiperiodic behavior is very important and attracting. However, in the literature there are a few models of autonomous systems with quasiperiodic dynamics, which allowed a physical realization. One of these systems is the Chua circuit, but this model is described by a piecewise-linear characteristic of nonlinearity and it has remained not enough explored in respect of quasiperiodic behavior in this system [8, 18]. Recently, in [19, 20, 21] a four-dimensional system - a modification of the Anishchenko-Astakhov generator with autonomous quasiperiodic dynamics was suggested for such studies. However, the research has been very limited by studying an autonomous system, when it generates a torus doubling [19, 20]. A system of two coupled generators of this kind was also studied in [21].

In order to get a complete understanding, the study of synchronization of quasiperiodic oscillations requires to consider three main directions:
i) synchronization of a resonance cycle on the torus;
ii) synchronization of a quasiperiodic regime with incommensurate frequencies;
iii) synchronization when the torus is destroyed.

The first problem was discussed in [10, 15] for a model of excited coupled elements. But more interesting and even new effects occur, when the oscillations in the autonomous system are non-regular, quasiperiodic or even weakly chaotic. This problem will be discussed in the present paper, i.e. we consider synchronization of autonomous quasiperiodic oscillations and torus which is being destroyed under external force. As an autonomous system, we will use a generator of quasiperiodic oscillations [22, 23]. This generator has the advantage to have the minimal dimension of phase space, for which an invariant torus can be generated and it can be realized as electronic circuit [23]. We will show that in this case new interesting features in the parameter plane appear. As external action we choose periodic sequences of short pulses. This choice simplifies the interpretation of our results, since between the pulses the dynamics of the system is autonomous. On the other hand, a pulsed action is important for various applications, e.g. in biophysics [24].

2 AUTONOMOUS GENERATOR OF QUASIPERIODIC OSCILLATIONS

As an autonomous system we use a generator of quasiperiodic oscillations, which was recently suggested and described in [22, 23]. It represents the "hybrid" of a self-generator with a hard excitation and a relaxation generator. The corresponding equations have the form:

$$\begin{aligned} &\ddot{x} - (\lambda + z + x^2 - \beta x^4)\dot{x} + \omega_0^2 x = 0, \\ &\dot{z} = \mu - x^2. \end{aligned} \tag{1}$$

Here ω_0 ($\omega_0 > 0$) is main frequency of the generator, the parameter λ ($\lambda \geq 0$) characterizes negative friction and controls subcritical Hopf bifurcation in self-generator of hard excitation, β ($\beta > 0$) is the response for the saturation of oscillations at large amplitudes. The parameter λ enters to the equation as well as z, which characterizes the state of a charging relaxation element, and its evolution in time is controlled by the second equation. System (1) has two independent time scales that allows for a two-frequency quasiperiodic dynamics. The first time-scale is the period of oscillations of the self-generator $T = \frac{2\pi}{\omega_0}$ and the second one is the characteristic recovery time of the state storage element $\tau = \mu^{-1}$. In [22] it was shown that this system can create quasiperiodic oscillations, but also periodic and chaotic regimes. In Fig. 1 some examples of phase portrait of system (1) for the Poincaré section made at $x = 0$ and the Fourier spectrum of periodic, quasiperiodic and chaotic regimes are shown. Figures a), d) correspond to periodic oscillations: in Fourier spectrum one can see main pike which response to frequency of self-generator ω_0. In Fig. 1 b), e) we see quasiperiodic oscillations. Phase

