A simple autonomous quasiperiodic self-oscillator

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Abstract

In this note a simple example of an autonomous three-dimensional system is considered demonstrating quasiperiodic dynamics because of presence of two coexisting oscillatory components of independently controlling and, hence, generally incommensurate frequencies. Attractor in such a regime is a two-dimensional torus. Numerical illustrations of the stable quasiperiodic motions are presented. Some essential features of the dynamical behavior are revealed; in particular, charts of dynamical regimes on parameter planes are considered and discussed.

Key words: quasiperiodic, torus, Lyapunov exponent, attractor, oscillator
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Quasiperiodic oscillations represent a wide-spread type of dynamical behavior in many dynamical systems in nature and technology [1]. In very general circumstances, they may be thought of as oscillations composed of two or more components characterized by incommensurate frequencies.

Importance of understanding quasiperiodic dynamics and of scenarios of their transformation under parameter variation may be stressed, in particular, in the context of the Kolmogorov – Arnold – Moser theory [2], proving persistence of the quasiperiodicity in conservative systems close to integrable. To some extent, these conclusions may be related as well to some dissipative systems. One more argument concerns the onset of turbulence, or, more general, the onset of chaos in systems of different physical nature. Landau and Hopf suggested a scenario of transition to turbulence via successive birth of superimposed modes with incommensurate frequencies [3,4]. Latter this concept was criticized by Ruelle and Takens [5], who argued for the onset of chaos already after a few steps of the mode birth process, because of appearance of a strange attractor. However, the quasiperiodic motions and their bifurcations remain relevant in this respect, at least, for understanding initial stages of the transitions. Finally, it is worth mentioning a relation of the problem to a field widely discussed in neuroscience and bio-chemistry in concern with mechanisms of bursting. Some of these mechanisms lead just to quasiperiodic behavior associated with a torus attractor [6,7].

Accurate analysis of details of the quasiperiodic dynamics, and of transformations of the respective regimes under parameter variation, makes it desirable to have simple low-dimensional models manifesting this kind of behavior. Some aspects may be studied in an easily produced class of models with quasiperiodic forcing by external periodic or quasiperiodic signal [1] Nevertheless, it seems reasonable to develop additionally examples of simple low-dimensional autonomous models with intrinsic quasiperiodic dynamics to have a basis for productive studies in the outlined directions. Unfortunately, a very restricted number of such models were advanced in literature [7,10,11,12,13,14,15]. In Ref. [10] the authors consider a system governed by equations with non-smooth (piecewise-linear) coefficients. Several systems of quasiperiodic attractors in autonomous systems of dimension N=3 and 4 are considered in Refs. [11,12,13]. It is worth noting also recent papers [14,15] studied a physically realizable system built on a base of modification of the co-called generator with inertial nonlinearity having the phase space dimension N=4.

In this note we suggest a simple example of self-oscillator, which demonstrates quasiperiodic dynamics at appropriate selection of parameters and is characterized by a minimal dimension necessary to have a possibility of the phenomenon, namely, N=3. The system we discuss may be easily implemented e.g. as an electronic device.

The main idea of the model we suggest is borrowed from the well known relaxation oscillator. It contains a storage element represented by a capacitor, which is charged gradually through a resistor from the voltage source, and then discharged rapidly through a "threshold" element, such as a neon lamp. This element has to have in fact two distinct thresholds, one for ignition, and another, lower the extinguishing voltage**.** When the decreasing voltage across the capacitor reaches the extinguishing level, the element switches off, and the capacitor starts charging again, repeating the cycle.

Now, let us suppose that the threshold device is replaced by a self-oscillatory element controlling by an output variable of a storage element in the same way as explained in the previous paragraph. For the self-oscillator to be a threshold element, we must assume it to possess a kind of hard excitation. So, in respect to the control parameter it must have a threshold of excitation of oscillations (the ignition threshold) and some lower parameter level for drop of the oscillations (the extinguishing threshold). Then, the relaxation

[1] In this way, an important discovery has been done; namely, a possibility of occurrence of strange nonchaotic attractors was sustained [6,7].

oscillations accompanied with the cycles of charge and discharge of the storage element will take place. The whole scheme has two characteristic time scales, one associated with the relaxation oscillations, and another with the basic period of oscillations in the self-oscillatory element. The respective frequencies may be chosen to have an arbitrary ratio, so, the system is expected to produce the self-sustained quasi-periodic oscillations.

