Review and Examples of Non-Feigenbaum Critical Situations Associated with Period-Doubling

S. P. Kuznetsov, A. P. Kuznetsov, and I. R. Sataev

Institute of Radio-Engineering and Electronics of RAS, Zelenaya 38, Saratov, 410019, Russian Federation

We review several critical situations, linked with period-doubling transition to chaos, which require using at least two-dimensional maps as models representing the universality classes. Each of them corresponds to a saddle solution of the two-dimensional generalization of Feigenbaum-Cvitanović equation and is characterized by a set of distinct universal constants analogous to Feigenbaum's α and δ. We present a number of examples (coupled Hénon-like maps, coupled driven oscillators, coupled chaotic self-oscillators), which manifest these types of behavior.

An important aspect of the problem of turbulent dynamics in spatially extended systems is the question: how does the spatio-temporal chaos originate from simple regular regimes as we vary one or more control parameters? The breakthrough in understanding the onset of chaos in low-dimensional systems was Feigenbaum's discovery of the period-doubling universality and the renormalization-group (RG) approach [1,2]. The one-dimensional non-invertible iterative maps represent the simplest class of systems, which exhibit the Feigenbaum type of behavior. However, the period-doubling transition to chaos with the same universal quantitative regularities occurs in many multi-dimensional dissipative nonlinear systems [3].

For a spatially extended system, as long as the Feigenbaum theory is applicable, it allows understanding the onset of regimes of only restricted complexity, associated with certain spatial forms, which are governed by one variable and described in terms of a one-dimensional iterative map. When new modes consequently come into play in a course of parameter variation on a road to developed spatio-temporal chaos, effective dimension of the dynamics increases, and description in terms of the one-dimensional maps inevitably becomes insufficient. In this paper, we review several situations associated with period doubling, which require at least two-dimensional maps as models for representation of the dynamics. These situations may arise in the context of multi-parameter analysis of transition to chaos in multidimensional systems.

Generalizing concept of "scenario" for a multi-parameter case, we may think of some configuration of domains of distinct regimes in the parameter space. Generic one-parameter Feigenbaum scenario occurs at the critical surface, a limit of a sequence of the period-doubling bifurcation surfaces. Behaviors that are more special may occur at some curves and points on this surface. In the multi-parameter analysis, we are obliged to consider them too, as phenomena of codimensions two and three, respectively. As believed, these critical situations, like the Feigenbaum one, allow RG analysis, which must reveal the intrinsic quantitative regularities.

To analyze types of critical behavior intrinsic to two-dimensional maps due to presence of an additional dimension of phase space, we use a two-dimensional generalization of the renormalization equation of Feigenbaum – Cvitanović [4,5]. In assumption that a coordinate system in the two-dimensional phase space is selected in such way that the rescaling transformation, performed in a course of the procedure, is diagonal ($X \to X/\alpha$, $Y \to Y/\beta$), the equations read

$$\begin{aligned} g_{k+1}(X,Y) &= \alpha g(g(X/\alpha,\, Y/\beta),\, f(X/\alpha,\, Y/\beta)), \\ f_{k+1}(X,Y) &= \beta_k f(g(X/\alpha,\, Y/\beta),\, f(X/\alpha,\, Y/\beta)), \end{aligned} \tag{1}$$

where α and β are some constants, specific for each type of critical behavior. A pair of functions $\{g_k(X,Y), f_k(X,Y)\}$ defines appropriately normalized evolution operator for 2^k units of discrete time dynamics under consideration. One can apply this procedure called the RG transformation repeatedly to obtain a sequence of the evolution operators for larger and larger time scales.

A critical situation usually corresponds to convergence of the operator sequence to some definite limit, *a fixed point of the RG transformation*, or, as alternative, to *a periodic point* or *a cycle*. However, the last possibility is not conceptually different, because in the case of period p one can speak of a fixed point of the RG transformation composed of p steps of the original construction.

The next step in the RG analysis consists in consideration of small perturbations of the solution associated with the critical situation under study. It gives rise to eingenvalue problem for a set of functional equations obtained from linearization of the RG transformation (1) near a fixed point or a periodic solution. Among the eigenmodes one should select the relevant ones, which have $|\nu|>1$ (they are responsible for asymptotic behavior of the solution at subsequent repetition of the RG transformation), and exclude modes associated with infinitesimal variable changes. The number of relevant modes n corresponds to codimension of the critical point.

Critical situations of higher codimensions deserve accurate study and classification because they represent "organizing centers" of the parameter space structure, where domains of all relevant characteristic dynamical regimes of the system are concentrated locally. An important task of the theory is also constructing of model systems, the simplest representatives of the universality classes, which would play for them the same role as the one-dimensional quadratic map for the Feigenbaum scenario.

