On the Criticality of the FQ-Type in the System of Coupled Maps with Period-Doubling

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The critical behavior of the system of two coupled invertible maps with period-doubling was investigated. We obtain that the critical behavior of the FQ-type exists in the invertible systems only when partial systems are coupled in a special way: with dissipative coupling. The coordinates of the critical (FQ) point in the system of dissipatively coupled Hénon maps were found. The scaling in the parameter plane near the FQ-point was demonstrated.

Key words: critical behavior, chaos, coupled systems

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1 Introduction

The investigation of coupled systems is now a very popular topic in the nonlinear dynamics. There are a lot of papers dedicated to the questions of synchronization (see [1,2] for example), spatio-temporal dynamics in the coupled maps lattices [3,4] and so on. But in the most part of those papers the partial systems were taken to be identical. The system of non-identical coupled maps was investigated in [5]. It was the system of two coupled logistic maps

$$\begin{aligned} x_{n+1} &= 1 - \lambda x_n^2 - C y_n^2, \\ y_{n+1} &= 1 - A y_n^2 - B x_n^2, \end{aligned} \tag{1}$$

where the coupling parameters B and C were fixed ($B = 0.375$, $C = -0.25$). In this system the new type of critical behavior (called the FQ-type) was discovered and the corresponding scaling constants, universal multipliers and other characteristics were calculated [5-8].

It seems that the FQ-type of criticality is typical for non-invertible coupled systems with period-doubling. But it is not evident that the invertible systems will demonstrate this type of critical behavior as a phenomenon of codimension 2 because it is known that some typical for non-invertible systems types of critical behavior can't be observed in the invertible systems with the same numbers of parameters (some examples are given in [9,10]). This question seems to be rather important because an invertible system always may be interpreted as a Poíncare map of some system of differential equations, so it seems to be more realistic then discrete map. In our paper we'll try to obtain the critical behavior of the FQ-type in the invertible systems.

2 "Natural" transition to invertible maps

Now we'll try to construct the invertible system in which the critical behavior of the FQ-type occur. It seems very natural that such system should be as close to (1) as possible, and the most natural way of constructing such system is to use Hénon maps instead of logistic maps because the Hénon map [11] is the most similar to logistic map invertible system. So, lets consider the system:

$$\begin{aligned} x_{n+1} &= 1 - \lambda x_n^2 - C u_n^2 - b y_n, \\ y_{n+1} &= x_n, \\ u_{n+1} &= 1 - A u_n^2 - B x_n^2 - b v_n, \\ v_{n+1} &= v_n. \end{aligned} \tag{2}$$

We'll take the values of coupling constants the same as in (1) and fix the parameter of dissipation b equal to 0.2. At the Fig.1 one can see the structure of the parameter plane (λ,A) for system (2) (up) and for system (1)(down). The main structure of these parameter planes is very similar, in particular, both Feigenbaum scenario and transition to chaos through the destruction of quasiperiodic motion may be observed. But also one can see rather essential differences (see Fig. 2).

In particular, in the region of quasiperiodical motion of system (2) there are Arnold tongues which have the unusual shape similar to the ring. Unfortunately, it is very difficult to investigate these tongues more detail due to rather large period and small size of them.

To understand if there is the FQ-type of criticality in the system (2) we should try to locate the critical point and to obtain the numerical estimates of the corresponding scaling constants. Below we'll consider some numerical methods which we used in this work.

