Complex Generalization of Approximate Renormalization Group Analysis and Mandelbrot Set. Thermodynamic Analogy

O.B. Isaeva and S.P. Kuznetsov

Saratov Branch of Institute of Radioengineering and Electronics of RAS,
38 Zelenaya st., Saratov, 410019, RUSSIA
E-mail: isaevaob@info.sgu.ru

Analogy between an approximate version of Feigenbaum renormalization group analysis in complex domain and the phase transition theory of Yang-Lee (based on consideration of formally complexified thermodynamic values) is discussed. It is shown that the Julia sets of the renormalization transformation correspond to the approximation of Mandelbrot set of the original map. New aspects of analogy between the theory of dynamical systems and the phase transition theory are uncovered.

Key words: bifurcation, chaos, RG method

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1 Introduction

Transition between magnetic and non-magnetic phase of matter is intensively studied for a long time [1, 2, 3, 4]. At low temperatures, elementary atomic magnets tend to be parallel providing coherent (magnetic) phase of the system. With increase of the temperature, thermal fluctuations break this order drastically. If the temperature is higher than the phase transition critical point (Curie point), then chaotic orientation of elementary magnets takes place (non-magnetic phase). However, near the critical point, the elementary magnets still keep order on certain range of distances and over certain time intervals, which decrease with growth of the temperature. Thus, at the critical point, areas of coherence and areas of thermal fluctuations coexist. Areas of fluctuations in the magnetic phase or areas of coherence in the non-magnetic phase may be distingueshed only within a certain range of scales. For large enough scales the system looks completely ordered (as at zero temperature) or chaotic (as at

infinite temperature). If one considers the same system at a definite temperature but within various scales, it looks as the system at different temperatures. The transformation of scales corresponds to the temperature renormalization.

Let we have a cubic lattice of N atoms with interatomic distance a at temperature T. If one considers this system at rough enough scale, at which the elementary cell has the characteristic size na and contains n^3 atoms, the system looks as a lattice of N/n^3 atoms at the renormalized temperature $T' = R(T)$ (R is a function of renormalization). At the moment of the phase transition, one can observe the thermal fluctuations within any scales. These fluctuations are self-similar. The idea of self-similarity has been offered by Kadanoff [5] and further advanced by Wilson to the renormalization method [6] (see also [2]).

Concepts of universality and scaling near the critical point and a renormalization group (RG) method have been transferred to nonlinear dynamics by Feigenbaum [7, 8]. The analogy be-

tween transition to chaos and phase transitions consist in increase of temporal scales with approach to a critical point, self-similarity, an opportunity of application of RG analysis, and in existence of universal critical indexes and scaling factors, which are determined by the most general requirements to the type of system [9, 10, 11, 12].

In 1952 Lee and Yang advanced an approach in the phase transition theory based on consideration of analytical properties of some thermodynamic values, such as partition function and free energy, depending on the temperature considered formally as a complex variable [13, 14].

The partition function is defined as follows:

$$
\begin{aligned}
Z_N(T, H) &= \sum_{|s_i|} \exp\left(-\frac{E(s_i)}{kT}\right) \\
&= \sum_{|s_i|} \exp\left(\frac{J}{kT}\sum_{i,j} s_i s_j + \frac{H}{kT}\sum_i s_i\right),
\end{aligned}
\tag{1}
$$

where E is a total energy of a configuration of the system, H is an external magnetic field, T is a temperature, k is the Boltzmann constant, s_i - is the spin variable, taking place at i-th cell of the lattice. Spins interact via magnetic field with the nearest neighbours ($J > 0$ corresponds to ferromagnetic interaction, $J < 0$ – antiferromagnetic). Partition function can be transformed to

$$
Z_N = e^{-N/2} \sum_{n=0}^{N} p_n z^n,
\tag{2}
$$

by the variable change $z = \exp(-2H/kT)$. It is known, that at the phase transition the divergence of the thermodynamic potential (free energy)

$$
f(H, T) = \lim_{N \to \infty} -\frac{kT}{N} \ln Z_N
\tag{3}
$$

takes place.

