# About the Typical Structures and Chaos Border in the Parameter Plane of Non-Autonomous Discrete Maps with Period - Doubling

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The method of Lyapunov charts drawing is used for investigation of typical structures in the parameter plane of non-autonomous period-doubling maps. The new type of structures formed by super-stable regimes lines is obtained in the parameter plane of maps driven by hierarchically-organized signal. The behaviour of the chain of unidirectionally coupled logistic maps is also investigated and the qualitative difference between the behaviour of second and third maps is obtained.

**Key words:** non-autonomous period-doubling maps, chain of unidirectionally coupled logistic maps, hierarchically-organized signal

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## 1 Introduction

In this work we investigated the behaviour of period-doubling systems driven by different external signals. We chose the non-autonomous logistic map (1)

$$x_{n+1} = \lambda - {x_n}^2 + \varepsilon y_n \tag{1}$$

as a representative of the class of unimodal period-doubling systems and two-parameter cubic map (2)

$$x_{n+1} = a - bx_n + {x_n}^3 + \varepsilon y_n \tag{2}$$

as a representative of the class of bimodal period-doubling systems. Here $y_n$ is external signal and $\varepsilon$ - its amplitude. Binary sequences were taken as external signal because they could present all main classes of signals: regular (periodic sequences), noisy (binary noise) and hierarchically-organized (self-similar sequences [1]). We used the method of Lyapunov charts drawing [2] to carry out our investigation: each point of the parameter plane was grey shadowed according to the value of Lyapunov exponent at this point. The correspondence between its values and grey shades was as follows: zero value - white colour, positive values - definite grey colour, negative values from zero to very large negative values ("minus infinity")- grey colours from white to black. During our investigation the main attention was paid to the form of chaos border and typical structures in the parameter plane.

## 2 Logistic map driven by different external signals

The Lyapunov charts were drawn for logistic map (1) driven by periodic sequence, binary noise and two self-similar sequences: the logistic signum-sequence and "rabbit sequence" [1] (Fig 1).

We can see that in the case of periodic signal there are smooth border of chaos and well-known

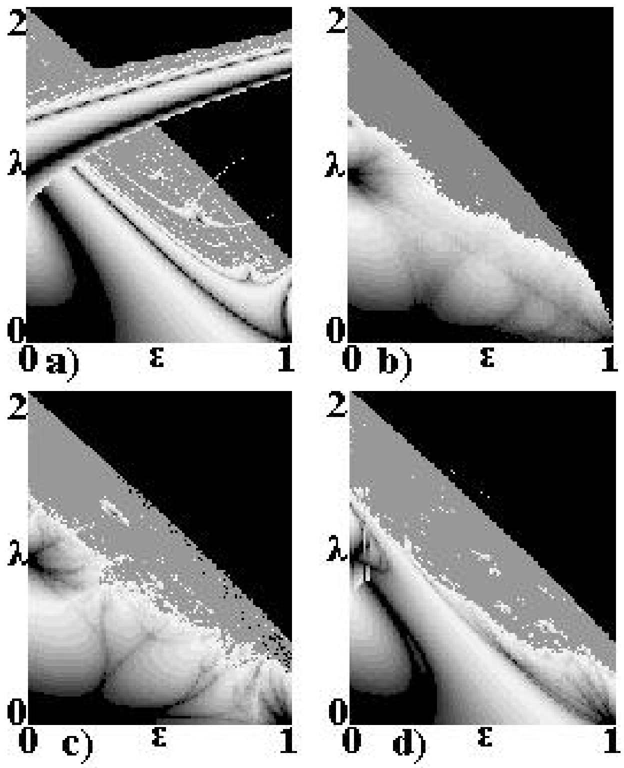


FIG. 1. The Lyapunov charts for logistic map driven by periodic signal (a), binary noise (b), "rabbits sequence" (c) and logistic signum-sequence (d).

"crossroad area" structures [3] in the parameter plane (Fig 1a).

The case of noisy signal is quite different (Fig. 1b). There are no sharp structures in the parameter plane, and the chaos border is not smooth, it is "diffuse". It can be shown that the Lyapunov chart demonstrates scaling properties in this case (Fig. 2).