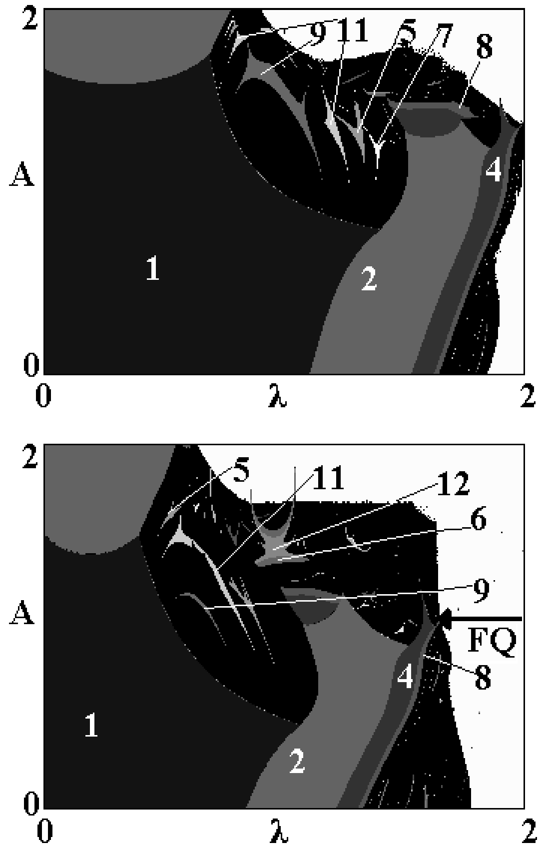

FIG. 1. The structure of the parameter plane of systems (2) (up) and (1) (down). The areas where the stable cycles exist are colored with different colors. The periods of cycles are written. The regions of nonperiodical behaviour (chaos and quasiperiodical motion) are colored with black, the regions of global instability are colored by white. The FQ-point at the parameter plane of (1) is marked by black narrow.

3 The numerical methods

If we'll have a good look at Fig. 1 we'll see that the basic element of the parameter plane is as shown at the Fig. 3. The region where the stable cycle with period n exists is bounded by the lines of period-doubling and Neimark bifurcations which intersect at some point. We'll call it "period-doubling terminal (or PDT) point" because the period-doubling line terminates here. If the FQ-type of criticality exists in the system then the sequence of PDT-points accumulates to

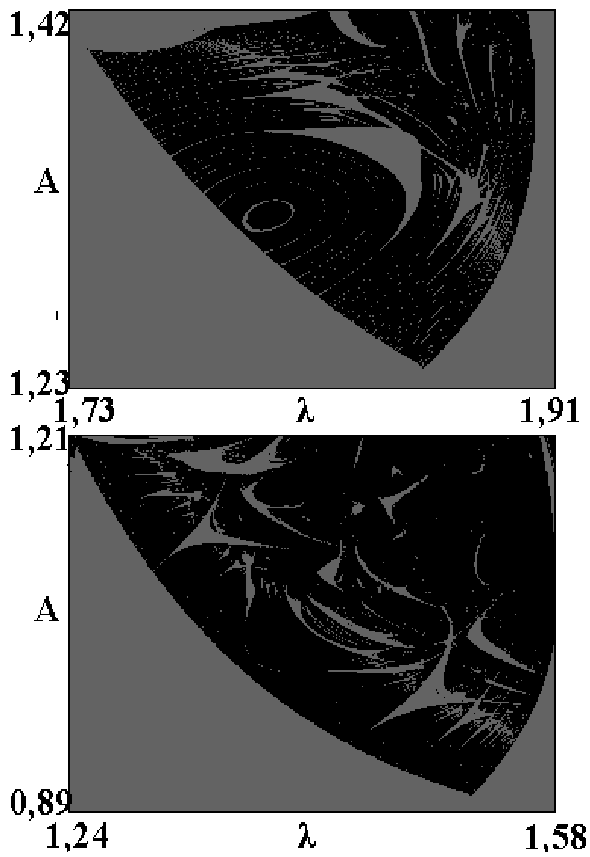

FIG. 2. The fine structure of the area of quasiperiodicity for the system (2) (up) and (1) (down). The regions of periodical motion are colored with grey, the regions of quasiperiodical motion - with black.