Obviously, the scheme we outline may be easily realized as an electronic device. However, in this note we do not intend to describe accurately such a device; rather our goal is to suggest as simple as possible explicit equations possessing the quasi-periodic dynamics and torus attractor. An appropriate set of equations reads

$$\begin{aligned} &\ddot{x} - (\lambda + z + x^2 - \tfrac{1}{2}x^4)\dot{x} + \omega_0^2 x = 0, \\ &\dot{z} = \mu - x^2, \end{aligned} \tag{1}$$

where x is a dynamical variable of the self-oscillatory element, and ω_0 is its basic operational frequency. The multiplicative factor at the first derivative $\dot{x}$ contains a parameter λ characterizing strength of the positive feedback in the self-oscillator, a nonlinear term x^2 that stimulates excitation of the oscillator, and a term x^4 that ensures saturation at larger amplitudes. The nonlinearity corresponding to combination of the last two terms gives rise to the hard excitation of the self-oscillator, also referred to as a subcritical Andronov – Hopf bifurcation.[2] The variable z may be thought as a value characterizing an instant state of the storage element. The time evolution of z is governed by the second equation. At small amplitudes of the self-oscillator ($|x| << 1$), the variable z simply undergoes linear growth, with the rate determined by the μ parameter. At large amplitudes, the mean value of the right-hand part of the equation becomes negative, and the value of z starts to decrease. In the first equation, the variable z is addend to the λ parameter. Because of hard excitation nature of the self-oscillator, it is characterized by distinct thresholds of excitation and extinguishing of the oscillations. So, the model will behave just like the relaxation oscillator we start with. Two independently controlled time scales are represented by the characteristic time of restoring the energy supply, $\tau \sim \mu^{-1}$, and the basic period of the self-oscillations, $T = 2\pi/\omega_0$. The qualitative arguments will be complemented with numerical computation in the rest part of the article.

As observed in computations, at certain parameters the model (1) manifests stable regime of pulsed generation. An example of such regime at $\lambda = 0$,

[2] It is worth mentioning a use of a system with subcritical Andronov – Hopf bifurcation to generate bursting dynamics [16], although the authors undertake more complicated modification involving delayed feedback.

$\omega_0 = 2\pi$, $\mu = 0.3$ is illustrated in Fig.1. In the time evolution of the system, four periodically repeating stages may be distinguished. The interval ***A*** corresponds to a stage of charge of the storage element; it means that the variable z grows (being negative), while the oscillations in the self-oscillatory element decay. As the variable z becomes positive, the threshold of self-excitation is reached, and we enter the stage ***B***. Here the amplitude in the self-oscillatory element grows and reaches some high level, corresponding to a limit cycle which would take place in the system with stationary energy supply. Note that the process starts not from zero amplitude, but from the rest small oscillations surviving the previous decay stage. In the stage ***C*** the decrease of the variable z takes place because of a relatively large average value of the x^2 term in the second equation. At some moment, the variable z becomes negative, but the self-oscillations of high amplitude persist due to the hard excitation nature of the oscillator, until a threshold of extinguishing of the oscillation is reached. The system enters the stage ***D*** of disappearance of the self-oscillatory state, drop and decay of the oscillations. Then, the process composed of the same stages A, B, C, D repeats again and again. At larger value of μ parameter, the stages of decay of oscillations become less pronounced, and in the three-dimensional phase space of the system $(x, \dot{x}, z)$ one can observe an attractor looking as a well-shaped torus (see Fig.2). Fourier spectra for these regimes are shown in Fig.3. They look precisely as expected for the quasiperiodic oscillations and contain a discrete set of equally distant frequency components. Their level decrease to the left and to the right from the main spectral line corresponding to the basic operational frequency of the self-oscillatory element. Distance to the nearest neighbor satellites corresponds to the frequency of the relaxation oscillations.