Obviously, zeros of the partition function (roots of the algebraic equation $Z_N = 0$) are the candidates for the phase transition points. However, these roots should be positive and real, which is impossible as all factors $p_n > 0$. In Yang-Lee theory, the temperature and the magnetic field are considered as complex variables [4, 13, 14]. Complex zeros of the partition

function (called the Yang-Lee zeros), gather near the real axis. It becomes possible only in the thermodynamic limit, the asymptotics of infinite number of atoms. When the number of atoms is finite, there is a finite number of zeros in the complex plane. With increase of number of atoms, the set of zeros becomes dense and nestles to the real axis more and more. In the thermodynamic limit the Yang-Lee zeroes form a fractal set, that cross the real temperature axis at the point of the phase transition. To find the Yang-Lee zeros it is appropriate to apply the Wilson renormalization method, which corresponds to a consecutive decrease of the number of degrees of freedom of the partition function. It is necessary to find the transformation of the N'-particle partition function to the N-particle one ($N' < N$). This transformation is not reversible. Thus, to get zeros of Z_N it is necessary to have zeros of $Z_{N'}$, and then to construct their backward images using the RG transformation. Having repeated this procedure many times one comes finally to the trivial two-atom partition function. If zeros of Z_2 belong to a basin of attraction of a fixed point of the RG transformation, then zeros of Z_N in the thermodynamic limit will coincide to the boundary of the basin of attraction in the complex plane. As shown by Derrida [15], this boundary, which is the Julia set of the RG transformation, is identical to the set of Yang-Lee zeros. Unfortunately, the renormalization transformation may be performed analytically only for simple class of models – the hierarchical lattices (for example, one and two-dimensional model with Ising spins [16], hierarchical lattices with Potts spins [17, 18]).

Yang-Lee theory appears to be fruitful for understanding the phase transitions. It seems that a similar approach to the analysis of transition to chaos in dynamical systems would be useful for deeper understanding of the analogy with the phase transitions, and for development of the new criteria describing complexity of behavior of nonlinear systems. In the present paper we develop such approach basing on the approximate RG analysis of transition to chaos through the period multiplication bifurcations.

2 Approximate RG analysis

In 1978 Feigenbaum discovered universality of cascade of the period-doubling bifurcations and described it on a basis of the RG method. The simplest example representing the Feigenbaum universality class is quadratic map

$$x_{n+1} = f_\lambda(x_n) = \lambda - x_n^2, \qquad (4)$$

where x is a real dynamical variable, and λ is a real parameter. This map has fixed points, which can be found as roots of the equation $x_* = \lambda - x_*^2$. If $\lambda > \Lambda_1 = 3/4$, a cycle of period 1 (that is the fixed point) loses its stability. This Λ_1 is the parameter, at which the first period-doubling bifurcation occurs (the multiplier of the fixed point is $\mu = f_\lambda'(x_*) = -2x_* = -1$). Values of λ for sequent bifurcations can be found by means of approximate RG method [19]. Let's apply the map (4) two times:

$$x_{n+2} = \lambda - \lambda^2 + 2\lambda x_n^2 - x_n^4 \qquad (5)$$

and neglect the last term, the fourth power of x_n. Then, by the scale transformation

$$x_n \to x_n/\alpha_0, \qquad \alpha_0 = -2\lambda. \qquad (6)$$

this map can be rewritten in the form $x_{n+2} = \lambda_1 - x_n^2$, which differs from (4) only by renormalization of λ

$$\lambda_1 = \varphi(\lambda) = -2\lambda(\lambda - \lambda^2). \qquad (7)$$

Thus, the operator of evolution for the double interval of discrete time can be reduced to the original operator by the renormalization transformation (7). Repeating this procedure with scale factors $\alpha_1 = -2\lambda_1, ...,$ one can obtain a sequence of the same form

$$x_{n+2^m} = \lambda_m - x_n^2, \qquad \lambda_m = \varphi(\lambda_{m-1}). \qquad (8)$$