The scaling constants are Feigenbaum's constant $\delta = 4,669...$ for parameter $\lambda$ and $\mu = 6,619...$ for parameter $\varepsilon$. The second constant was obtained in [4].

The case of self-similar signal demonstrates some interesting peculiarities (Fig.1 c,d). The chaos border consists of a big amount of fragments of different size, so it is not smooth,

but it is not so diffuse as in the case of noise.

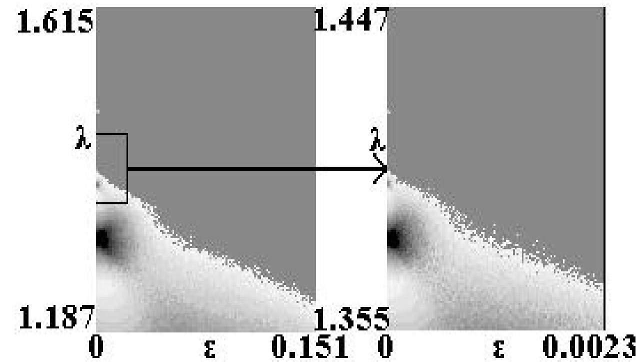


FIG. 2. Scaling in the parameter plane of noisy driven logistic map. Right picture corresponds to rectangle on the left picture.

Then, the lines of super-stable regimes (which looks like black lines in the light areas) in this case ramify and form the complex structure looking like a system of infinitely ramifying beams similar to the pictures on a leaf. These structures are especially well seen on the magnified fragments of Lyapunov charts (Fig 3.).

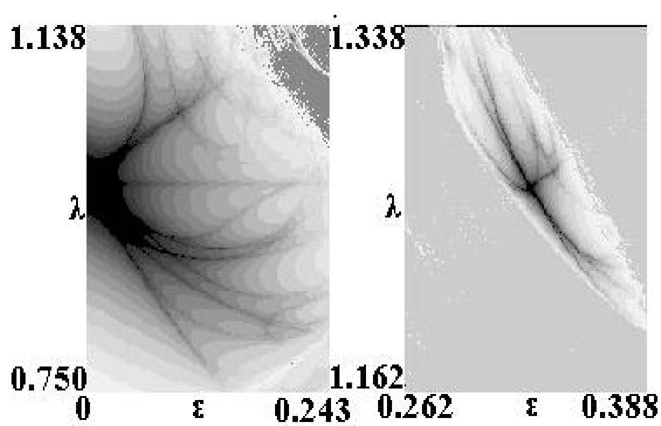


FIG. 3. Magnified fragments of Lyapunov chart for logistic map driven by "rabbits sequences". One can see branching structures of super-stable regimes lines.

We can see these structures in the Lyapunov charts for both self-similar sequences but they are more clearly expressed in the case of "rabbits sequence" (Fig.1 c). Also there are "islands" of regular regimes in the chaotic area, which are penetrated by these branching structures too.

Also we drew Lyapunov charts for two-parameter cubic map (2) when $y_n$ was binary noise in one case and "rabbits sequence" in another case.
Some of these charts are presented in Fig.4.

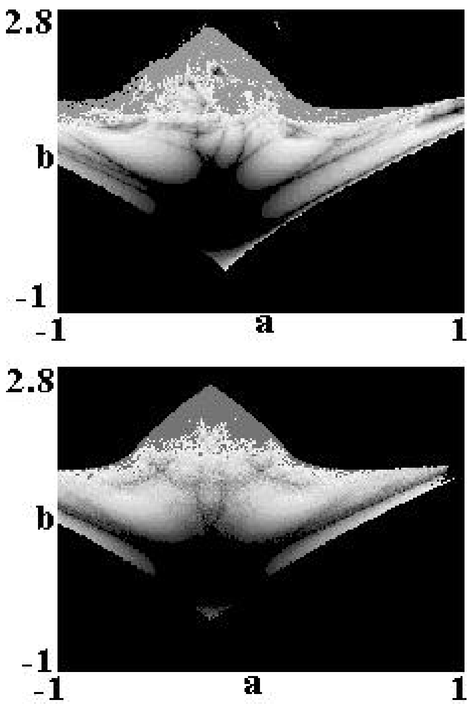


FIG. 4. Lyapunov chart for map (2) driven by "rabbits sequence" (up) and binary noise (down). Signal's amplitude $\varepsilon = 0.5$

These charts demonstrate the same peculiarities that the charts for logistic map, in particular, branching structures of super-stable regimes lines, so we can suggest that these peculiarities are typical for systems driven by self-similar sequences.