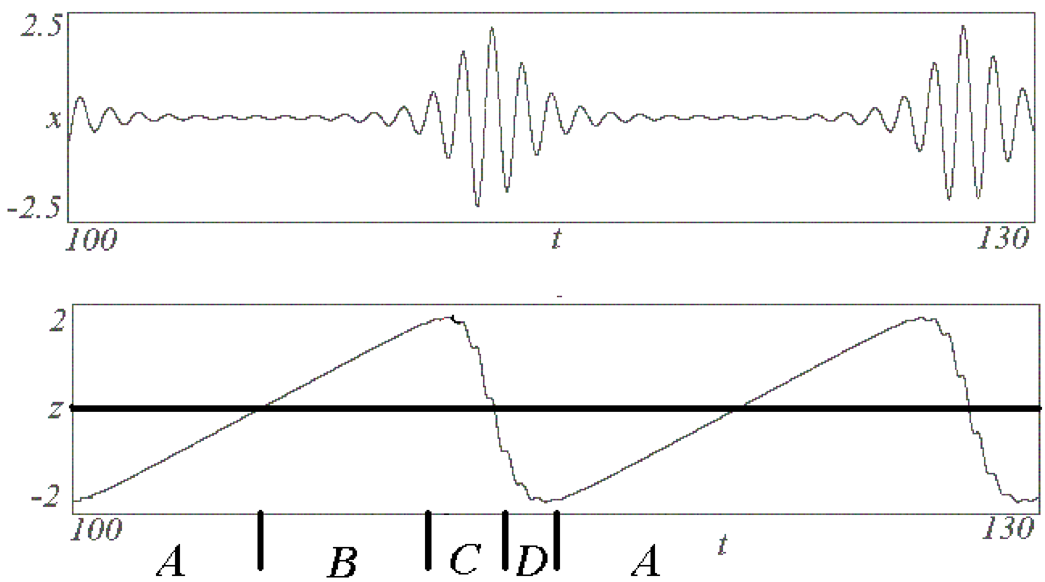

Figure 1. Dynamical variables versus time in model (1) at $\lambda = 0$, $\omega_0 = 2\pi$, $\mu = 0.3$.

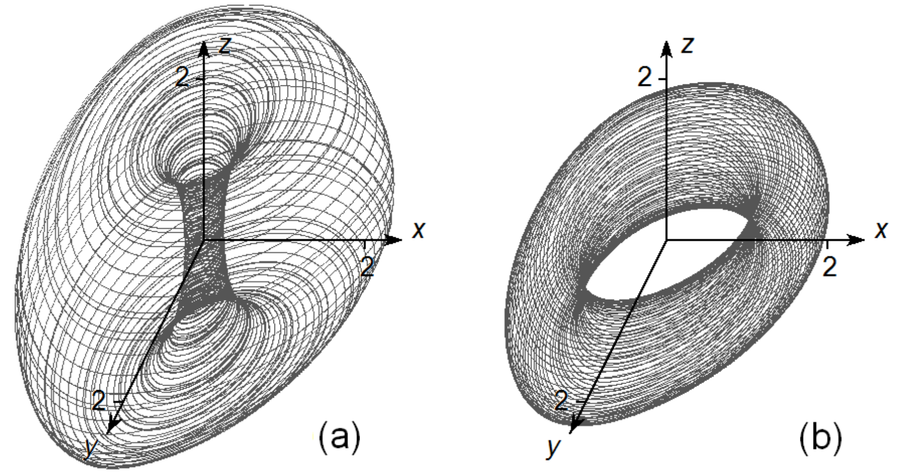

Figure 2. Portraits of attractor in the three-dimensional phase space of variables (x, $y = \dot{x}/\omega_0$, z) for the model (1) with λ=0, $\omega_0 = 2\pi$ at μ=0.5 (a) and μ=0.9 (b)