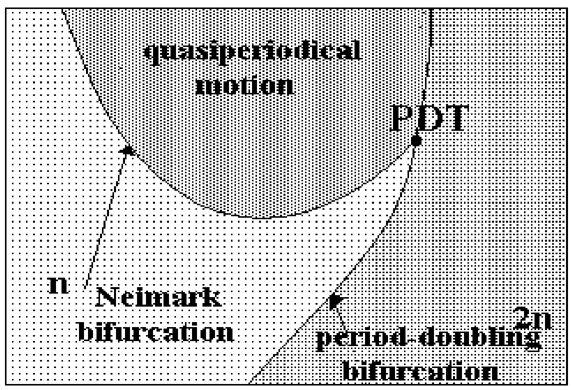

FIG. 3. The basic pattern of the parameter plane's structure for systems (1) and (2)

the critical point of the FQ-type. It is remarkable that in every infinitely small neighborhood of this point we can observe both the period-doubling and quasyperiodicity. (The name "FQ" originated from this fact: it means "Feigenbaum+quasyperiodicity").

It is clear that at the PDT-point both multipliers of n-cycle are equal to -1, so it is possible to obtain this point by solving numerically corresponding equations. Also we can estimate scaling constants and scaling vectors for the parameter plane as eigenvalues and eigenvectors of so called scaling matrix (see [12]) which may be found from the following expressions:

$$\begin{aligned}\Delta\lambda_{n-1} &= \delta_{11}^{(n)}\Delta\lambda_n + \delta_{12}^{(n)}\Delta A_n \\ \Delta A_{n-1} &= \delta_{21}^{(n)}\Delta\lambda_n + \delta_{22}^{(n)}\Delta A_n \\ \Delta\lambda_n &= \delta_{11}^{(n)}\Delta\lambda_{n+1} + \delta_{12}^{(n)}\Delta A_{n+1} \\ \Delta A_n &= \delta_{21}^{(n)}\Delta\lambda_{n+1} + \delta_{22}^{(n)}\Delta A_{n+1},\end{aligned} \tag{3}$$

where $\Delta\lambda_n = \lambda_{n+1} - \lambda_n, \Delta A_n = A_{n+1} - A_n$, and λ_n and A_n are the coordinates of the PDT-point of period n. (Here the elements of scaling matrix obtained from the coordinates of PDT-points for the cycles of periods $n-1$, n and $n+1$ were denotes as $\delta_{ij}^{(n)}$.)

Another way to obtain the FQ-point is eigenvalue-matching method [13-15]. It is based on the fact that at the critical point all cycles asymptotically have the same multipliers. So the sequence of points in the parameter plane where multipliers for cycles of period n and 2n are equal must converge to the critical point. Scaling matrix may be obtained from equation

$$\Delta_n = \Gamma_n^{-1}\Gamma_{2n}, \tag{4}$$

where

$$\Gamma_n = \begin{pmatrix} \frac{\partial Sp_n}{\partial\lambda} & \frac{\partial Sp_n}{\partial A} \\ \frac{\partial Det_n}{\partial\lambda} & \frac{\partial Det_n}{\partial A} \end{pmatrix}. \tag{5}$$

(All derivatives in matrix (5) are taken in the point where the multipliers of the cycles of period n and 2n are equal.)

Table 1

N	λ	A
1	1.40443503	0.79643742
2	1.86175463	1.23944066
4	1.97335903	1.35074553
8	1.99056056	1.36629930
16	1.99624542	1.37214443
32	1.99617992	1.37183054
64	1.99670754	1.37245687
128	1.99678555	1.37254857
256	1.99680475	1.37257100
512	1.99680427	1.37257030
1024	1.99680724	1.37257388

Table 2

N	δ_1	k_1	δ_2	k_2
32	4.893315	1.052708	-1.767793	0.813615
64	5.773516	1.061852	-1.948423	0.791786
128	5.992605	0.848376	-53.67671	0.793495
256	8.651061	0.840215	1.672860	0.874160
512	1.91+2.41i		1.91-2.41i	
1024	7.851900	0.867895	-0.965509	0.826408
2048	35.83928	0.823423	4.211473	0.829215

4 Searching for the FQ-point in the system (2)

First it should be noted that both methods give rather good results when they are applied to the system (1), but the eigenvalue-matching method gives better results. Below there are the results of these methods applied to system (2).