## 3 Fractal signal and its properties

We have seen that the most interesting behaviour is demonstrated by systems driven by hierarchically-organised signals.
Let us consider a special signal (3):

$$y_{2n} = \beta(1 + y_n); y_{2n+1} = -\alpha(1 + y_n); y_0 = \frac{\beta}{(1 - \beta)} \quad (3)$$

to investigate this case more closely.
This signal was introduced in [5] and called "fractal" there, because the phase portrait for the variable $y_n$ was a two-scale Cantor's set with the parameters $\alpha$ and $\beta$.
Let us investigate the logistic map (1) driven by fractal signal (3). We obtain Lyapunov charts at the $(\lambda, \varepsilon)$ plane for different values of parameter $\alpha$. Parameter's $\beta$ value was fixed for simplicity ($\beta = 0$). These charts are shown in Fig.5.

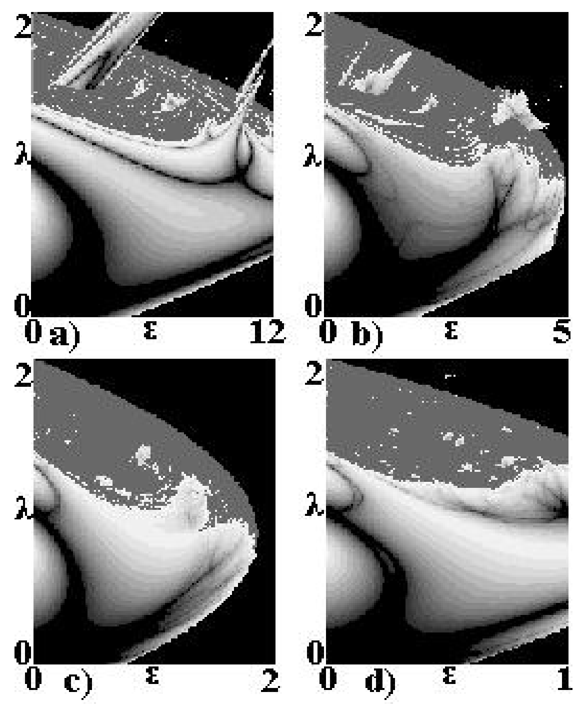


FIG. 5. Lyapunov charts for logistic map driven by fractal signal; $\beta = 0; \alpha = 0,1$ (a); $\alpha = 0,5$(b); $\alpha = 0,8$(c); $\alpha = 1$(d).

One can see that they are quite different from each other. When $\alpha$ is small (e.g. $\alpha = 0,1$) the chaos border is smooth at any values of $\varepsilon$, and there are structures, similar to "crossroad area", in the region of large $\varepsilon$. So, in this case the structure of fractal signal at small $\alpha$ is similar to periodic signal. When $\alpha$ becomes larger ($\alpha = 0,5$), the picture changes. The chaos border in the region of small $\varepsilon$ is still smooth, but in the region of large $\varepsilon$ "crossroad

area" structures are distorted and the chaos border becomes "diffuse". And when $\alpha$ becomes larger ($\alpha = 1$) the chaos border becomes "diffuse" even at small $\varepsilon$ and instead of "crossroad area" structures there are branching structures of super-stable regimes lines similar to that in the case of logistic map driven by self-similar sequences. So, the chaos border changes from smooth to "diffuse" when varying the signal's parameter $\alpha$.
We obtain the Lyapunov charts in the $(\lambda, \alpha)$ plane to investigate this phenomenon more closely (Fig.6).