Fixed points of these maps correspond to the 2^m-cycles of the original map ($m = 1, 2, 3, ...$). It is easy to see, that all these cycles, as well as the fixed point of the map (4), become unstable at $\lambda_m = \Lambda_1 = 3/4$. Solving a chain of the equations

$$\Lambda_1 = \varphi(\Lambda_2), \Lambda_2 = \varphi(\Lambda_3), ..., \Lambda_{m-1} = \varphi(\Lambda_m) \quad (9)$$

Table 1.				
	λ_*	α	δ	μ
Rigorous	1.401	-2.802	4.669	-1.569
Approximate	1.37	-2.74	5.73	-1.54

we get the corresponding sequence of bifurcation values of parameter λ (with $\lambda \approx \Lambda_m$ the 2^m-cycle of (4) arises). From iteration diagram of Fig.1 it is evident, that this sequence converges with $m \to \infty$ to a definite limit Λ_∞, the fixed point of the RG transformation. It satisfies the equation $\Lambda_\infty = \varphi(\Lambda_\infty)$, thus $\Lambda_\infty = (1+\sqrt{3})/2 \approx 1.37$. The scaling factors also converge to the limit: $\alpha_m \to \alpha$, where $\alpha = -2\Lambda_\infty \approx 2.74$. The multipliers (Floquet eigenvalues of the 2^m-cycles) converge to the universal value $\mu_m \to \mu = \sqrt{1 - 4\Lambda_\infty} \approx -1.54$.

From transformation (8) it is possible to obtain the law of convergence of the bifurcation sequence:

$$\Lambda_m = \varphi(\Lambda_\infty) + \varphi'(\Lambda_\infty)(\Lambda_{m+1} - \Lambda_\infty) = $$
$$= \Lambda_\infty + \delta(\Lambda_{m+1} - \Lambda_\infty) \qquad (10)$$

where $\delta = \varphi'(\Lambda_\infty) = 4 + \sqrt{3} \approx 5.73$ is a constant, characterizing the convergence to the critical point.

In table 1 we summarize the values of critical indexes (critical point, scale factor, parameter scaling constant and universal multiplier) obtained by means of the exact and approximate RG analysis. The correspondence between them is well enough.

3 Complex RG transformation and its Julia set: The critical phase transition line

Let us regard the variable x and parameter λ of map (4) as complex numbers. Then it becomes clear, that Feigenbaum's universal scaling laws correspond to a special case of general scaling properties of Mandelbrot sets. The Mandelbrot
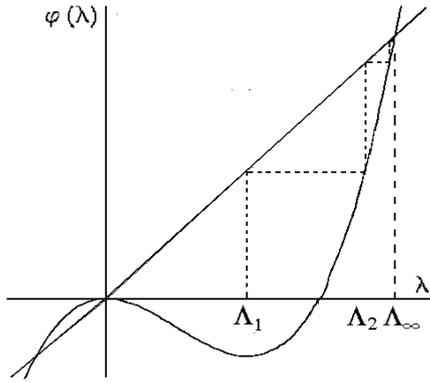
FIG. 1. Iteration diagram of the RG transformation (7). Dashed line designates the backward iterations starting at the first period doubling bifurcation point ($\Lambda_1 = 3/4$) and mapping to the further bifurcation points Λ_m.

FIG. 2. Correspondence between the Mandelbrot set (a) and the Feigenbaum bifurcation tree (b). Mandelbrot set is built up on a plane $(\mathrm{Re}\lambda, \mathrm{Im}\lambda)$ for the complex quadratic map (4). Bifurcation tree is shown in the plane (λ, x_n) for the map (4). Gray color in panel (a) indicates areas of periodic dynamics (the periods of cycles are marked); black color designates points corresponding bounded in the phase space chaotic dynamics; white color means escape to infinity.

set shown in Fig.2, can be determined as [4]

$$M = \{\lambda \in \mathbf{C} : \lim_{n \to \infty} f_\lambda^n(0) \neq \infty\}. \qquad (11)$$

Thus, M is the set of values of complex parameter, for which the trajectories launched from the extremum of the map (4) $0 \to \lambda \to \lambda - \lambda^2 \to , , ,$ remain in a finite domain. Mandelbrot set includes parameter values corresponding to periodic trajectories of the map (the main cardioid and a number of the round "leaves", painted by gray color), and a set of parameter values corresponding to bounded chaotic dynamics (black fractal pattern). Feigenbaum's cascade of period-doublings takes place along the real axis (see the picture of the bifurcation tree in the Fig.2).