The sequence of PDT-points converges rather well (see Table 1) to the point with coordinates $\lambda_c = 1.99681\ldots$ and $A_c = 1.37275\ldots$, but it is not monotonic. For example, the parameter values for the PDT-point for the cycle of period 32 are less then for the cycle of period 16. Also there are some essential differences in a fine structure of parameter plane (see Fig. 4).

In the Table 2 one can see scaling constants and vectors obtained from PDT-sequence. (For scaling vectors the tangent coefficient is shown).

It is clear that these values even approximately are not close to the typical for the FQ-type values $\delta_1 = 6.32631925\ldots$ and $\delta_2 = 3.44470967\ldots$. The eigenvalue-matching method gives similar results (see Tables 3 and 4).

This results allow us to conclude that there are no FQ-type of criticality in the system (2).It seems that it is due to the influence of third relevant eigenvalue of RG-equation corresponding to the FQ-type $\delta_3 = -1.900\ldots$. This eigenvalue does not affect the dynamics of (1) due to the symmetry of the system, but in the system (2) there are no such symmetry.

5 Dissipatively coupled Hénon maps

It is possible to avoid the influence of third eigenvalue by introducting the special type of coupling, so called dissipative coupling (see [5,6]). We can introduce the dissipative coupling between two autonomous maps $x_{n+1} = f(x_n)$ and $y_{n+1} = g(y_n)$ as

$$\begin{aligned} x_{n+1} &= f(x_n) + C(f(x_n) - g(y_n)), \\ y_{n+1} &= g(y_n) + B(g(y_n) - f(x_n)), \end{aligned} \tag{6}$$

where B and C are the coupling constants. This coupling tends to equalize the values of dynamical variables of partial systems.

When partial systems are Hénon maps, this procedure results in system (7):

$$\begin{aligned} x_{n+1} &= 1 - \lambda x_n^2 - b(1-C')y_n - Cu_n^2 - bC'v_n, \\ y_{n+1} &= (1-C')x_n + C'u_n, \\ u_{n+1} &= 1 - Au_n^2 - b(1-B')v_n - Bx_n^2 - bB'y_n, \\ v_{n+1} &= (1-B')u_n + B'x_n, \end{aligned} \tag{7}$$

Table 3

N	λ	A
32	1.99691244	1.37270417
64	1.99680041	1.37256554
128	1.99681349	1.37258140
256	1.99680210	1.37256763
512	1.99680838	1.37257526

Table 4

N	δ_1	k_1	δ_2	k_2
32	10.92251	0.619983	3.572088	0.822044
64	3.84+0.45i		3.84-0.45i	
128	10.49683	0.769194	3.687652	0.837312
256	4.853185	0.869885	1.806063	0.819520
512	8.941400	0.804633	3.671438	0.827040

Table 5

N	λ	A
1	1.38086158	0.76747515
2	1.86757505	1.24659522
4	1.97122318	1.34817659
8	1.99205487	1.36816214
16	1.99586986	1.37171421
32	1.99658765	1.33237640
64	1.99679763	1.37259965
128	1.99686780	1.37268023
256	1.99688667	1.37270236
512	1.99689167	1.37270830
1024	1.99689329	1.37271025
2048	1.99689375	1.37271079
FQ	1.99681...	1.372575...

Table 6

N	δ_1	k_1	δ_2	k_2
64	6.344390	2.178323	2.622498	0.818234
128	9.113578	0.737851	4.223072	0.840365
256	6.033608	0.528929	3.848202	0.824688
512	5.767523	0.870097	2.314523	0.822454
1024	7.548862	8.016119	3.956383	0.826543

where we use some definitions: $C' = C\frac{\lambda - B)}{A\lambda - CB}$, $B' = B\frac{A-C}{A\lambda - CB}$ and the coupling parameters were fixed as follows B=0.375, C=-0.25, b=0.2. (The parameters were changed so that system becomes (1) with the same values of coupling parameters when b=0.)