1. Conservative Period-Doubling Criticality in the Context of Dissipative Dynamics

Soon after the works of Feigenbaum, several authors noticed that infinite sequences of period-doubling bifurcations occur not only in dissipative but also in conservative systems [6,7]. In contrast to the dissipative case, the convergence rate is a distinct universal factor, $\delta\approx 8.72$. An appropriate version of the RG analysis was developed [6,7,4]. We prefer to separate terminologically this type of critical behavior from the classic Feigenbaum class and call it *the Hamiltonian period-doubling criticality* (H-type).

This critical behavior may occur in situations like motion of charged particles in vacuum in electric and magnetic fields, or in systems of celestial mechanics with gravitational interaction. If one wish to approach H-type criticality, say, in experiments with a forced nonlinear oscillator actualized as a mechanical device or an electronic circuit, a straightforward idea is to exclude the energy loss. In this case, in principle, we may speak only of more or less satisfactory approximation for a conservative system. Alternatively, we may try to arrange H-type of criticality not in a conservative, but in a self-oscillatory system. In this case, H type will appear not due to vanishing dissipation, but due to compensation of dissipation from external non-oscillatory source of energy. In this case the H criticality appears as phenomenon of codimension 2: we must control two parameters, one responsible for strength of nonlinearity, and another for the energy balance in the system.

Let us consider a van der Pol oscillator driven by a sequence of short pulses of period T, and assume that amplitudes of the kicks depend on an instantaneous value of the dynamical variable as $F(x)$. The dynamical equation reads

$$\ddot{x}-(\varepsilon-\mu x^2)\dot{x}+x=\sum_m F(x)\delta(t-mT)\,. \qquad (2)$$

In assumption that parameters ε, μ, and amplitude F are small, between the kicks we can use a method of slow amplitudes. In this assumption, one can derive an explicit stroboscopic Poincaré map. For simplicity, let us set $T=(4k+1)\pi/2$ and select a concrete function $F(x)=1-Ax^2$. Then, the map takes a form

$$x_{n+1}=B(1-Ax_n^2-y_n)\left[1+C\left(x_n^2+(1-Ax_n^2-y_n)^2\right)\right]^{-1/2},\ y_{n+1}=Bx_n\left[1+C\left(x_n^2+(1-Ax_n^2-y_n)^2\right)\right]^{-1/2}, \qquad (3)$$

where $B=\exp\frac{1}{2}\varepsilon T$, $C=\mu T(\exp\varepsilon T-1)/4\varepsilon T$, and n numerates steps of discrete time. Note that in a limit $\varepsilon\to 0$, $\mu\to 0$ we have B=1, C=0, and the map (3) reduces to the area-preserving Hénon map.

Figure 1 shows a chart of dynamical regimes for the model (3) at certain fixed μT. The horizontal axis corresponds to parameter, which controls the Andronov – Hopf bifurcation of a limit cycle birth in the autonomous van der Pol oscillator, and the vertical to parameter of nonlinearity in the kick amplitude dependence.

At large negative ε, far from the Andronov-Hopf bifurcation, the oscillator behaves as a linear system, and nonlinearity enters into play only due to the kick amplitude dependence on x. In this domain, the map is equivalent (up to a variable change) to the Hénon map and manifests transition to chaos via the Feigenbaum period doubling cascade. At positive ε, the oscillator becomes active, and quasiperiodic behavior due to beating of its own oscillations and of periodic kicks arises.

If we increase ε and follow the Feigenbaum critical line, it terminates at some point. Accurately, location of this point may be estimated as a limit of the sequence of terminal points for the curves of subsequent period-doubling bifurcations. At those points the respective periodic orbits have two Floquet multipliers equal to (-1). As a limit, we get the critical point

$$\varepsilon T_c=0.4036684037636\ldots,\ A_c=4.083016502041\ldots \qquad (4)$$

The best way to check belonging of the critical point associated with period doubling to a supposed universality class, consists in computation of multipliers for orbits of period 2^k with large integer k. This is convenient, in particular, because multipliers are invariant in respect to selection of a coordinate system in the phase space. The multipliers must tend to the universal values obtained from the RG analysis.