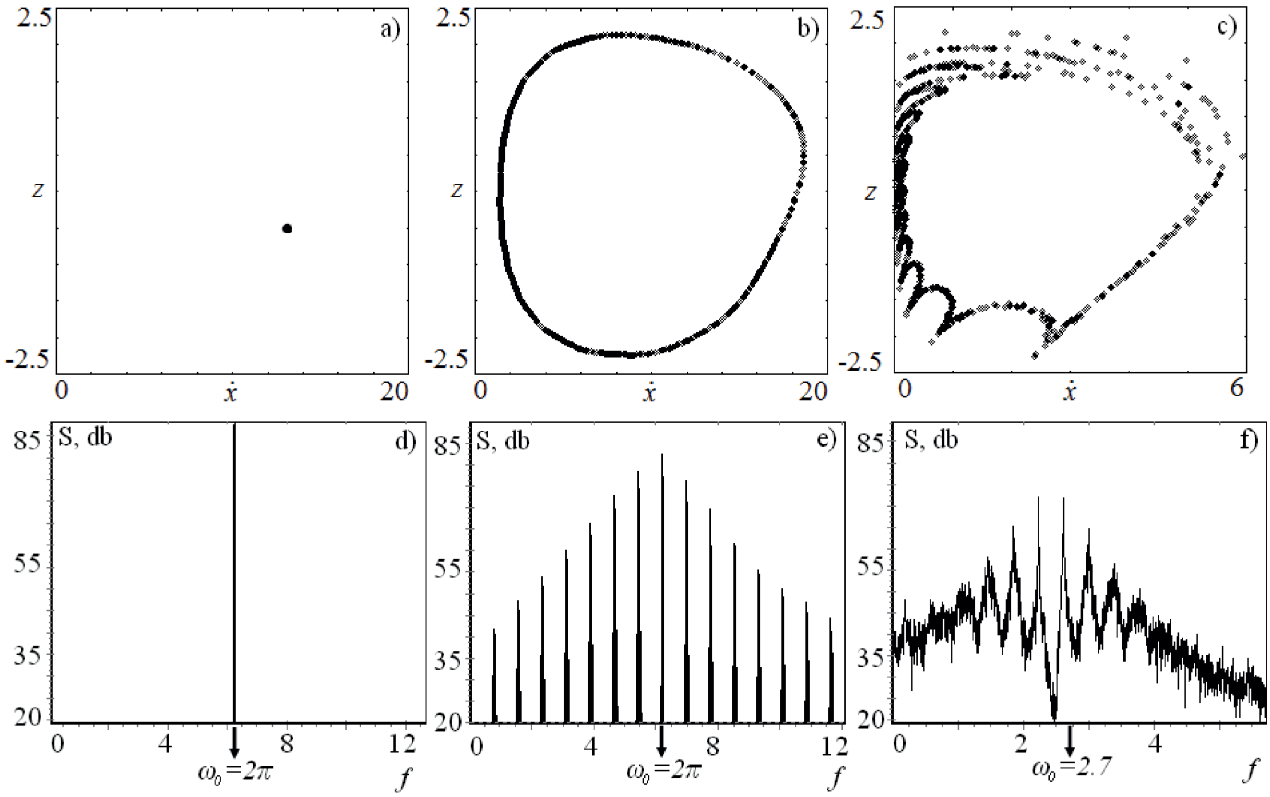

Figure 1: The phase portrait in the Poincaré section and Fourier spectrum for system (1): a) periodic oscillations $\lambda = 0$, $\beta = 0.25$, $\omega_0 = 2\pi$, $\mu = 2.2$; b) quasiperiodic oscillations $\lambda = 0$, $\beta = 0.25$, $\omega_0 = 2\pi$, $\mu = 0.9$; c) chaotic oscillations $\lambda = 0$, $\beta = 0.5$, $\omega_0 = 2.7$, $\mu = 0.3$.

portrait has form of closed invariant curve. Fourier spectrum contains a discrete set of equally distant frequency components. Their level decrease to the left and to the right from the main spectral line corresponding to the basic operational frequency of the self-oscillatory element. Distance to the nearest neighbor satellites corresponds to the frequency of the relaxation oscillations. At least, Fig. 1 c), f) correspond chaotic oscillations. We would like to note that this chaotic attractor is result of destruction of torus that is why the form of phase portrait in Poincaré section is similar to invariant curve. Also in Fourier spectrum one can see main pike corresponding to ω_0 and pike-satellites, but it has noisy component. Later we consider all these cases excited by a periodic pulse force.

3 SYNCHRONIZATION OF QUASIPERIODIC OSCILLATIONS UNDER EXTERNAL FORCING

Let the external signal in the form of periodic sequence of delta-pulses acts on the model (1):

$$\begin{aligned} &\ddot{x} - (\lambda + z + x^2 - \beta x^4)\dot{x} + \omega_0^2 x = A \sum \delta(t - nT), \\ &\dot{z} = \mu - x^2. \end{aligned} \tag{2}$$

Here A is the amplitude of the external signal, and T is its period, $\delta(t-nT)$ the Dirac delta-function and n the number of pulses.

For the following discussion it is very important to start by considering synchronization of the limit cycle before the onset of a Neimark-Sakker bifurcation which creates an invariant torus. This case corresponds to the follwing set of parameters $\beta = 0.25$, $\omega_0 = 2\pi$, $\mu = 2.2$. In Fig. 2 a chart of dynamical regimes of system (2) on the parameter plane period vs amplitude of the external force (T, A) is presented. The color palette was chosen in correspondence with the period of the regime in the stroboscopic section, i.e. through the period of the external action T, and the gray color refers as quasiperiodic and chaotic regimes. We observe a set of Arnold tongues, among them tongues of period-1 (it corresponds to one fixed point in the stroboscopic section) have the largest size. The set of these tongues responds to resonances on modes of the external force. The distance between these

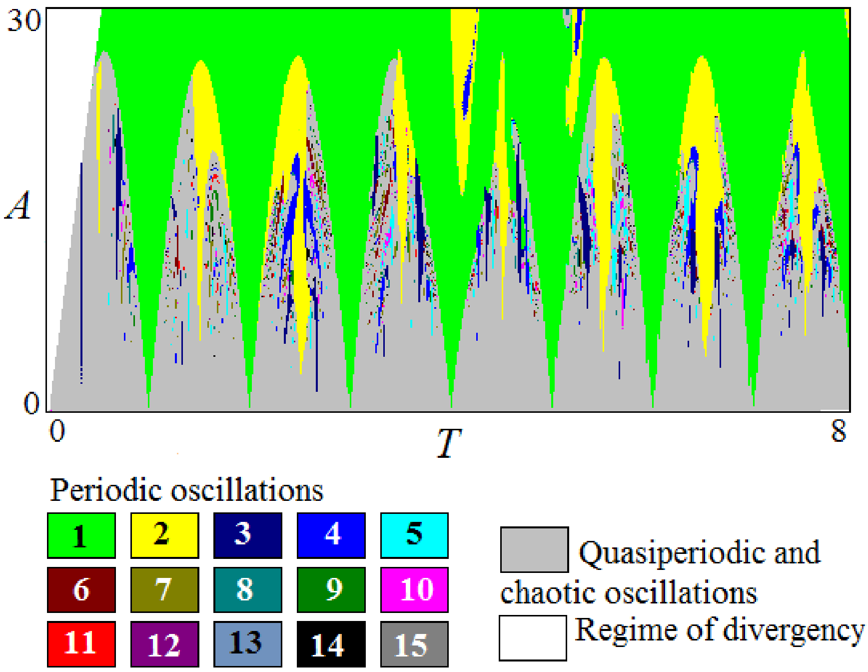

Figure 2: The chart of dynamical regimes of excited by pulses generator (2) in case when limit cycle is realized. $\lambda = 0$, $\beta = 0.25$, $\omega_0 = 2\pi$, $\mu = 2.2$. Under chart there is a palette corresponding colors, which will be also used in Fig. 3 and 7.