In the Fig.5 one can see the structure of the parameter plane for system (7). It is clear that its structure is practically the same that for system (1) except the fact that near the axes there are the regions of global instability. These regions don't influence the dynamics near the FQ-point, but it is interesting that the transition from period 1 to global instability here occur through the region of quasiperiodical motion with Arnold tongues.(see Fig.6) Lets find the FQ-point in the system (7) with the help of the above-listed numerical methods. The coordinates of PDT-points are in Table 5 and the values of scaling constants and vectors - in Table 6.

It is clear that the PDT-sequence shows good convergence to the point $\lambda_c = 1.99689\ldots$, $A_c = 1.3782711\ldots$, and the values of scaling constants are close to the typical for the FQ type. Lets also pay attention to the fact that eigenvector corresponding to the constant δ_2 demonstrates rather good convergence unlike the first eigenvector. The same situation occurred in the system (1). So we can say that there are no special direction in the parameter plane corresponding to the scaling constant δ_1. (See [6] for the reasons of this fact).

In the tables 7 and 8 one can see the results of eigenvalue-matching method applied to the sys-

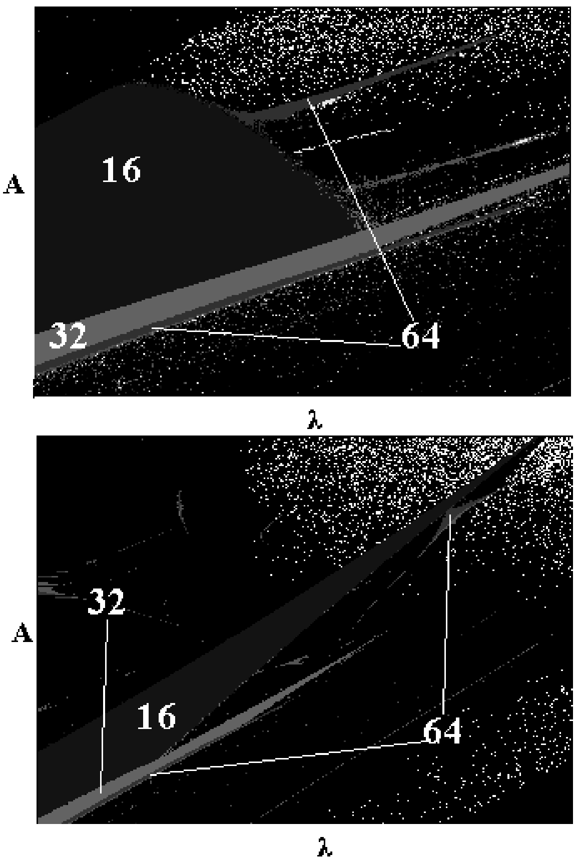

FIG. 4. The fine structure of the parameter plane for system (2) (up) and (1) (down). One can see that the shape of the region of period 16 differs very much and the Arnold tongues have rather unusual shape for system (2) unlike the system (1).

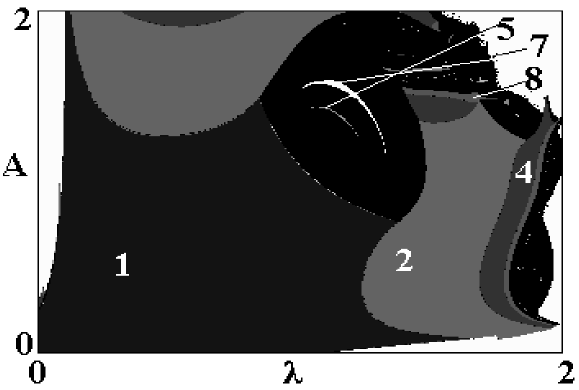

FIG. 5. The structure of the parameter plane of system (7).