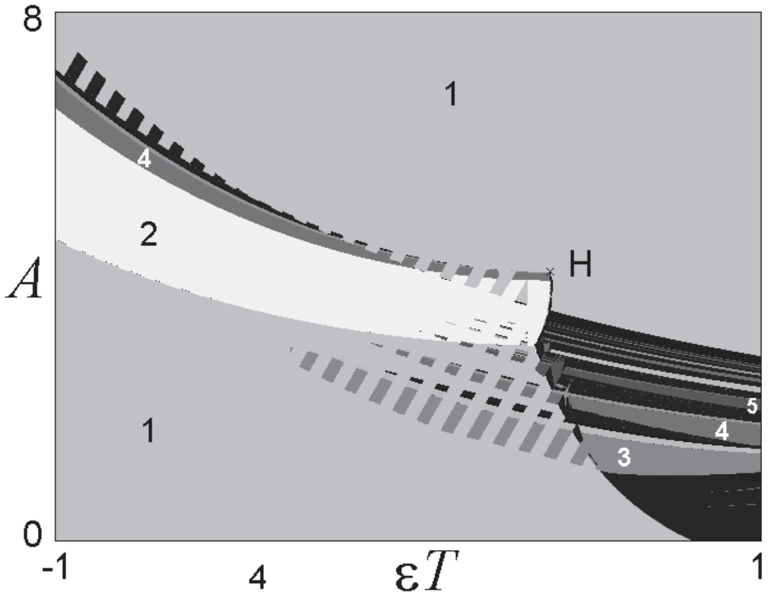

Figure 1. Chart of regimes for the map (3) on the plane of parameters εT and A at constant μT. Horizontal axis corresponds to the parameter, which controls the Andronov – Hopf bifurcation in the autonomous van der Pol oscillator, and the vertical axis to the parameter, which controls degree of nonlinearity in the kick amplitude dependence. Gray tones designate periodic behaviors with periods labeled by numbers, black corresponds to chaos. Strips denote areas of multistability, the alternating tones designate regimes associated with the distinct coexisting attractors. Critical point H is marked with this letter. The value of μT =3.246832310801 selected to have C=1 at the critical point.

For the critical point under consideration the results are summarized in Table 1. Observe fast convergence to the universal values expected for the H-type critical point from the RG analysis [5] (the last row in the table). Also, as seen from the table, a product of two multipliers for higher periods of cycles tends to 1 with high precision, which corresponds to the conservative nature of the dynamics in asymptotic of large time scales.

Table 1. Multipliers of cycles of period $p=2^k$ and their products at the critical point of H type in the Hénon – van der Pol map (3)

Period	μ_1	μ_2	$\mu_1\mu_2$
4	-2.058910	-0.4864611	1.0015799
8	-2.057285	-0.4859759	0.9997908
16	-2.057504	-0.4860392	1.0000278
32	-2.057475	-0.4860309	0.9999963
64	-2.057477	-0.4860325	1.0000005
128	-2.057461	-0.4860359	1.0000000
RG	-2.0574783	-0.4860318	1

2. Bicritical Point in a Model with Unidirectional Coupling

Let us turn now to a special class of two-dimensional non-invertible maps, which allow decomposition onto subsystems with unidirectional coupling: $x_{n+1} = G(x_n),\ y_{n+1} = F(x_n, y_n)$ [8, 9]. In literature, such systems were discussed, in particular, as models of turbulence in open flows [10]. Systems with unidirectional coupling may be constructed artificially; for example, in electronics and optics such coupling may be designed easily in experiments. Recently, systems of this kind are studied in the context of problem of chaotic communication [11,12].

Already in the first work reported on the discovery of the bicritical behavior [8], beside theoretical considerations and computations, some experimental results were presented for a system of two periodically driven nonlinear RL-diode circuits. In the scheme, the unidirectional coupling was arranged by a special amplifier. By variation of two control parameters, which were amplitudes of external driving in both subsystems, in the experiment it was sufficiently easy to bring simultaneously both subsystems to the chaos threshold and get the bicritical situation.

Recently Kim and Lim [13] presented a detailed computational study for a system of driven nonlinear oscillators with unidirectional coupling:

$$\begin{aligned}
\dot{x}_1 &= y_1, \\
\dot{y}_1 &= -2\pi(\beta\Omega y + \Omega^2 - A\cos 2\pi t)\sin 2\pi x_1, \\
\dot{x}_2 &= y_2 + c(x_1 - x_2), \\
\dot{y}_2 &= -2\pi(\beta\Omega y + \Omega^2 - B\cos 2\pi t)\sin 2\pi x_2 + c(y_1 - y_2).
\end{aligned} \tag{5}$$

In these equations, variables with subscripts 1 and 2 relate to the master and the slave subsystem, respectively. As computed in Ref. [13], at fixed β=1, Ω=0.5, and coupling constant c=0.2, the bicritical point of the system (5) is located at A=A_c=0.798049182451, B=B_c=0.80237721.