However, in the complex plane one can observe many other accumulation points of other bifurcation cascades. One of such points, with scaling properies distinct from the Feigenbaum laws and intrinsic exclusively for complex analytical maps [20, 21] is associated with the cascade of period tripling [22]. Also, there exist critical points connected with cascades of period quadrupling, period 5-tupling, etc.

Thus, in the complex domain, a number of new scenarios of transition to chaos and other critical phenomena occurs. Let us consider the approximate RG transformation generalized on a com-
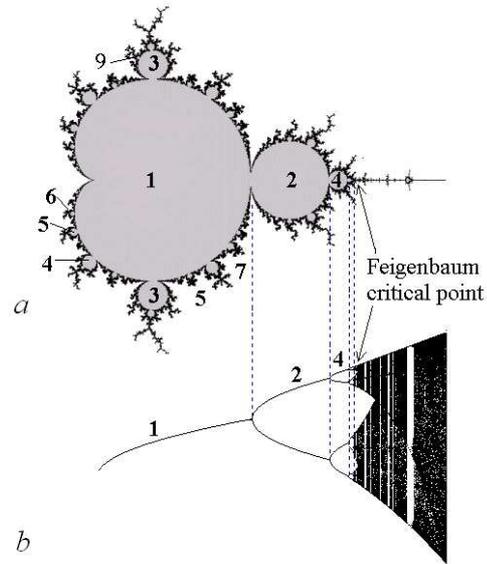
plex plane

$$\lambda_{n+1} = \varphi(\lambda_n) = -2\lambda_n(\lambda_n - \lambda_n^2). \qquad (12)$$

Starting from any point of a complex plane the trajectory of the map approaches the attractor at $\lambda = 0$ or escapes to the attractor at infinity. The borderline of these basins of attraction is the Julia set J of the of the complex mapping (12) [4]. Let us call it the *critical set* by analogy with the theory of phase transitions, where it is interpreted as the critical phase border. Julia set J is determined as the border of the set

$$P = \{\lambda \in \mathbf{C} : \lim_{n \to \infty} \varphi^n(\lambda) \neq \infty\}. \qquad (13)$$

In Fig.3 a diagram on the complex plane of initial values of λ of RG transformation (12) is represented. Black color indicates the basin of attraction P, that is the area bounded by the Julia set J. Shades of gray color designate areas

of different time of escape (i.e. dynamic distance up to attractor at infinity), i.e. the sets of points with various number of iterations necessary to escape out of a large enough ball.

It is evident that the Julia set in Fig.3 is quite similar to the Mandelbrot set of the map (4) (Fig. 2). Similarity between them can be explained as follows. Indeed, the escape of λ to infinity with iterations of RG transformation means the escape to infinity of iterations of the map (4). Distinctions can be explained by approximate character of RG transformation. Also, it is necessary to take into account that transformation (12) describes properties only for the period-doubling cascade of period-doublings, thus providing the similarity of sets M and J only near the Feigenbaum point. (The asymptotic similarity of the area of an analyticity of exact solution of the RG equations and Mandelbrot set near the Feigenbaum point was discussed earlier in works [23, 24, 25].) It is interesting to generalize these results for other sequences of period-multiplication bifurcation cascades.
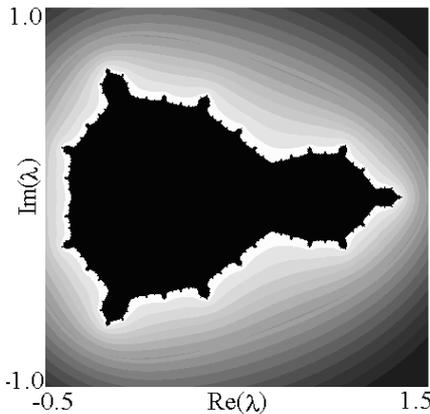

FIG. 3. Complex plane of initial values of λ for RG transformation (12) is represented. Black color designates the basin of attraction of origin bounded by the Julia set. Shades of gray color mark the area escape to infinity. Light colors correspond to the larger dynamical distance from the attractor at infinity.