tongues is approximately $T = 1$, that corresponds to the main period of oscillations of the generator at $\omega_0 = 2\pi$.

Now we analyze the case of synchronization of quasiperiodic oscillations. We change the parameter μ, and put it $\mu = 0.9$, which correspond to an autonomous quasiperiodic regime. In Fig. 3 a chart of the dynamical regimes and a magnified part are shown. The magnified part was chosen in such a way, that one can observe much more detailed features of the synchronization picture. In this case, there is the set of synchronization tongues of period-1 on the chart of dynamical regimes, which follows through approximately equal intervals by the period of the external force $T \approx 1$. Note, however, that these tongues are gathered in groups of, approximately, 7-8 tongues; this time interval corresponds to the second time-scale on the chart $T \approx 8$. For a clearer understanding of this result, in Fig. 3 b) the magnified part of chart of dynamical regimes, covering the first group of eight tongues of period-1, is shown.

The occurrence of regular regimes of period-1 implies the opportunity to stabilize quasiperiodic oscillations by an external signal, such that a complete synchronization of the system by an external signal is realized. This complete synchronization (in the chart it corresponds to the green color) has a "threshold" by the amplitude of the external force. It distinguishes this form of synchronization from synchronization in a regular regime. The main features in Fig. 3 are the following: at the beginning the "threshold" of complete synchronization is increased with increasing the period of the external force, but at $T \approx 4$ it starts to decrease. At $T \approx 8$ the "threshold" of synchronization goes to zero but does not reach it; it becomes very low, almost zero. After this, the sequence repeat not exactly, but it has the same structure.

To explain the obtained features, let us consider some time realization, which is shown in Fig. 4 a). The form of oscillations is typical for a quasiperiodic regime, where one can distinguish two characteristic time scales: T_0 is the main period of the oscillation of the generator, which at $\omega_0 = 2\pi$ equals $T_0 = 1$; T_{puls} is the period of beats of the quasiperiodic oscillations, which can be estimated as $T_{puls} \approx 7 - 8$[1]. Since the external force is a sequence of delta-pulses, then between the pulses the system is autonomous. In turn through each moment of the time T the coordinate $\dot{x}$ gets an additive, which equals to amplitude of pulse A.

[1] T_{puls} was estimated from Fig. 4 a), but also it can be estimated as $T_{puls} \approx \frac{2\pi}{\omega_0} \approx 6.977...$