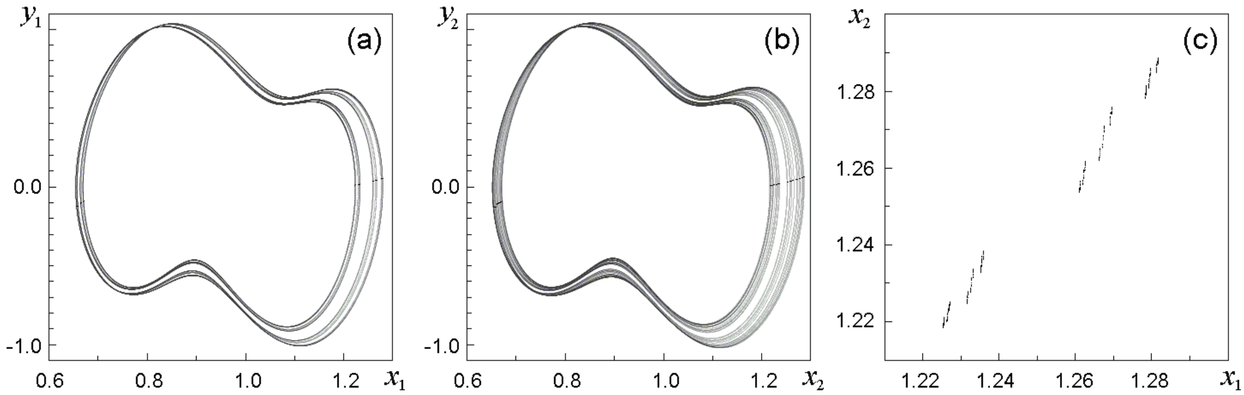

Figure 2. Phase portraits of the bicritical attractor for the system of driven nonlinear oscillators with unidirectional coupling of Kim and Lim (5) at β=1, Ω=0.5, c=0.2, A=0.798049182451, B=0.80237721. Diagrams (a) and (b) show two projections of the attractor, one onto the plane of variables of the master subsystem, and another for the slave system. Black dots on the portraits correspond to the cross-section with a hyper-plane t=0.35 (mod 1) (stroboscopic Poincaré section). Diagram (c) represents these points on the plane of x_2 versus x_1 to compare it with portraits of the attractors discussed above for the model maps.

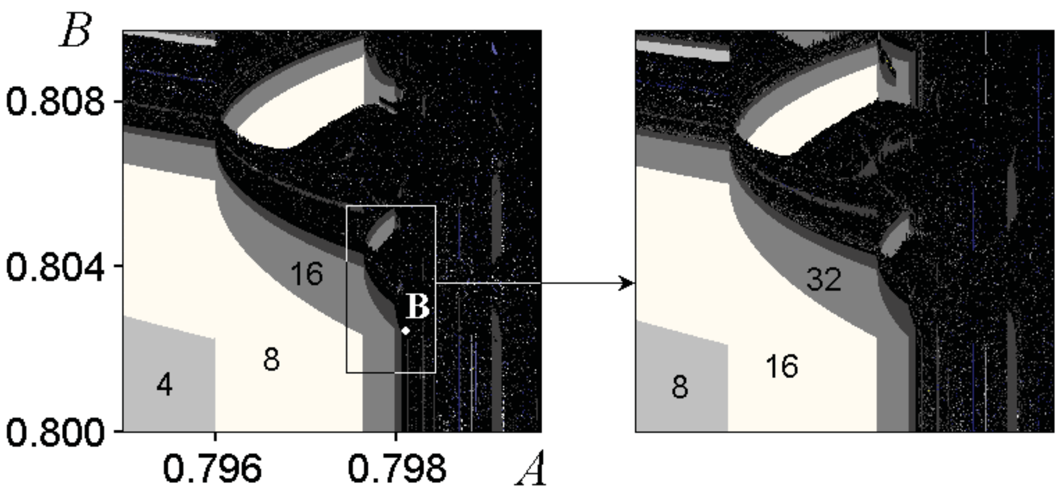

Figure 3. Parameter plane diagrams for driven nonlinear oscillators with unidirectional coupling of Kim and Lim (5), β=1, Ω=0.5, c=0.2. Areas of distinct periodic regimes are shown in gray scale, and periods are marked with numbers. The second picture is obtained by magnification of the small box from the first one with factors δ_1=4.6692 and δ_2=2.3927 along the horizontal and vertical axes, respectively.