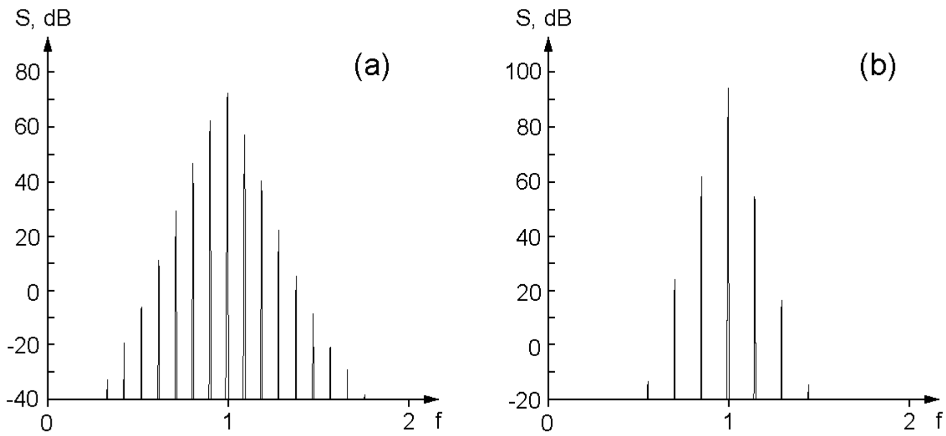

Figure 3. Fourier spectra of oscillations of the variable x on the attractor for the model (1) with λ=0, $\omega_0 = 2\pi$ at μ=0.5 (a) and μ=0.9 (b)

Let us turn to further numerical studies to be assured in the quasiperiodic nature of the observed regimes.

In Fig.4 (colored on-line) we show a chart of dynamical regimes at λ=0 on the plane of parameters determining the characteristic time scales of the system (ω_0, μ). In computations, the parameter plane was scanned with small enough step along each coordinate axis. At each point, a sustained regime of oscillations was analyzed to reveal its periodic or quasiperiodic nature with a use of the numerical construction of the Poincaré section. [3] Non-periodic regimes (including quasiperiodicity and chaos) are shown in gray. On this background, one can see colored areas (Arnold's tongues) corresponding to internal reso-

[3] As the surface for the Poincaré map construction a half-plane $x = 0$, $\dot{x} > 0$ was chosen in the phase space $(x, \dot{x}, z)$ of the system.