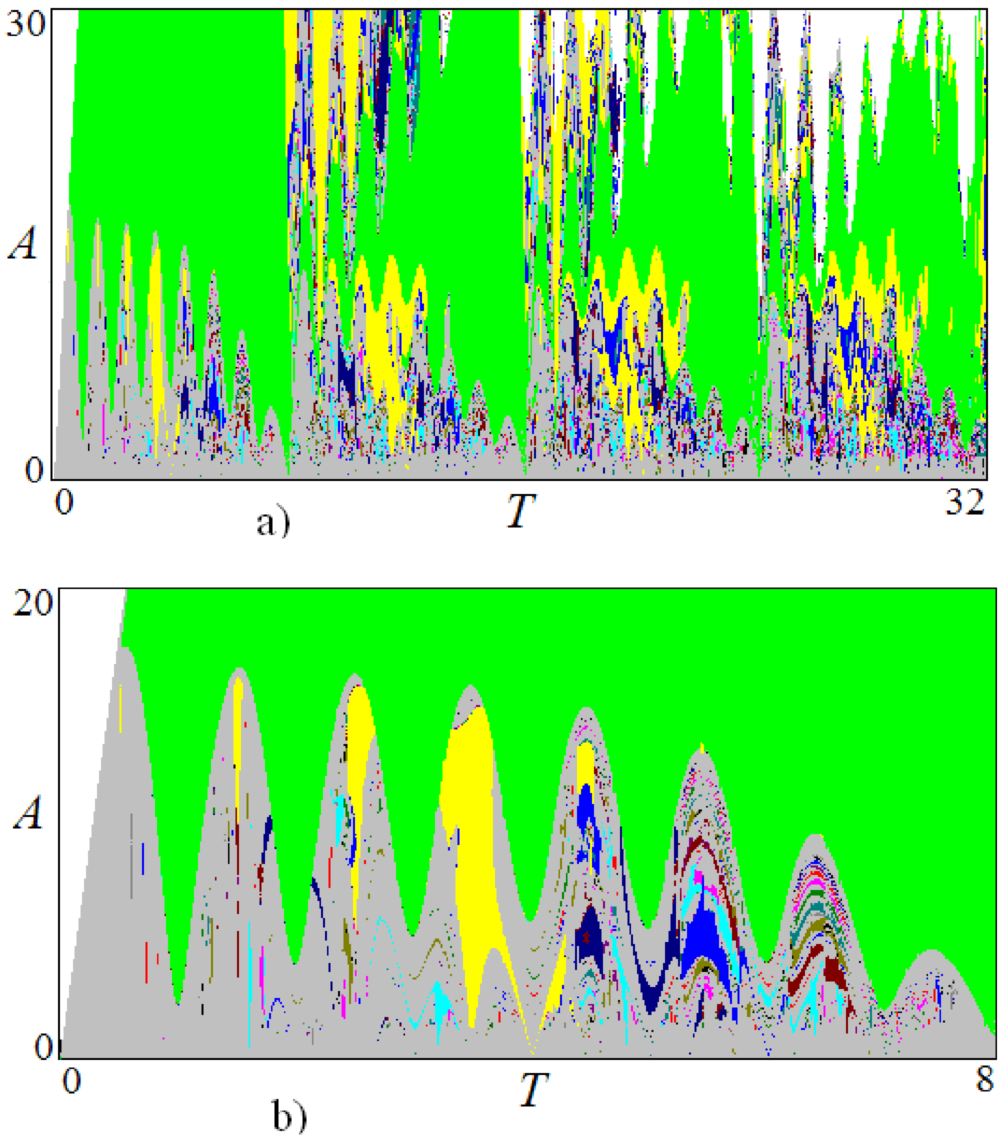

Figure 3: The chart of dynamical regimes of excited by pulses generator (2) and its magnified part in case when quasiperiodic regime is realized. $\lambda = 0$, $\beta = 0.25$, $\omega_0 = 2\pi$, $\mu = 0.9$.

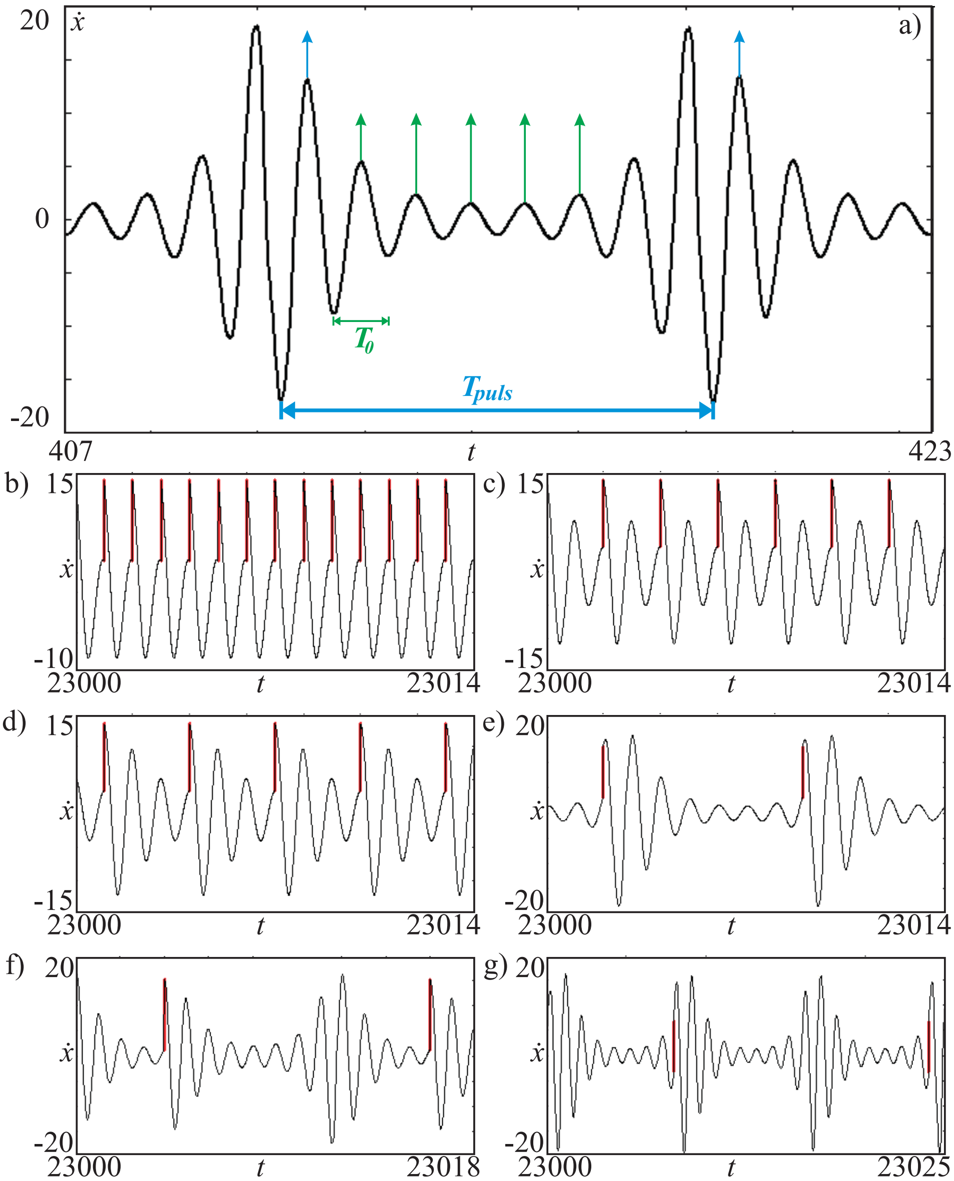

Figure 4: Figure a) illustrates different opportunity, leading to synchronization of quasiperiodic oscillations with the occurrence of a period-1 regime within the interval T_{puls}. Arrows show different variants of moments of actions and amplitudes of external action. b) - g) examples of time realizations demonstrating complete synchronization for non-autonomous system (2) at $A = 10$, b) $T = 1$; c) $T = 2$; d) $T = 3$; e) $T = 7$; f) $T = 12$, $A = 14$; g) $T = 16$

