Scaling at the Onset of Chaos in a Network of Logistic Maps with Two Types of Global Coupling

A.S. Ivanova [1] and S.P. Kuznetsov [2]

[1] *Department of Nonlinear Processes, Saratov State University, 83 Astrakhanskaya str., Saratov, 410026, RUSSIA*
E-mail: ivas@sequel.rsm.ru

[2] *Institute of Radio-Engineering and Electronics of RAS, Saratov Branch 38 Zelenaya str., Saratov, 410019, RUSSIA*
E-mail: kuznetsov@sgu.ru

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We study a network of logistic maps with two types of global coupling, inertial and dissipative. Features of the clusterization process are revealed and compared, which are associated with presence of each type of couplings. For the parameter region near the onset of chaos we outline scaling properties in dynamical behavior of the network and illustrate them on the Kaneko phase diagrams.

Key words: logistic map, global coupling, cluster, universality, scaling

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Recently a notable interest is attracted to systems of globally coupled elements [1-4].

In particular, such models are discussed in context of description of neural networks, multi-modal lasers, electronic and chemical systems. They are regarded as perspective for nontrivial application in information processing to reproduce, at least in some respects, analogous processes in biological systems.

One particular class of networks, rather simple but possessing many nontrivial peculiarities in dynamical behavior, was introduced by Kaneko [1]. It is a set of logistic maps, each coupled to each other with equal strength. Studies of Kaneko and other authors revealed reach phenomenology of dynamics of this model. In particular, a phenomenon called clusterization has been discovered – this is a spontaneous formation of groups of elements in process of dynamics of the network with instantaneous states exactly coinciding for elements relating to each group (cluster).

A possibility of coexistence of attractors with different number of clusters at the same parameter values of the network gives an opportunity for information storage; control of them may be used for information processing.

In the original work of Kaneko and in many subsequent researches, a special concrete type of coupling was assumed: it tends to equalize instantaneous states of the interacting elements and is called the dissipative coupling [5,6]. However, it is not the only possible type of interaction of the elements. As found from the renormalization group (RG) analysis of the onset of chaos in coupled period-doubling maps and in coupled map lattices, in general case a weak coupling introduced by smooth functions of state variables is a combination of *two* relevant types of coupling – inertial and dissipative [5-8]. Both types of coupling possess distinct properties in respect to the RG transformation and manifest different scaling regularities at the onset of chaos in coupled logistic maps. A necessity of consideration of globally coupled models with two types of coupling was mentioned e.g. in Refs.[9, 10], but neither convenient models, nor detailed studies of features of dynamics were reported there.

In the present paper we introduce a model of a network of logistic maps with dissipative and inertial global coupling. Some features of dynamics and structure of phase diagrams will be discussed and compared for the networks with these types of coupling. In particular, we outline properties of universality and scaling in dynamics of the globally coupled maps near the onset of chaos. Due to richness of dynamical phenomena, this region may be especially interesting for applications in the information processing.

Let us consider a network of coupled logistic maps governed by the following equations:

$$\begin{aligned} x_{n+1}(i) = (1-\varepsilon_2)f(x_n(i)) - \varepsilon_1 x_n(i) \\ +\tfrac{\varepsilon_1}{N}\sum_{j=1}^{N} x_n(j) + \tfrac{\varepsilon_2}{N}\sum_{j=1}^{N} f(x_n(j)), \end{aligned} \quad (1)$$

where $f(x) = 1 - \lambda x^2$ is a nonlinear function corresponding to the logistic map, n designates the discrete time, index i enumerates elements of the network, N is the total number of the elements, ε_1 and ε_2 are parameters of coupling.

Note that two last terms in Eq. (1) do not depend on the index i, in other words, they are the same for all elements of the network. Hence, they may be regarded as two mean fields responsible for two types of global coupling:

$$F_n^{(1)} = N^{-1}\sum_{j=1}^{N} x_n(j) \quad \text{and} \quad F_n^{(2)} = N^{-1}\sum_{j=1}^{N} f(x_n(j)). \quad (2)$$

As follows from previous works on coupled logistic maps [5-10], the term with $f(x)$ in Eq.(1) and the mean field $F^{(2)}$ represent the pure dissipative coupling. The linear term and the field $F^{(1)}$ correspond to a combination of inertial and dissipative coupling, but the dissipative contribution is rather small. Quantitatively, the coefficients ε_1 and ε_2 are expressed via the coefficients of inertial and dissipative coupling ε_I and ε_D as follows [6,8]:

$$\varepsilon_I = \varepsilon_1, \quad \varepsilon_D = \varepsilon_2 - 0.088\varepsilon_1. \quad (3)$$

In the system with both types of coupling we expect to observe the same phenomenon of clusterization discovered by Kaneko for the dissipative case. Indeed, the main reason – coincidence of conditions of motion for elements with equal instantaneous states – is valid irrespectively to the number of mean fields.