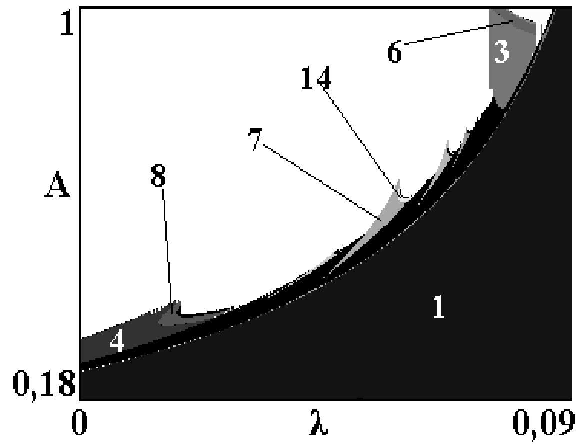

FIG. 6. The region of quasiperiodical motion on the parameter plane of system (7).

tem (7).

We can see that the coordinates of the FQ-point may be defined with more precision with this method ($\lambda_c = 1.9968939\ldots$, $A_c = 1.3727110\ldots$, and scaling constants demonstrate rather good convergence to the values $\delta_1 = 6.32631925\ldots$, $\delta_2 = 3.44470967\ldots$. So we can conclude that in the invertible system (7) the FQ-type of criticality exists.

6 Scaling near the FQ-point on the parameter plane

Now lets draw the illustrations of scaling on the parameter plane of system (7) near the FQ-point. We should define the special "scaling" coordinates (C_1, C_2) in such a way that the structure of the parameter plane will be invariant when scaling with constant δ_1 along the axis C_1 and constant δ_2 along the axis C_2. Let the axis C_2 to be collinear with the eigenvector k_2 corresponding to scaling constant δ_2; and the axis C_1 be collinear with the axis λ for simplicity because there are no definite eigenvector corresponding to scaling constant δ_1 and we can choose the direction of this axis arbitrary [6]. Then we can see that "natural"

Table 7

N	λ	A
16	1.99689769	1.37268669
32	1.99688586	1.37270094
64	1.99689574	1.37271326
128	1.99689414	1.37271127
256	1.99689376	1.37271081
512	1.99689395	1.37271105
1024	1.99689393	1.37271102
FQ	1.9968939...	1.3727110...

Table 8

N	δ_1	δ_2	k_1	k_2
32	6.372306	3.354398	-0.159957	0.823286
64	6.407103	3.472693	-0.484587	0.824998
128	6.287635	3.494486	-4.669529	0.825698
256	6.299577	3.374163	0.0251426	0.825207
512	6.386254	3.419344	0.192018	0.825159
1024	6.294248	3.461336	1.431319	0.825182

and "scaling" coordinates satisfy next formula:

$$A = A_c + C_2,$$
$$\lambda = \lambda_c + C_1 + k_2 C_2, \quad (8)$$

where $\lambda_c = 1.996839\ldots$ and $A_c = 1.3727110\ldots$ are the coordinates of the FQ-point (the scaling center) and $k_2 = 0.8452...$ is the tangent coefficient of the axis C_2 in the "natural" coordinates.

The structure of the parameter plane in the scaling coordinates is demonstrated at the Fig.7, and at the Fig. 8 the scaling is demonstrated in the close vicinity of the critical point. It is clear that the structure of the parameter plane reproduces very well when we rescale coordinates with constants C_1 and C_2. Also we can see that the fine structure of the parameter plane near the FQ-point is practically the same as the structure of the parameter plane near the FQ-point for system

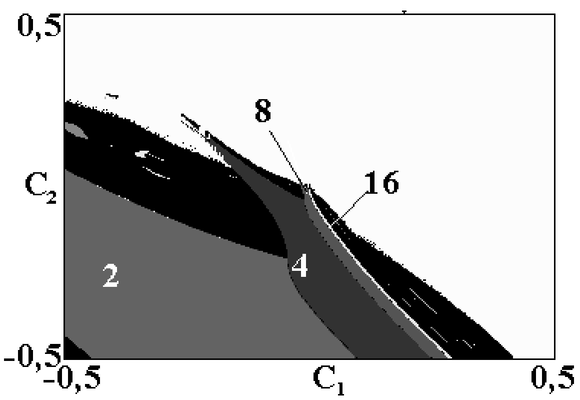

FIG. 7. The structure of the parameter plane of system (7) in the scaling coordinates (C_1,C_2) .