4   Complex cycles of RG transformation

Let us describe procedure of visualisation of critical set J with the help of a backward iterations. This method is based on known fundamental properties of the Julia sets of analytical maps [4]:

If there is a repeller point λ_* for the map $\lambda_{n+1} = \varphi(\lambda_n)$, then the set

$$J' = \{\lambda : \varphi^n(\lambda) = \lambda_*, n = 1, 2, ...\}, \qquad (14)$$

is dense in J. The fixed point can be found directly from an iterative polynomial, as a root of the equation $\varphi(\lambda_*) = \lambda_*$. If the fixed point satisfies a condition $|\varphi'(\lambda_*)| > 1$, then it is repeller.

Julia set is invariant both in respect to forward and backward iterations. Backward orbit of any point laying in some basin of attraction approaches closer to a border of this basin, i.e. to the Julia set.

Repeller points in our case are known to be $\lambda_* = (1 \pm \sqrt{3})/2$. The backward transformation also can be found analytically by solving equation (12) in respect to the variable λ_n (we easily do it with the help of software packet *Mathematica*)

$$
\begin{aligned}
\lambda_n^1 &= \tfrac{1}{3} + \frac{2^{2/3}}{3(4+27\lambda_{n+1}+3\sqrt{3\lambda_{n+1}(8+27\lambda_{n+1})})^{1/3}} \\
&\quad + \frac{(4+27\lambda_{n+1}+3\sqrt{3\lambda_{n+1}(8+27\lambda_{n+1})})^{1/3}}{3\times 2^{2/3}}, \\
\lambda_n^{2,3} &= \tfrac{1}{3} - \frac{1\pm\sqrt{3}i}{3\times 2^{2/3}(4+27\lambda_{n+1}+3\sqrt{3\lambda_{n+1}(8+27\lambda_{n+1})})^{1/3}} \\
&\quad - \frac{(1\pm\sqrt{3}i)(4+27\lambda_{n+1}+3\sqrt{3\lambda_{n+1}(8+27\lambda_{n+1})})^{1/3}}{6\times 2^{2/3}}.
\end{aligned}
$$
$$(15)$$

Also the Julia set J can be regarded as a set of all unstable cycles of every possible periods n [4, 26, 27]. With increase of n this set becomes more and more dense, and at $n \to \infty$ the distribution of the points on the complex plane yields the Julia set. Numerical calculation of an unstable cycle of period n is connected with constructing of every possible periodic sequences $\varepsilon_1, \varepsilon_2, ..., \varepsilon_n, \varepsilon_1, \varepsilon_2, ..., \varepsilon_n, ...$, where ε_i takes values 1, 2 or 3, which correspond to a choice of a root λ^1, λ^2 or λ^3 at the i-th backward iteration of (15). As a result of iteration procedure, the sequence

of values of λ converges to a cycle of period not exceeding n.

So, one of the unstable fixed points of RG transformation (12) is situated on the real axis and corresponds to the Feigenbaum transition to chaos. Beside it, there are unstable cycles of any other period. Elements of a cycle of period n of RG transformation (12) correspond to the accumulation points of bifurcations of period 2^n-tupling on the Mandelbrot set. This fact as well explains the similarity of the critical Julia set and the Mandelbrot set for the original map.