In Fig. 4 a) it is clearly shown, that complete synchronization to the frequency of the external force is possible when the pulses act at the moments $T = T_0$, $T = 2T_0$, $T = 3T_0$, ... , $T = nT_0$ (green arrows). Of course, the dynamics of system (2) is not one dimensional and it has two other variables, and the picture of synchronization is repeated not exactly but has an approximate character.

In Fig. 4 a) one can see, that with growing n in Eq. (2) the amplitude of the signal required to induce synchronization, at the beginning increases, and then decreases. The reason for this is the presence of beats with the period T_{puls}. At $T \approx T_{puls}$ the amplitude of the signal required for stabilization is almost equal zero. So we have a qualitative explanation of the structure of the parameter plane period vs amplitude of the external force within one group of tongues in Fig. 3.

Fig. 4 a) also illustrates, that at $T > T_{puls}$ the picture of synchronization repeats again (blue arrows). In this case, there is still not an exact reproduction, as the observed regimes with incommensurate frequencies, and autonomous oscillations do not reproduce itself accurately. As a result, there is a second group of 7-8 tongues, etc.

Fig. 4 b) - g) demonstrate complete synchronization of system (2) by pulses action with different period of external force, how it was described above. The pulses are marked out by red solid lines. Fig. 4 b) - e) correspond to the case of synchronization within period T_{puls} ($T < T_{puls}$); f) - g) are cases when $T > T_{puls}$.

System (1) in a quasiperiodic regime is characterized by two incommensurate frequencies. When then an external force is added, it becomes possible to realize quasiperiodic regimes with even three incommensurate frequencies, i.e. a three-frequency torus. Next we analyze three-frequency tori by means of the spectrum of Lyapunov exponents of the non-autonomous system (2) on the parameter plane of the external force. System (2) is characterized by four Lyapunov exponents Λ_n, however, since the system is non-autonomous, one of the exponents always equals zero, i.e. we have to consider only three Lyapunov exponents and neglect the always zero one. So, in dependence on the spectrum of Lyapunov exponents, the following regimes are possible in this system:

P, periodic, $0 > \Lambda_1 > \Lambda_2 > \Lambda_3$;
T_2, two-frequency torus, $\Lambda_1 = 0$, $0 > \Lambda_2 > \Lambda_3$;
T_3, three-frequency torus, $\Lambda_1 = \Lambda_2 = 0$, $\Lambda_3 < 0$;
C, chaos, $\Lambda_1 > 0 > \Lambda_2 > \Lambda_3$.