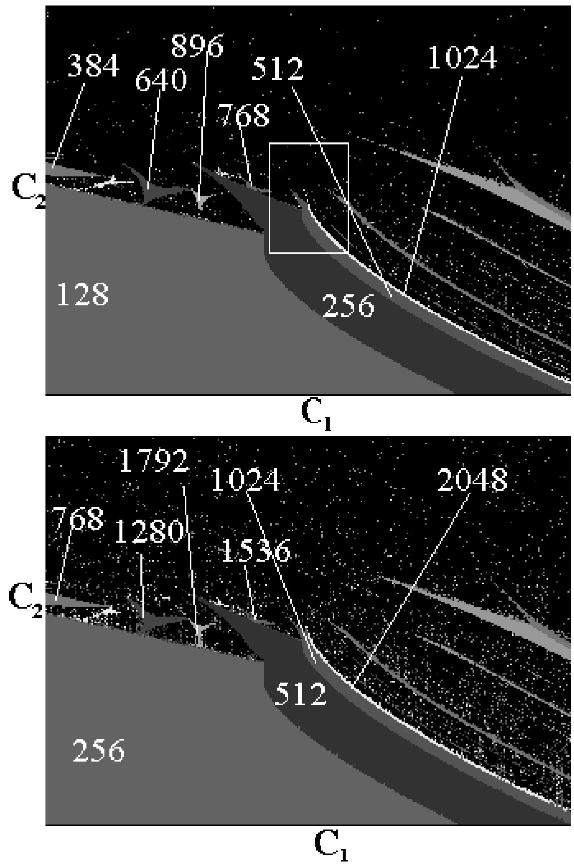

FIG. 8. The illustration of scaling in the vicinity of the FQ-point at the parameter plane of system (7). The below figure is the rescaled rectangle marked on the upper figure.

(1) (compare Figs. 8 and 9).

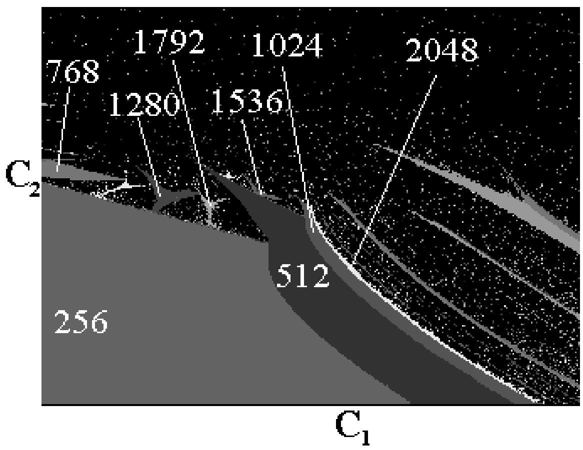

FIG. 9. The fine structure of the parameter plane of system (1) near the FQ-point. It is well seen that it is very similar to the structure of the parameter plane of system (7) near the FQ-point (compare with Fig.8).

7 Conclusion

So we demonstrated that the FQ-type of criticality exists not only in non-invertible but also in the invertible systems of coupled maps with period-doubling. To observe the behavior of this type in the system of invertible coupled maps with period-doubling as the phenomenon of codimension 2 we should introduce the dissipative coupling between the systems. In this case the structure of the parameter plane near the FQ-point in completely the same as in the system of coupled non-invertible maps and it is possible to demonstrate scaling near the FQ-point with the typical for the FQ-type scaling constants.

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