Diagrams (a) and (b) in Fig.2 show phase portraits of the bicritical attractor in two projections from the five-dimensional extended phase space. The first is a plane of variables for the master subsystem, and another of those for the slave system. The trajectories constituting the attractor are drawn in gray, and black dots correspond to moments of cross-section of an orbit with a hyper-plane t=const in the phase space (stroboscopic Poincaré section). Diagram (c) represents those points on the plane (x_2, x_1). It looks remarkable similar to portraits of the bicritical attractors for the model maps [9]. Figure 3 shows parameter plane charts locally near the bicritical point. The scaling property characteristic for a vicinity of the bicritical point is illustrated: Under magnification with factors δ_1=4.6692… and δ_2=2.3927… the structure of the domains in the parameter plane obviously looks similar.

3. Criticality of FQ type ("Feigenbaum+Quasiperiodicity")

In the first work reported about the FQ critical behavior [14], it was found in a system of two asymmetrically coupled one-dimensional maps

$$x_{n+1} = 1 - \lambda x_n^2 - C y_n^2,\ y_{n+1} = 1 - A y_n^2 - B x_n^2 \qquad (6)$$

by variation of λ and A with fixed B and C. For particular B=0.375 and C=–0.25 the FQ-point is placed at λ_c=1.654524590, A=A_c=1.030837593.

A more realistic model should be based on coupled two-dimensional dissipative invertible maps, say, Hénon maps, which may be interpreted as Poincaré maps for a hypothetical flow system. Recently, in Ref. [15] it was shown in computations that FQ criticality indeed presents in such a model.

More complex situation occurs if we try to build up a system of two coupled autonomous self-oscillators. In the case of three-dimensional partial systems, the formally constructed Poincaré map is five-dimensional, not reducible, in general, to two coupled two-dimensional maps. Apparently, presence of an additional dimension in the Poincaré map facilitates appearance of the third eigenmode in the solution of the RG equation, and it becomes necessary to have three control parameters to reach the critical situation FQ. A concrete example we have considered supports this assertion. This is a system of two Chua's circuits with dissipative coupling governed by equations

$$\begin{aligned}
\dot{x}_1 &= \alpha_1\big(y_1 - h(x_1)\big) + \varepsilon_1(x_2 - x_1), & \dot{x}_2 &= \alpha_2\big(y_2 - h(x_2)\big) + \varepsilon_2(x_1 - x_2),\\
\dot{y}_1 &= x_1 + y_1 - z_1 + \varepsilon_1(y_2 - y_1), & \dot{y}_2 &= x_2 + y_2 - z_2 + \varepsilon_2(y_1 - y_2),\\
\dot{z}_1 &= -by_1 + \varepsilon_1(z_2 - z_1), & \dot{z}_2 &= -by_2 + \varepsilon_2(z_1 - z_2),
\end{aligned}
\qquad h(x) = \begin{cases} (2x+3)/7, & x \le -1,\\ -x/7, & -1 < x < 1,\\ (2x-3)/7, & x \ge 1. \end{cases} \qquad (7)$$

A search for the FQ point by variation of two parameters, with rest fixed, was unsuccessful. On the other hand, by variation of three parameters, α_1, α_2, and b at fixed ε_1=–0.05, ε_2=0.2, the FQ point was detected and located at

$$\alpha_1 = 6.330061623840\ldots,\ \alpha_2 = 6.585930638394\ldots,\ b = 10.19802309657\ldots \qquad (8)$$

Table 2 gives evidence of the true FQ nature of this point. There we present pairs of senior multipliers for unstable periodic orbits coexisting at the critical point; $p=2^k$ designates a number of steps of the Poincaré map necessary to close the cycle. Observe evident convergence to the universal values obtained from the RG analysis.

Table 2. Multipliers of cycles at the critical point of FQ type in the coupled Chua circuit

P	μ_1	μ_2
16	–1.557415	–1.086792
32	–1.586435	–1.067858
64	–1.594082	–1.025180
128	–1.562911	–1.078976
256	–1.579819	–1.057080
RG	–1.579739	–1.057149

Figure 4 demonstrates another characteristic property of the FQ critical dynamics. It shows a portrait of attractor of the system (7) in projection onto a plane of two variables relating to the first partial system. A small part of the picture inside in a small rectangular is magnified, and a series of pictures demonstrates in more details the fractal-like "strips" constituting the attractor. Under magnification by factor α_1=–1.9000… structure of the "strips" reproduces itself in accordance with our expectations based on the results of the RG analysis. (It is rather difficult to extract another scaling factor α_2=–4.0081… from such computations because of fast shrinking of the respective details of the fractal attractor.)