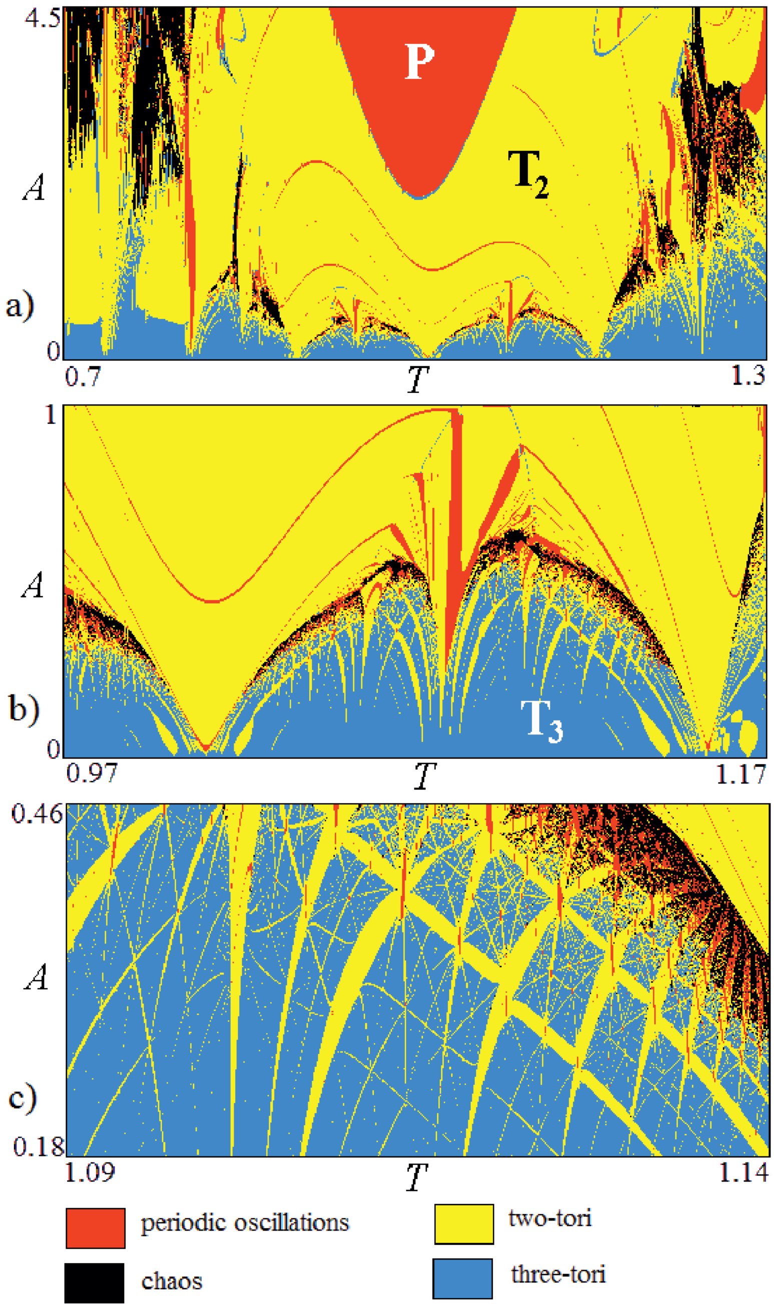

Figure 5: The chart of Lyapunov exponents in the vicinity of main tongues of complete synchronization of the generator (2) and its magnified parts. $\lambda = 0$, $\beta = 0.25$, $\omega_0 = 2\pi$, $\mu = 0.9$.

Fig. 5 presents the chart of regimes, determined with the described rules. The colors in Fig. 5 were chosen corresponding with the above four types of regimes, determined by the spectrum of Lyapunov exponents. The scale was chosen there to emphasize more interesting areas, in particular, near the main tongues of complete synchronization. In Fig. 5 a) at small amplitudes of the external signal, we observe three-frequency quasiperiodic oscillations. In the magnified part of the chart, Fig. 5 b), it becomes clear that in the area of three-frequency oscillations the set of two-frequency tongues are embedded. And the largest tongues have thinner "satellites". With a further zoom, Fig. 5 c), an Arnold resonance web [2] [27] is visualized, i.e. there is a set of narrow bands of two-frequency regimes of the parameter plane. At the intersections of these bands a small island of periodic regimes of high order occurs.

In Fig. 6 the dependence of the three Lyapunov exponents on the amplitude of the external pulses at $T = 1.44$ is shown. In Fig. 6 a)-6 d) the three-dimensional phase portraits in a stroboscopic section are shown for different values of the amplitude of the perturbations. At small amplitudes there is an interval, where the two largest Lyapunov exponents equal zero, which corresponds to a quasiperiodic regime with three incommensurate frequencies. For a decreasing amplitude of the perturbation, we observe the following evolutions: at large enough amplitudes ($A \approx 6$) two-frequency tori are realized. For a decreasing amplitude this torus undergoes a torus-doubling bifurcation and a doubled two-frequency torus is realized. Then near this doubling-torus a three-frequency torus occurs (Fig. 6 c), 6 d)).

At the amplitude $A > 0.5$, there is a quasiperiodic regime with two incommensurate frequencies which is characterized by one zero and two negative Lyapunov exponents and one can see such kind of bubbles. It is well known, that negative Lyapunov exponent characterizes degree of compression of neighbor phase trajectories in different directions. The presence of such bubbles indicates that at varying of amplitude of external force the degree of compression in two stable directions is changing. The bubbles is alternated with intervals, where both the smallest exponents is equal. It is mean that in system has place equal compression of phase trajectories in two

[2]Usually term "Arnold resonance web" is used in the theory of conservative chaos for description corresponding resonance structures in phase space [25, 26]. The way of construction Arnold resonance web using Lyapunov exponents on the plane of base frequencies in paper [27] was applied to dissipative systems. In our paper we talk about Arnold resonance web of dissipative systems