To analyze dynamics of the globally coupled network in dependence on the parameters let us follow the approach of Kaneko based on the concept of *phases*.

Been given a point in the parameter space $(\lambda, \varepsilon_1, \varepsilon_2)$ we consider an ensemble of identical independent networks, each with its own, randomly chosen initial conditions. Performing a sufficiently large number of iterations, we estimate a number of clusters been formed in each representative of the ensemble. Accounting statistics of the final states, we classify the phase as coherent (one-cluster state dominates in the ensemble), ordered (states with few clusters are of the highest probability), partially ordered (clusters with small and large numbers of elements appear with comparable probabilities), turbulent (number of clusters is of order of the total number of elements in the network). On the phase diagrams of Fig.1 the respective phases are indicated by symbols C, O, PO, and T. Panel (a) relates to the case of inertial coupling, this is cross-section of the parameter space by plane $(\lambda, \varepsilon_1 = \varepsilon_I, \varepsilon_2 = 0.088\varepsilon_I)$. Panel (b) corresponds to dissipative coupling and reproduces the diagram computed by Kaneko. It represents a cross-section of the parameter space by plane $(\lambda, \varepsilon_1 = 0, \varepsilon_2 = \varepsilon_D)$ and is presented here for comparison. (Here we restrict ourselves by frames of a rough approach to the definition of phases, as in the original work of Kaneko, because it is sufficient to reach our aims and to demonstrate scaling. However, it has been shown recently that detection of presence or absence of clusterization, in a sense of exact coincidence of instantaneous states of elements, especially in chaotic domain, require much more subtle numerical technique and analysis [4].)

Observe that in the domain of subcritical dynamics of individual elements the network with inertial