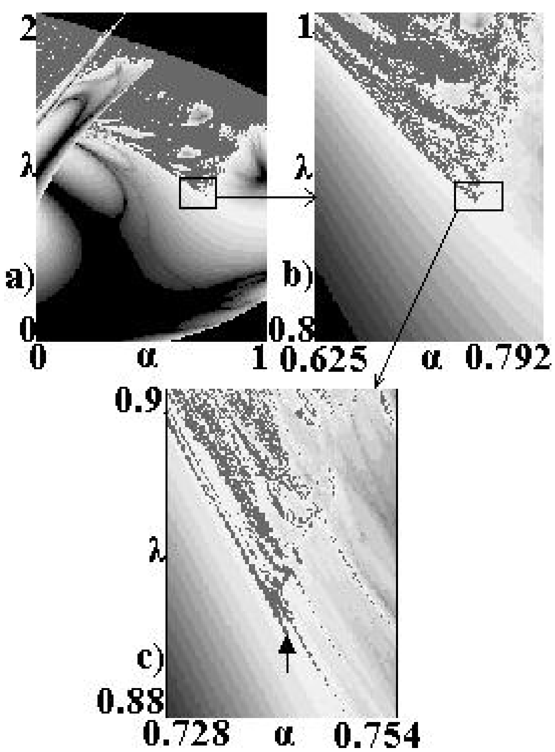


FIG. 6. The Lyapunov chart for logistic map driven by fractal signal (a) and its magnified fragments (b and c). The point of "criticality crisis" is marked with an arrow. $\varepsilon = 1; \beta = 0$. The correspondence between color and value of Lyapunov exponent is different on different pictures.

Here we can see that the chaos border is smooth until parameter $\alpha$ is smaller then some value $\alpha_{CC}$. When the parameter exceeds this value, the chaos border sharply becomes "diffuse". We have called this phenomenon the "criticality crisis" because we suggest that the way of transition to chaos changes sharply at this point. Of course, the value $\alpha_{CC}$ should depend on the parameter $\beta$. We have defined this dependence graphically by drawing magnified fragments of Lyapunov charts and defining the co-ordinates of corresponding point (see Fig.6 for example). In our work this procedure was applied to charts in $(\lambda, \alpha)$ plane for different $\beta$ and in $(\lambda, \beta)$ for different $\alpha$. The obtained points were plotted on the $(\alpha, \beta)$ parameter plane (Fig.7). One can

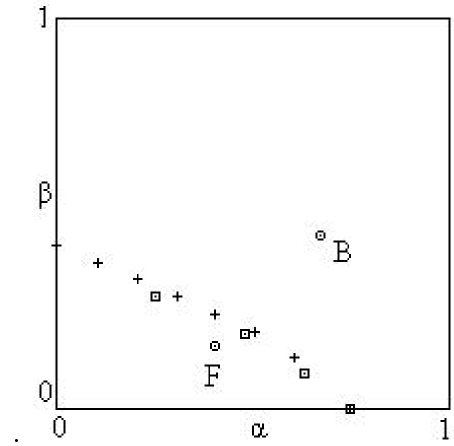


FIG. 7. The line corresponding to criticality crisis in the parameter plane of fractal signal.Point obtained by applying the described procedure to $(\lambda, \alpha)$ plane are marked with squares, and to $(\lambda, \beta)$ plane - with crosses. Point F corresponds to the approximation of Feigenbaum's signal, point B - bicritical signal.

see that they form rather smooth line. We can call it "criticality crisis line" because if the point corresponding to the fractal signal lies below this line, the chaos border will be smooth, at least at the small amplitudes. But when this point lies above this line, the chaos border will be "diffuse" at any values of amplitude.

## 4 The chain of unidirectionally coupled logistic maps

It is well known that autonomous logistic map demonstrates Feigenbaum's scenario of transition to chaos. At the critical value of parameter

$\lambda = \lambda_c = 1,401155189...$ the system demonstrates Feigenbaum's critical behaviour.