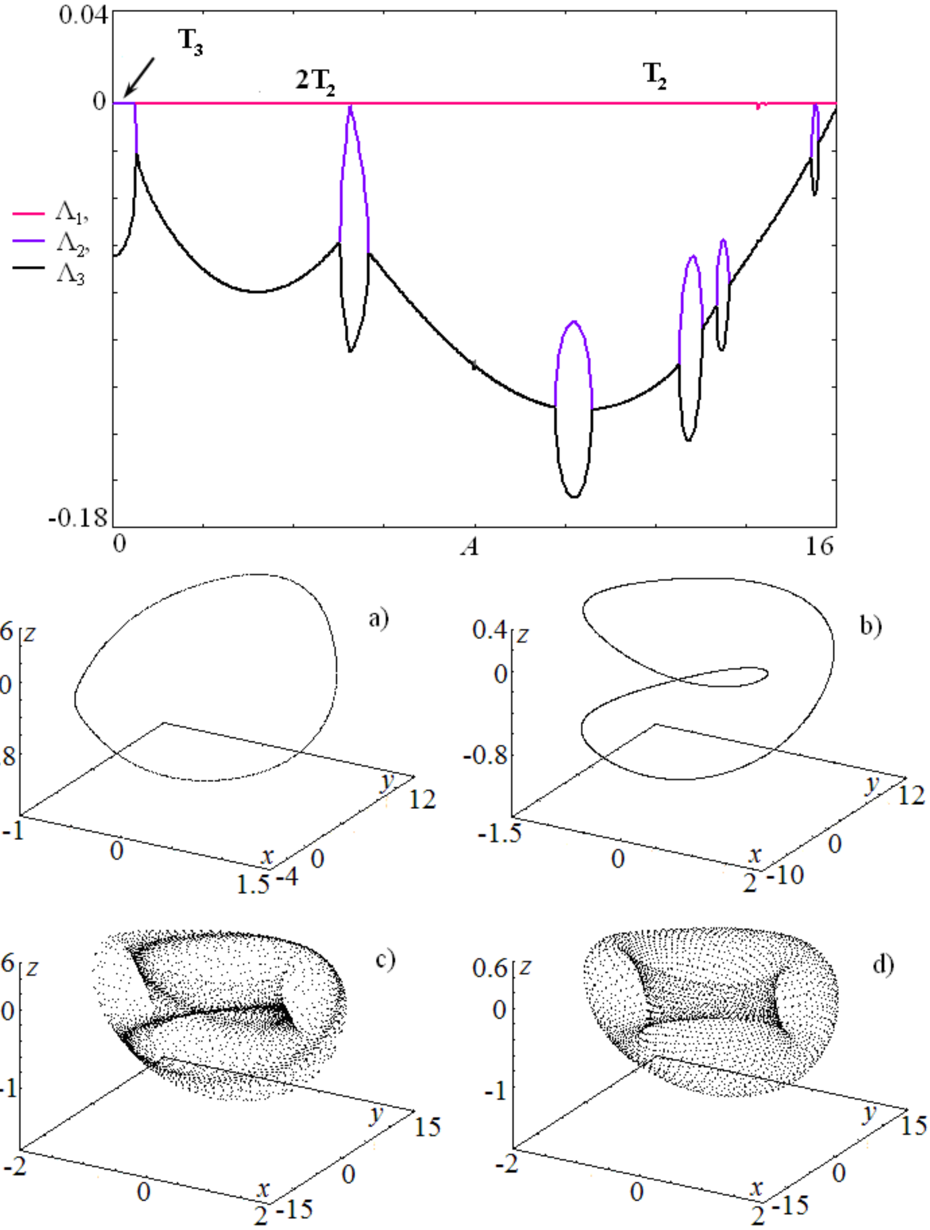

Figure 6: The plot of the three Lyapunov exponents for system (2) and the phase portrait in stroboscopic sections, $\lambda = 0$, $\beta = 0.5$, $\omega_0 = 2\pi$, $\mu = 0.9$, $T = 1.44$;) $A = 7$, b) $A = 0.52$, c) $A = 0.48$, d) $A = 0.25$.

directions. Then compression in the one of the directions is increased, and in another direction compression is reduced. The classical dependence of the Lyapunov exponents, character for bifurcation of doubling of two-frequency torus is the next: two the largest Lyapunov exponents equal zero, the third and all following exponents is negative. If we change parameters the third exponent is increasing and at the bifurcation point become equal zero, and after that it again decrease and become negative. As result of this bifurcation two-frequency double torus birth, and on the phase portrait in stroboscopic section appears additional loop. This transition we can observe in Fig. 6 (a and (b. At the larger amplitudes one can see, that two the smallest Lyapunov exponents have similar dynamics, however, the third exponent not reach zero value, and we do not have a qualities changing of behavior, and in stroboscopic section we have smooth invariant curve.

4 SYNCHRONIZATION OF TORUS LOOSING SMOOTHNESS UNDER EXTERNAL FORCING

Finally let us turn to the problem of synchronization by an external force in the chaotic regime, which occurs as a result of torus which is being destroyed. This is a very important problem connected with the nature of this chaotic synchronization.

In Fig. 7 the chart of dynamical regimes for the system excited by pulses (2) in a chaotic regime are shown. As one can see, in the interval $T < 15-16$ there is a structure similar to the case of synchronization of quasiperiodic oscillations. This fact can be explained, because chaos that occurs in the autonomous system is based on a two-frequency torus loosing smoothness. That is why the dynamics of this systems has two characteristic time-scales: $T_0 = 2.4$ and $T_{puls} = 15 - 16$, which have formed the corresponding picture.

With an increasing period of the external pulses, in the interval $T > T_{puls}$ the "threshold" of synchronization is increasing, the bottom of tongues of synchronization are destroying, and chaotic regimes inside the area of period-1 occur.

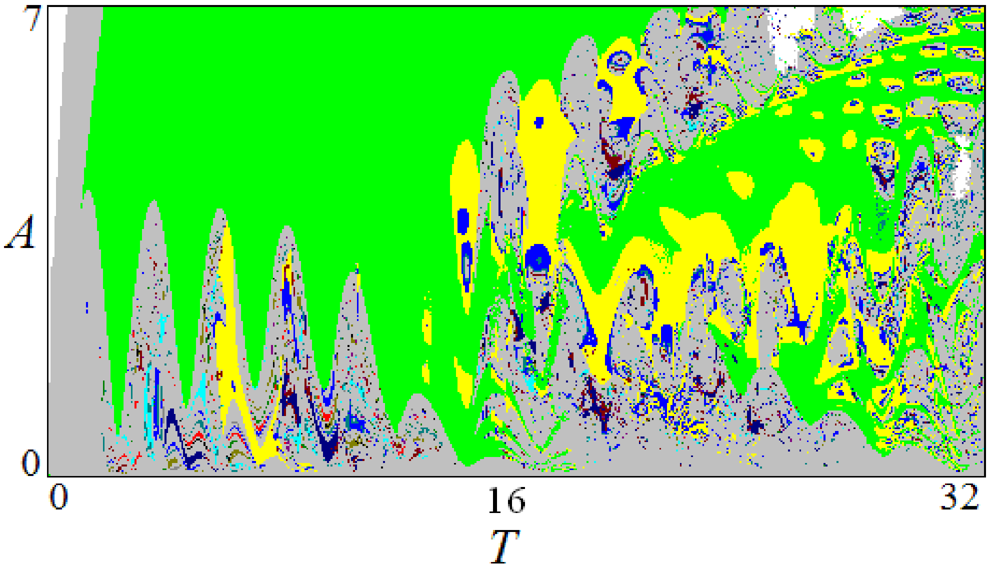

Figure 7: The chart of dynamical regimes of excited by pulses generator (2) in case when chaotic regime is realized. $\lambda = 0$, $\beta = 0.5$, $\omega_0 = 2.7$, $\mu = 0.3$.