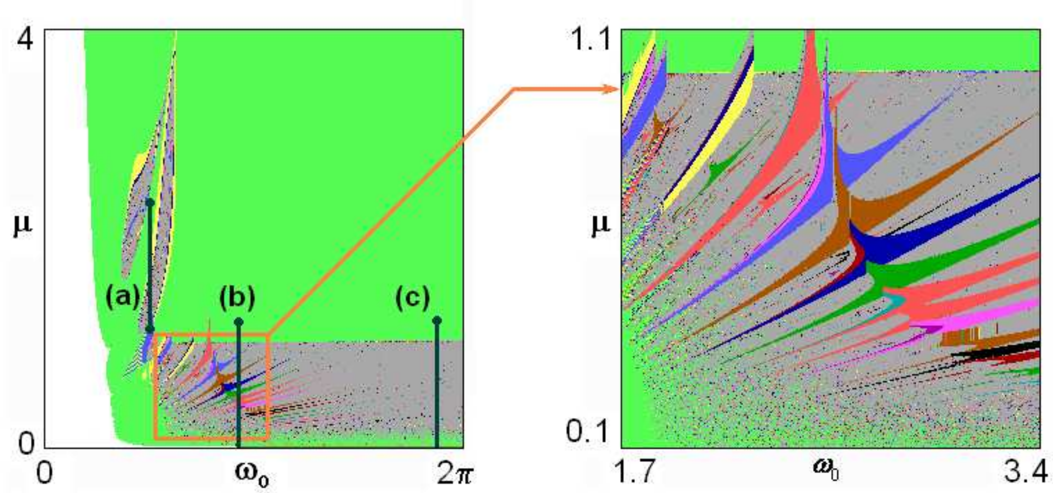

Figure 4. Chart of dynamical regimes on the plane of parameters determining the characteristic time scales of the system, (ω_0, μ) at λ=0 and its magnified fragment. Non-periodic regimes (including quasiperiodicity and chaos) are shown in gray. On this background one can see areas (Arnold's tongues) corresponding to internal resonances of two basic oscillatory motions intrinsic to the system. They are colored in dependence of the ratio of the basic frequencies, which is in this case rational. Light green color designates regimes of frequency ratio 1:1. White area in the left part of the chart designates a divergence of the motion to infinity.

nances of two basic oscillatory motions intrinsic to the system. The colors are chosen in dependence on a ratio of the basic frequencies being rational in this case. Light green color designates the regimes of frequency ratio 1:1. Apparently, a border of appearance of the tongues and quasiperiodicity on a road from small to larger values of ω_0 at μ<1 corresponds to the Neimark – Sacker bifurcation of stability loss of a limit cycle associated with the resonance 1:1.

Figure 5 shows plots for three Lyapunov exponents of the system computed with the Benettin algorithm [17] in dependence on parameter μ responsible for characteristic time of the storage element. The dependences correspond to roads along segments of straight lines drawn in Fig.4 and marked with the same letters as panels in Fig.5.

In regions of periodic regimes, one Lyapunov exponent is zero (up to numerical errors), and other two are negative. Situations of presence of two zero exponents while the third is negative are interpreted as quasiperiodic. Also, one can observe chaotic behavior with a positive exponent, a zero one, and a negative one. They occur at the upper edges of the Arnold tongues.

Transition to chaos via destruction of the quasiperiodic oscillations observed in the range 1<ω_0<3 with decrease of parameter μ is illustrated with pictures in the Poincaré section in Fig.6. Under decrease of the parameter μ one can see a birth of ergodic torus from the limit cycle (panels f, e), formation of a