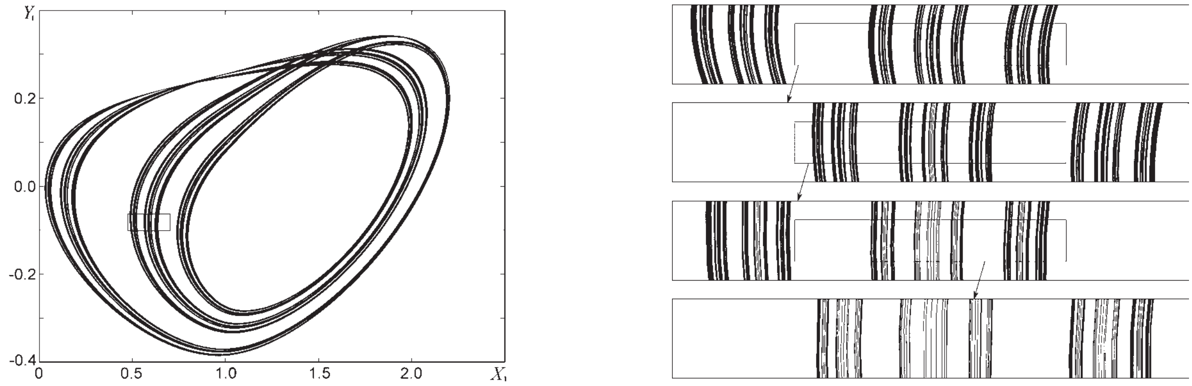

Figure 4. Portrait of attractor of the model (7) at the critical point FQ in projection onto the plane of two variables relating to the first partial system. A small fragment of the picture inside in a small rectangular is shown separately. Under subsequent magnification by factor $|\alpha_1|$=1.9000… the structure of "strips" constituting the attractor reproduces itself on each second step of enlargement (account negative sign of $|\alpha_1|$).

4. Criticality of C type

Let us consider a standard form of the fold mapping [16] $(u, v) \to (u^2, v)$ and compose it with a general affine transformation $(u, v) \to (A + Bu + Cv,\ D + Eu + Fv)$, where $A,B,\ldots F$ are parameters. Then, by a variable and parameter change it is reduced to a map

$$x_{n+1} = a - x_n^2 + by_n,\ \ y_{n+1} = -x_n^2 + d \cdot y_n\,. \tag{9}$$

As shown in Ref. [17], this map manifests some special critical behavior associated with a period-two saddle orbit of the Feigenbaum – Cvitanovic equation. At fixed b=–0.6663 it is located at a_c=0.24990280, d_c=0.45290288.

The following example, as we believe, is of principal significance, although relates to an artificially constructed model map.

One of the most widely discussed scenarios of the onset of turbulence comes back to Landau and Hopf [18, 19] and consists, as they suggested, in subsequent birth of oscillatory components with incommensurate frequencies, or, in language of more modern nonlinear dynamics, in subsequent birth of attractors represented by tori of higher and higher dimensions. In accordance with latter argumentation of Ruelle and Takens [20], after few first bifurcations a strange chaotic attractor will be born instead of the higher-dimensional torus. In any case, this picture contains an intermediate stage of bifurcation of Neimark – Sacker, the onset of torus from the limit cycle [21, 22].

Let us construct a model map, which can demonstrate all bifurcation relevant for the problem of stability loss of a limit cycle, including the Neimark – Sacker bifurcation. In linear stability analysis of dynamics in terms of Poincaré section near the limit cycle one obtains a linear map, which may be written in appropriately chosen variables as $x_{n+1} = Sx_n - y_n$, $y_{n+1} = Jx_n$, where S and J are trace and determinant of the Jacobian matrix defined over one period of the cycle. Next, we introduce nonlinearity in the map "by hands", in a hope that the most common features of the bifurcation transitions will be caught in the constructed map. Namely, we set [23]

$$x_{n+1} = Sx_n - y_n - (\varepsilon y_n^2 + x_n^2),\ \ y_{n+1} = Jx_n - (y_n^2 + x_n^2)/5\,. \tag{10}$$

Domain of stability of the fixed point at the origin has a form of triangle on the parameter plane (S, J) with sides [24, 25]: 1–S+J=0 (one multiplier equals 1, the saddle-node bifurcation), 1+S+J=0 (one multiplier equals −1, the period-doubling bifurcation), and J=1 (two complex conjugate multipliers have unit modulus, the Neimark - Sacker bifurcation), see Fig.5a.

In Fig.5b we present chart of dynamical regimes for the map (10) on the parameter plane (S,J) at fixed ε=0.535. One easily recognizes the stability triangle. On the topside, the Neimark – Sacker bifurcation takes place of birth of motion spiraling around the former fixed point. Concrete nature of the regime depends on the rotational number linked with argument of the complex multiplier at the bifurcation. In the region upper the bifurcation border one can see tongues of periodic regimes and domains of quasiperiodicity between them.