5 CONCLUSION

We have studied the dynamics of a model of a generator of quasiperiodic oscillations under periodic pulse action. When the autonomous system demonstrates quasiperiodic oscillations, the parameter plane of the non-autonomous system consists of small-scale and large-scale structures, connected with two incommensurate time scales of quasiperiodic oscillations of the autonomous system. Also for this case the existence of three-frequency tori has been shown. We have revealed a characteristic picture of two-frequency and three-frequency tori and also a resonance Arnold web has been found.

When analyzing the chaotic regime, we have observed a partial, but not full, destruction of the small- and large-scale structures. In this case three-frequency tori are destroyed, but two-frequency, periodic and chaotic regimes are still existing.

This research was supported by the grants of RFBR No. 14-02-00085 and of RF President program for leading Russian research schools NSh-1726.2014.2. N.V.S. thanks IRTG 1740 (DFG) for supporting her visit to The Potsdam Institute for Climate Impact Research.

References

[1] A. Pikovsky, M. Rosenblum, J. Kurths, *Synchronization: A Universal Concept in Nonlinear Science* (Cambridge University Press, Cambridge, England, 2001).

[2] P.S. Landa, *Nonlinear Oscillations and Waves in Dynamical Systems* (Kluwer Academic Publishers, Dordrecht, 1996).

[3] L.Glass and M.C. MacKey, *From Clock to chaos* (Princeton University Press, Princeton, 1988).

[4] A.G. Balanov, N.B. Janson, D.E. Postnov, O. Sosnovtseva, *Synchronization: from simple to complex* (Springer, 2009)

[5] J.A. Glazier and A. Libchaber, IEEE Trans. Circuits and Systems, *35*, 790 (1988).

[6] E.M. Izhikevich, Int.J. of Bif. and Chaos, *10*, 1171 (2000).

[7] R. Straube, D. Flockerzi, S.C. Müller and M.J.B. Hauser, Phys. Rev. E, *72*, 066205 (2005).

[8] T. Matsumoto, L. Chua, R. Tokunaga, IEEE Transactions on Circuits and Systems, *34*, 240 (1987).

[9] P.M. Battelino, Phys. Rev. A, *38*, 1495 (1988).

[10] V. Anishchenko, S. Nikolaev and J. Kurths, CHAOS, *18*, 037123 (2008).

[11] Yu. P. Emelianova, A.P. Kuznetsov, I.R. Sataev, L.V. Turukina, Physica D, *244*, 36 (2013).

[12] A.P. Kuznetsov, I.R. Sataev, L.V. Turukina, Comm. Nonlin. Sci. Num. Sim., *16*, 2371 (2011).

[13] M. Ivanchenko, G. Osipov, V. Shalfeev, J. Kurths, Physica D, *189*, 8 (2004).

[14] A.P. Kuznetsov, N.V. Stankevich and L.V. Turukina, Physica D, *238*, 1203 (2009).

[15] V. Anishchenko, S. Astakhov and T. Vadivasova, Europhysics Letters, *86*, 30003 (2009).

[16] C. Baesens, J. Guckenheimer, S. Kim, R.S. MacKay, Physica D, *49*, 387 (1991).

[17] V. Maistrenko, A. Vasylenko, Yu. Maistrenko, E. Mosekilde, Int. J.of Bif. and Chaos, *20*, 1811 (2010).

[18] M.S. Baptista and I.L. Caldas, Phys.Rev.E, *58*, No.4, 4413 (1998).

[19] V.S. Anishchenko, S.M. Nikolaev, Tech. Phys. Lett., *31*, 853 (2005).

[20] V. Anishchenko, S. Nikolaev and J. Kurths, Phys. Rev. E, *73*, 056202 (2006).

[21] V.S. Anishchenko, S.M. Nikolaev, J. Kurths, Phys. Rev. , *76*, 040101 (2007).

[22] A.P. Kuznetsov, S.P. Kuznetsov, N.V. Stankevich, Commun. Nonlin.Sci.Num.Sim., *15*, 1676 (2010).

[23] A.Kuznetsov, S.Kuznetsov, E.Seleznev, and N.Stankevich, in Nonlinear dynamics of Electronic System (VDE-Verlag,Berlin Offenbach, 2012) pp.100-102.

[24] V.V. Klinshov, V.I. Nekorkin, JETP Lett., *87*, 78 (2008).

[25] J. Laskar, Physica D, *67*, 257 (1993).

[26] C. Froeschle, E. Lega, M. Guzzo, Celestial Mechanics and Dynamical Astronomy, *95*, 141 (2006).

[27] H. Broer, C. Simó, R. Vitolo, Reprint from the Belgian Mathematical Society, 767 (2008).