The sets of unstable cycles of periods $6-11$ are shown in Fig.4 (a-f). One can see that with increase of n, the full picture of critical set becomes more and more distinguishable. The analogy between elements of the cycles and Yang-Lee zeros, and also between limit of the infinite period of cycles n and a thermodynamic limit in the theory of phase transitions is evident.
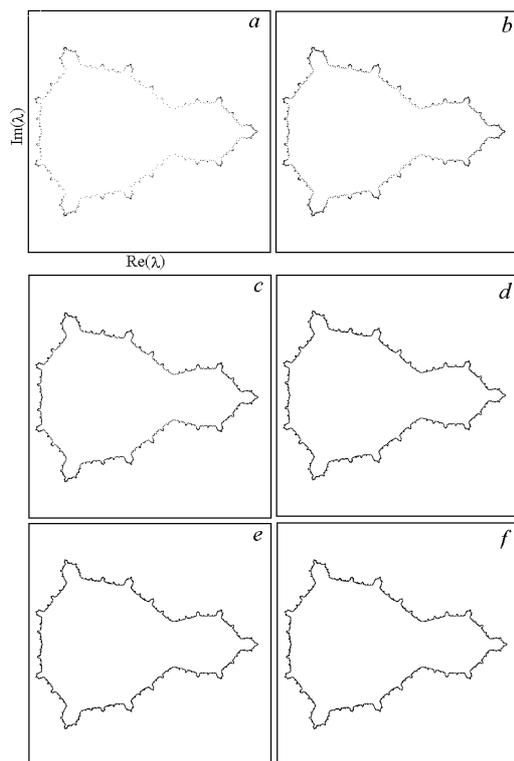

FIG. 4. Sets of all unstable cycles with periods $n = 6$ (a), 7 (b), 8 (c), 9 (d), 10 (e), 11 (f) obtained by backward iterations of RG transformation (12).

5 Electrostatical analogy – the potential of critical set and its properties near the transition to chaos

The phase transition critical point corresponds to intersection of the real axis with the critical line in the complex plane, formed in the thermodynamical limit by the set of zeros of partition function. This critical point should provide the jump of a derivative of the thermodynamic free energy (in the case of I-type phase transition) or of the second derivative (in the case of the II-type phase transition). Let us notice that the distribution of free energy depending on temperature can be interpreted as the distribution of the electrostatic potential created in 2D space by the set of charges, located at the partition function zeros, and in thermodynamic limit – the potential of the corresponding limiting distribution of charge on the critical Julia set.

The electrostatic potential for critical Julia set can be calculated from the Hubbard and Douady [4] formula

$$U = \frac{1}{3^n} \lim_{n \to \infty} \ln |\varphi^n(\lambda)|. \qquad (16)$$

This formula arises from reasons of existence of conformal mapping of the Julia set to a circle. The factor is not so important. In our case it is defined by the third order of the map (12): By one iteration, the potential of the point increase three times. At infinity, the potential of the Julia set coincides with potential of the charged disk $U_0 = (1/3^n) \ln |z|$.

We have examined distribution of the electrostatic potential and its derivatives, both in complex area, and on the real axis. In Fig.5 the set of several equipotential lines of homogeneously charged 2D object limited by Julia set of map (12) are shown. This figure characterizes the distribution of potential in the complex plane. In Fig.6 the distribution of the electrostatic normal component of a field (derivative of the potential) along the real axis is shown. The fact of existence
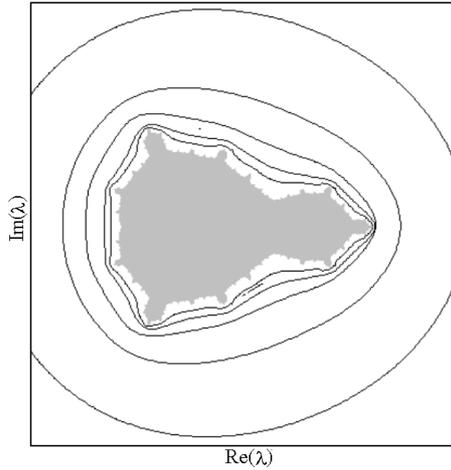
FIG. 5. Julia set of the map (12) and its equipotential lines, demonstrating distribution of electrostatic field at the areas of escape to infinity on the complex plane.