Let us add the second unidirectionally coupled logistic map, so the whole system will be described with map (4):

$$x_{n+1} = 1 - \lambda {x_n}^2, y_{n+1} = 1 - A{y_n}^2 - B{x_n}^2 \quad (4)$$

If we fix the parameter $\lambda = \lambda_c$, the second equation in (4) may be interpreted as logistic map driven by Feigenbaum's critical signal. This system is rather well investigated by different methods, including the renormalization group analysis, in particular, in works [6,7]. Varying parameter A with fixed parameter B We can observe transition to chaos when varying parameter A with fixed parameter B . The corresponding type of critical behaviour was called bicritical. The critical value of A depends on a coupling constant B. In particular, when $B = 0,375, A_c = 1,124981403...$ [6].

On the Fig.8(AB) the Lyapunov chart for the second subsystem of (4) is shown. One can see that the chaos boundary (so called bicritical line) is smooth, at least when B is not very big.

Then let us add the third unidirectionally coupled map, so the whole system will be described with a map (5):

$$x_{n+1} = 1 - \lambda {x_n}^2, y_{n+1} = 1 - A{y_n}^2 - B{x_n}^2,$$

$$z_{n+1} = 1 - C{z_n}^2 - D{y_n}^2 \quad (5)$$

We fixed the coupling parameter B=0,375 and the parameters of first and second systems $\lambda = \lambda_c$ and $A = A_c = 1,124981403...$, so the third subsystem was driven by bicritical signal. The Lyapunov chart for this case is shown in the Fig.8(CD).

One can see that in contrast with Fig.8(AB) there is no smooth border of chaos in this case, i.e. the dependence of critical value $C_c$ on the coupling parameter D is not smooth.

So, the behaviour of second and third systems in the chain of unidirectionally coupled logistic maps is qualitatively different. We can try to explain this phenomenon with the help of fractal signal's properties.

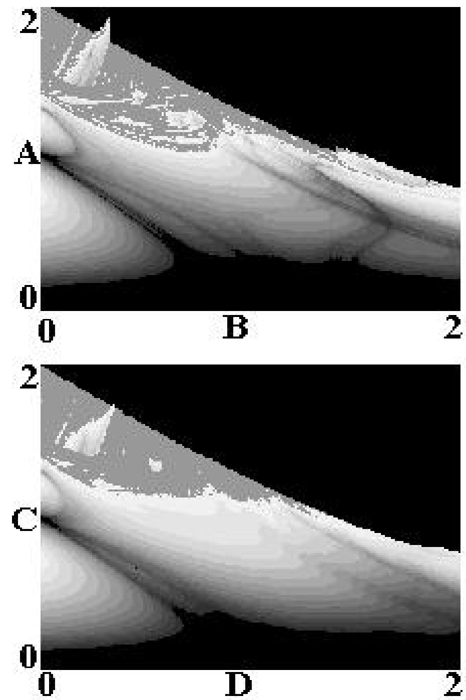


FIG. 8. Lyapunov charts for the second (AB) and the third (CD) subsystems of (4).

It is shown in [5] that this signal can serve as an approximation of different critical attractors when we choose its parameters $\alpha$ and $\beta$ in a proper way. In particular, it will approximate Feigenbaum's attractor when we choose $\alpha = \frac{1}{|a_F|} = 0,399155...$ and $\beta = \alpha^2 = 0,159325...$ and we will approximate bicritical attractor when we choose $\alpha = \frac{1}{|a_B|} = 0,664311...,\beta = \alpha^2 = 0,441309....$

Let us plot corresponding points on the signal's parameter plane $(\alpha, \beta)$ (point $F$ and $B$ in Fig.7). We can see that the point $F$ corresponding to Feigenbaum's signal lies below this line and point $B$ corresponding to bicritical signal - above the line. This fact elucidates the previous results.

Thus, in our work the logistic map driven by different signals was investigated with the help of Lyapunov charts drawing method. We revealed, that branching structures of super-stable regimes lines appeared in the case of self-similar influence and they seemed to be typical. Also we revealed the phenomenon of "criticality crisis" and corresponding to it line in the parameter plane of fractal signal.

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