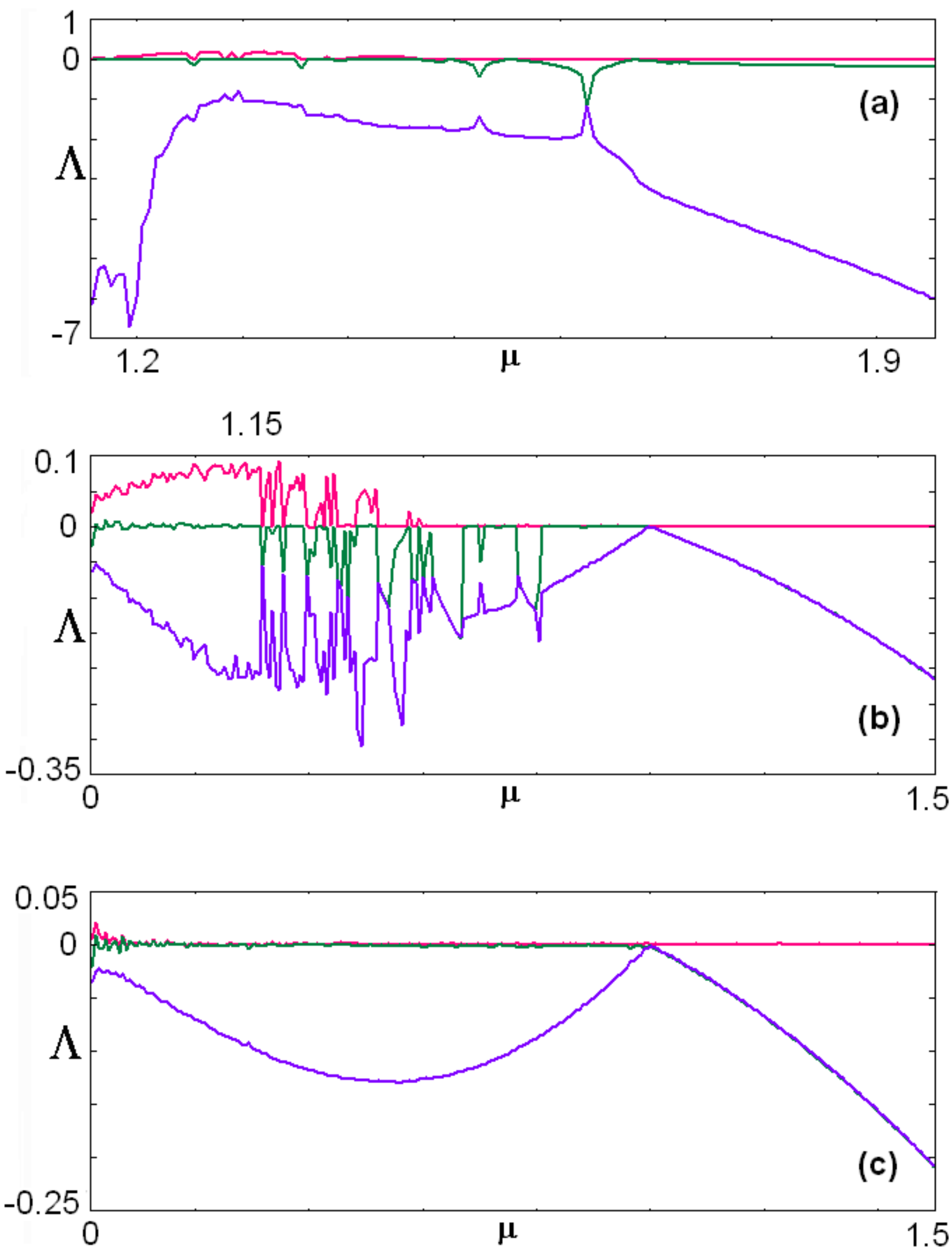

Figure 5. Lyapunov exponents of the system computed with the Benettin algorithm [17] in dependence on parameter μ at $\omega_0 = 1.5$(a), $\omega_0 = 2.7$ (b), $\omega_0 = 2\pi$ (c).

stable resonance periodic motion on the torus (panel d), and then appearance of chaotic attractor (panels c,b,a).

At large values of ω_0, a proportion of observable in computations periodic and chaotic regimes in the range $\mu<1$ decreases radically, and quasiperiodic behavior dominates, see a plot of the Lyapunov exponents in panel (c) of Fig.5.

To conclude, in this article we presented a simple model of an autonomous generator of quasiperiodic oscillations with two independent frequencies pos-