of the jump of the derivative of the potential at the critical point proves to be true. Near the other point of crossing of the critical set with the real axis the first derivative of potential behaves continuously, but has a break providing jump of the second derivative. This point has to be interpreted as the II-type phase transition. As appears, the dependence of potential close to the Feigenbaum critical point behave asymptotically as $U(\mathrm{Re}\lambda) = (\mathrm{Re}\lambda - \mathrm{Re}\lambda_*)^\gamma$, where the power factor $\gamma \simeq 0.63$ (see Fig. 7).
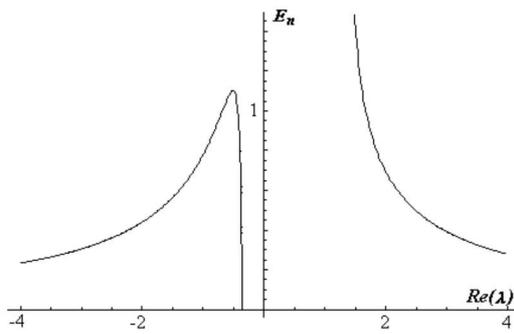

FIG. 6. Distribution of normal component of intensity of the electrostatic field of the critical set along the real axis λ.

According to these results, we can assert that the potential of the Julia set of the RG transformation is a certain criterion of ordering of
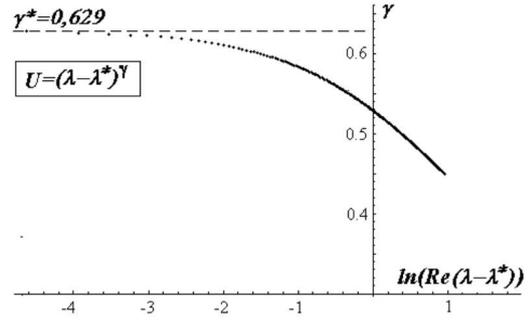

FIG. 7. Distribution of inclination of a double logarithmic curve $\ln U(\mathrm{Re}\lambda)$ vs $\ln(\mathrm{Re}\lambda - \mathrm{Re}\lambda_*)$ along real axis λ near the Feigenbaum critical point λ_*.

chaotic dynamics near to the point of transition to chaos, like the free energy is connected with the order parameter (magnetization) of thermodynamic system. One of known analogue of the order parameter is Lyapunov exponents [2]. However sometimes it is more effective to use the electrostatic potential for the description of dynamics, for example when the trajectories escape to infinity.

6   Conclusion

In the present paper the complex variable version of approximate RG method for period-doubling bifurcation cascade is considered. It is shown, that the Julia set of renormalization transformation of parameter of complex logistic map is the approximate version of Mandelbrot set of this map. This similarity is explained by the fact, that Julia set is a set of every possible unstable cycles of RG transformation, and elements of these cycles correspond to accumulation points of various bifurcations of period 2^n multiplication ($n \to \infty$), located on the boundary of Mandelbrot set dense everywhere.

The obtained outcomes are interpreted in a view of analogy with theory of phase transitions, namely the Yang-Lee theory, based on investigation of properties of thermodynamic values depending on complex temperature. It is necessary to note, that the elements of considered unstable cycles of RG transformation which is equiv-

alent to points of Julia set of this transformation, in thermodynamic analogy corresponds to so-called Yang-Lee zeros, defining in a thermodynamic limit the borderline of phase transition. It is shown, that at the points of transition to chaos the jump of an electrostatic field of a critical Julia set is observed. Within the framework of considered analogy, the jump of a derivative of a free energy at phase transition points takes place. Thus, it is possible to define a new criterion of transition to chaos. The electrostatic potential can be regarded as order parameter for the transition.

We conclude that complex generalization of approximate RG method appears to be useful to advance understanding of the critical phenomena at threshold of chaos and for development of analogy with the theory of phase transition, which can give new approaches to investigation of these phenomena.

7 Acknowledgements

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