Let us consider one of the tongues, that of period 4, in more details. Diagram (c) shows this tongue and its neighborhood with magnification. Observe that the period-doubling bifurcation curves inside the tongue visibly stick into its edge. Computations confirm that there is a sequence of terminal points for the period-doubling bifurcation curves at the edge of the synchronization tongue, which converges to a limit point located at

$$S=S_c=-0.548966\ldots,\ J=J_c=1.547188\ldots \qquad (11)$$

This is a critical point of C-type. To give evidence of it on the quantitative level, we present in Table 3 numerical data on multipliers of cycles of period 2^k computed at this point.

A remarkable feature of dynamics at the critical point C derived from the RG analysis is presence of the critical quasiattractor, a countable infinite set of coexisting stable cycles of period proportional to 4^k, k=0,1,2,… (Notice that the multipliers for these cycles in the Table are less then 1 in modulus.) In computations, it is possible to get al least several first representatives of this family of attractors.

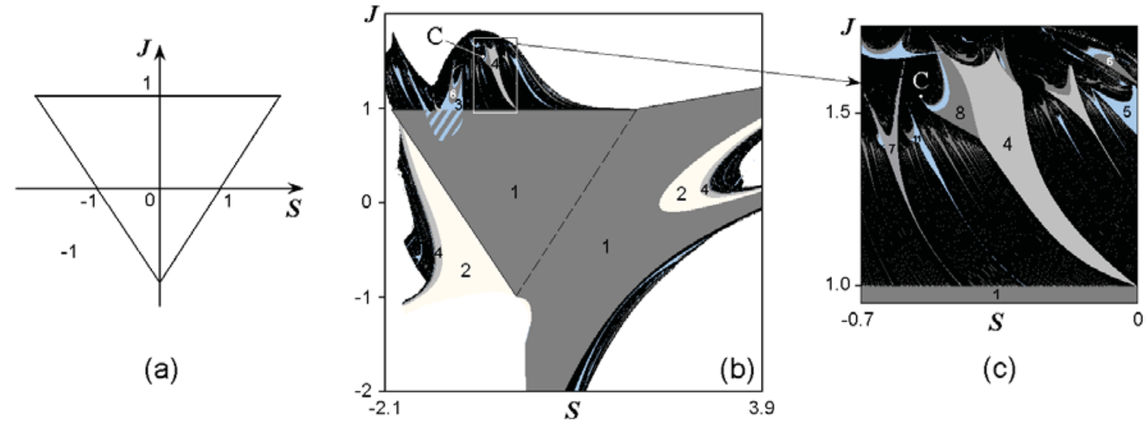

Figure 5. Parameter plane for the model map (10): (a) triangle of stability for the fixed point at origin; (b) chart of dynamical regimes and its magnified fragment (c). Gray scales are used to show areas of periodic dynamics. Black designate chaos, quasiperiodicity or unrecognized high-period regimes. Stripped area indicates coexistence of different attractors. Critical point C located at the period-doubling accumulation point at the edge of synchronization tongue is marked in diagrams (b) and (c).

Table 3. Multipliers of cycles of period $p=2^k$ at the critical point of C type in the model map (10)

p	$\mu_2^{(2)}$	$\mu_1^{(2)}$	$\mu_2^{(1)}$	$\mu_1^{(1)}$
64	1.179719	–0.874220		
128			0.859691	–0.695732
256	1.175752	–0.855538		
512			0.850658	–0.722936
1024	1.172441	–0.847454		
2048			0.847450	–0.725255
RG	1.174459	–0.848865	0.847450	–0.725255

Observe nice correspondence of multipliers to the universal values known from the RG analysis (the last row of the Table). As we found, critical points of the same nature occur as well inside some other tongues above the Neimark – Sacker bifurcation.

As follows from this example, in the multiparameter analysis of transition to turbulence via quasiperiodicity (scenario of Landau – Hopf – Ruelle – Takens), already on a stage of birth of the second incommensurate frequency, one can expect presence of critical points of C-type with intrinsic nontrivial features of dynamical behavior, including coexistence of a countable set of attractive periodic orbits.

The authors acknowledge support from RFBR grant No 03-02-16074.

REFERENCES

[1] M. J. Feigenbaum. The universal metric properties of nonlinear transformations. J. Stat. Phys. **21**, 1979, №6, 669-706.

[2] E.B. Vul, Ya.G. Sinai, K.M. Khanin. Feigenbaum universality and thermodynamic formalism. Russ. Math. Surv., **39** (3), 1984, 1-40.