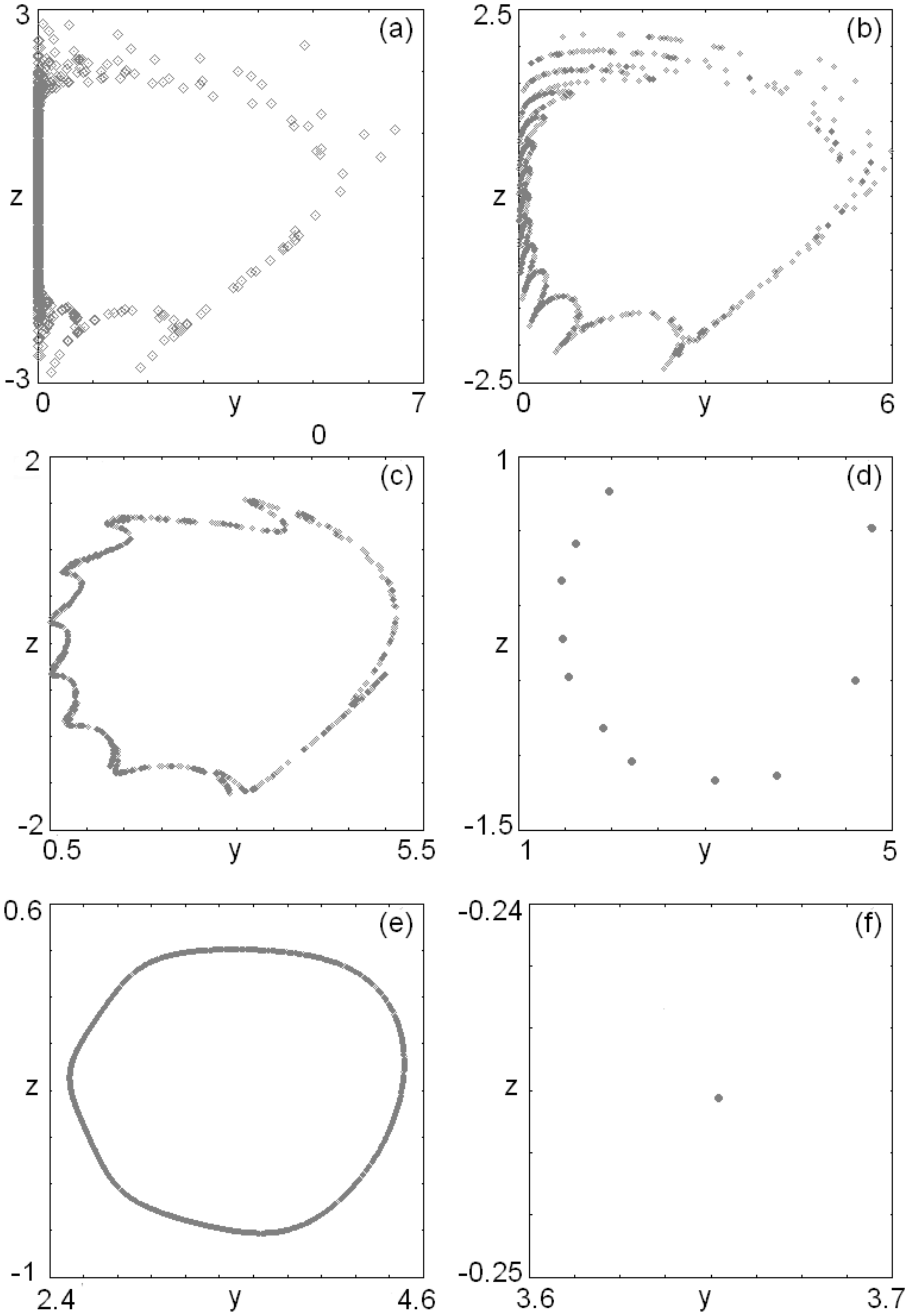

Figure 6. Attractors of the model (1) shown in the Poincaré section by a half-plane $x = 0$, $y > 0$ at μ=0.1 (a), μ=0.3 (b), μ=0.5 (c); μ=0.7 (d); μ=0.9 (e); μ=1 (f). Other parameters are λ=0, $\omega_0 = 2.7$.

sessing torus attractor embedded in the three-dimensional phase space. The system may be easily realized, e.g. as an electronic device. Beside the quasiperiodicity itself, we demonstrate in computations a number of associated dynamical phenomena, like internal resonances corresponding to Arnold tongues in parameter plane and transitions from quasiperiodicity to chaos.

In some respect the system we consider resembles the Hindmarsh - Rose model known in neuroscience (e.g. [18]). Like that, it consists of an oscillatory subsystem capable to the sub-critical Andronov-Hopf bifurcation and a first-order relaxation-type subsystem. Beside some differences, like a concrete structure of the first-order subsystem, in our model it effects the oscillator directly through the parameter controlling the Andronov-Hopf bifurcation, while in the Hindmarsh - Rose model such action is accompanied by a shift of the equilibrium point and, respectively, with rearrangement of the local structure of the phase space near it. In all publications we aware of, in consideration of the Hindmarsh - Rose model the attention is concentrated on slow-fast dynamical phenomena including spiking and bursting, not on the quasiperiodic motions and torus attractors. A question of possibility of such phenomena deserves attention, but it is a matter of a careful special research.

The model we introduce may be useful in theoretical, numerical and experimental studies of many aspects of quasiperiodic dynamics in autonomous systems and related topics. Also, it seems attractive methodologically and applicable in educational courses.

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