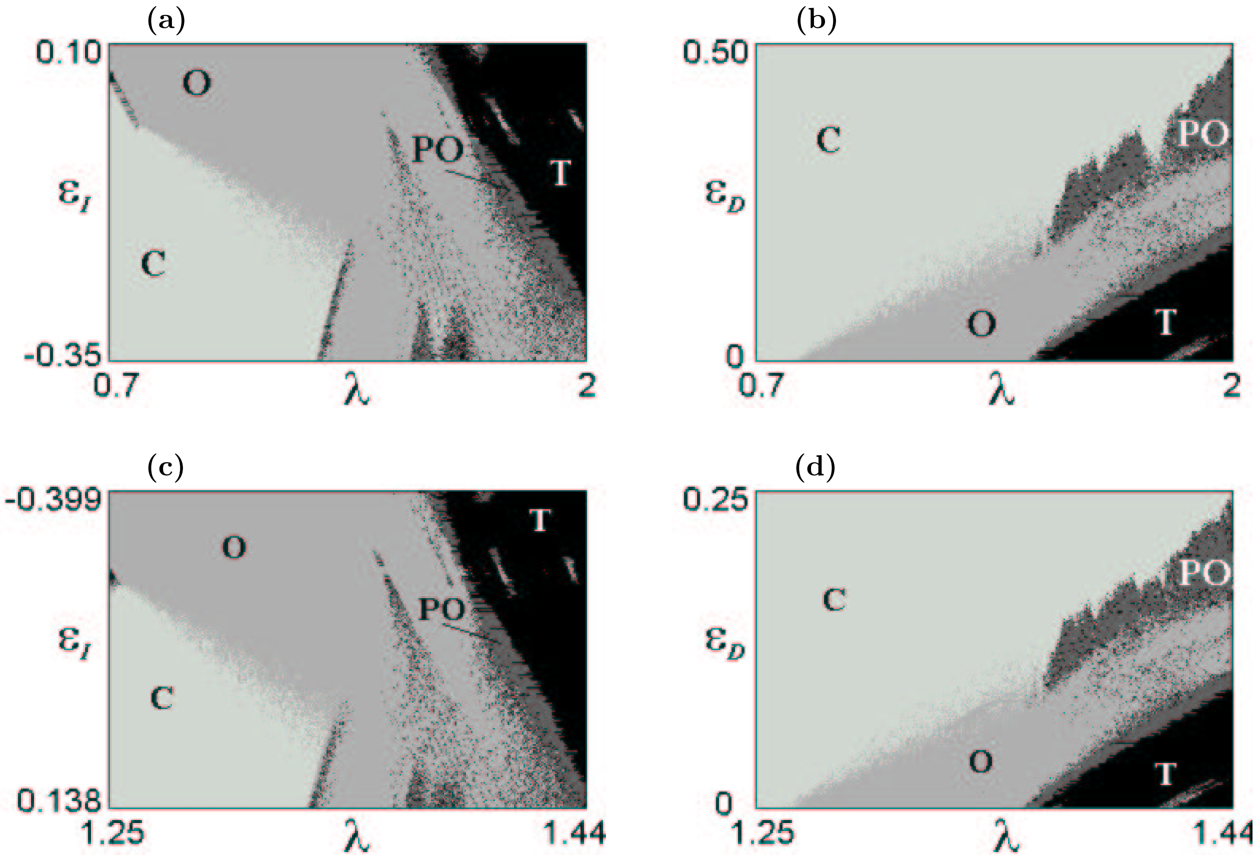

FIG.1. Kaneko's phase diagrams in cross-section of the parameter space of the globally coupled network of logistic maps by planes of pure inertial coupling (a,c) and pure dissipative coupling (b,d). The bottom pictures are built with the use of scaling rules formulated in the main text. Observe similarity of the diagrams (a) and (c), (b) and (d). The symbols designate Kaneko's phases: C – coherent, O – ordered, PO – partially ordered, T – turbulent.

coupling demonstrates more reach dynamics than in the case of dissipative coupling. Presence of phases O and PO indicates that formation of multiple clusters and multistability takes place here, in domain of regular dynamics. Perhaps, this implies that the networks with inertial coupling should be considered as more perspective candidates for applications in information processing than those with dissipative coupling.

In Refs.[5-10] it was stated that coupled maps with inertial and dissipative coupling near the onset of chaos demonstrate a property of scaling. The network with global coupling may be thought as a set of elements coupled pairwise. Hence, analogous scaling regularities must be valid for the global coupling, and the statement may be formulated as follows.

Let us suppose that at the parameters λ, ε_1, ε_2 we detect some phase for the ensemble of systems (1) with initial conditions randomly chosen from interval $|x| < C$. Then, for ensemble with random initial conditions from interval $|x| < C/\alpha$, at the point $\lambda + (\lambda - \lambda)/\delta$, ε_I/α, $\varepsilon_D/2$, the same kind of phase will be observed, but with doubled time scale of motion. Here λ_c is the parameter value of the period-doubling accumulation in an individual element of the network (the logistic map), $\alpha = -2.5029\ldots$ and $\delta = 4.6692\ldots$ are the Feigenbaum universal constants.

Straightly speaking, the above statement is true only asymptotically: smaller the neighborhood of the critical point $(\lambda_c, 0, 0)$, more accurate the scaling property holds. But actually it works well even in sufficiently large scales. It may be seen from comparison of the phase diagrams of Fig 1 (a) and (c), (b) and (d). The bottom pictures present results obtained with rescaling of the parameters and initial conditions in accordance with the formulated rules.

In, conclusion, we have introduced a model network of logistic maps with two types of global coupling. We argued that the Kaneko classification of phases suggested for the network of dissipatively coupled maps remains valid for our generalized model. In contrast to the dissipative coupling, the inertial one ensures reach dynamics (multi-stability, states of multiple clusters – the phases O and PO) not only in supercritical (chaotic) domain of the individual elements, but also in subcritical domain and at the border of chaos, where the dynamics is more regular. This circumstance indicates that the networks with inertial coupling are more interesting for possible applications in information processing than those with dissipative coupling.

For the first time we have demonstrated scaling regularities in phase diagrams of the models with global coupling. Note significance of this observation. As follows from RG argumentation developed earlier for coupled maps and for coupled map lattices, the scaling property appears as an attribute of the universality class. It means that analogous dynamical regimes and the same regularities will be intrinsic not only to our logistic map model, but for globally coupled networks composed of any other period-doubling elements that relate to the Feigenbaum universality class (say, of forced nonlinear dissipative oscillators) [11,12]. So, the results are expected to be common for globally coupled networks of different physical nature (electronics, biology, economics, etc.).

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