[3] P.Cvitanovic, ed. Universality in Chaos. Adam Hilger, 2nd Edition (1989), 631pp.

[4] J.-P.Eckmann, H.Koch, P.Wittwer. Existence of a fixed point of the doubling transformation for area-preserving maps of the plane. Phys. Rev., **A26**, 1982, 720-722.

[5] A.P.Kuznetsov, S.P.Kuznetsov, I.R.Sataev. A variety of period-doubling universality classes in multi-parameter analysis of transition to chaos. Physica **D109**, 1997, 91-112. See also: http://sgtnd.narod.ru/science/alphabet/eng/index.htm.

[6] J.M. Greene, R.S. MacKay, F. Vivaldi, M.J. Feigenbaum. Universal behavior in families of area preserving maps. Physica **D3**, 1981, No 3, 468-486.

[7] M. Widom and L.P. Kadanoff. Renormalization group analysis of bifurcations in area-preserving maps. Physica D5, 1982, 287-292.

[8] B.P.Bezruchko, Yu.V.Gulyaev, S.P.Kuznetsov, and E.P.Seleznev. New type of critical behavior of coupled systems at the transition to chaos. Sov. Phys. Dokl. Vol. **31**(3), 1986, 258-260.

[9] A.P.Kuznetsov, S.P.Kuznetsov, I.R.Sataev. Bicritical dynamics of period-doubling systems with unidirectional coupling. International Journal of Bifurcation and Chaos, 1991, Vol.1, No 4, p. 839-848

[10] I. S. Aranson, A. V. Gaponov-Grekhov and M. I. Rabinovich. The onset and spatial development of turbulence in flow systems, 33, 1988, 1-20.

[11] T. Heil, J. Mulet, I. Fischer, C. R. Mirasso, M. Peil, P. Colet, and W. Elsäber. ON/OFF Phase Shift Keying for Chaos-Encrypted Communication Using External-Cavity Semiconductor Lasers. IEEE Journal of Quantum Electronics, **38**, 2002, No 9, 1162-1170.

[12] S. Wang, J. Kuang, J. Li, Y. Luo, H. Lu and G. Hu. Chaos-based secure communications in a large community. Phys. Rev. **E66**, 2002, 065202 (4 pages)

[13] S.-Y. Kim and W. Lim. Bicritical scaling behavior in unidirectionally coupled oscillators. Phys. Rev. **E63**, 036223 (2001).

[14] S.P.Kuznetsov, I.R.Sataev. New types of critical dynamics for two-dimensional maps. Phys.Lett. **A162**, 1992, No 3, p.236-242.

[15] A.P.Kuznetsov, A.V.Savin, S.-Y.Kim. On the Criticality of the FQ-Type in the System of Coupled Maps with Period-Doubling. Nonlinear Phenomena in Complex Systems, 7:1, 2004, 69-77.

[16] V.I. Arnol'd, G.S. Wassermann. Catastrophe Theory. Springer-Verlag, 1992, 150 pp.

[17] S.P.Kuznetsov, I.R.Sataev. Period-doubling for two- dimensional non-invertible maps: Renormalization group analysis and quantitative universality. Physica **D101**, 1997, pp.249-269.

[18] L.D. Landau. On the problem of turbulence. Doklady Akademii Nauk SSSR, **44**, 1944, 311 (in Russian).

[19] E. Hopf. A mathematical example displaying features of turbulence. Communications on Applied Mathematics, **1**, 1948, 303-322.

[20] D. Ruelle and F. Takens. On the nature of turbulence. Commun. Math. Phys., **20**, 1971, 167-192.

[21] J. Guckenheimer and P.Holmes. Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields. Springer – Verlag, NY, 1997, 459 pp.

[22] Y.A. Kuznetsov. Elements of Applied Bifurcation Theory. Springer-Verlag. New York, 1995, 591 pp.

[23] A.P. Kuznetsov, A.Yu. Kuznetsova, I.R. Sataev. On critical behavior of a map with Neimark – Sacker bifurcation at destruction of the phase syncronization at the limit point of the Feigenbaum cascade. Izvestija VUZov – Prikladnaja Nelineinaja Dinamika (Saratov), **11**, 2003, No 1, 12-18. (in Russian).

[24] J.M.T. Thompson and H.B. Stewart. Nonlinear Dynamics and Chaos. Geometrical Methods for Engineers and Scientists. John Wiley and Sons, Chichester – NY – Brisbane – Toronto – Singapore, 1987, 376 pp.

[25] S.P. Kuznetsov. Dynamical Chaos. Moscow, Fizmatlit, 2001, 295 pp. (in Russian).