S. P. KUZNETSOV
Saratov Division of Institute
of Radio-Engineering and Electronics
Russian Academy of Sciences
Zelenaya 38, Saratov, 410019, Russia

Max-Planck-Institut für Physik Komplexer Systeme
Nöthnitzer Straße 38, 01187 Dresden, Germany

E-mail: kuznetsov@sgu.ru

GENERALIZED DIMENSIONS OF THE GOLDEN-MEAN QUASIPERIODIC ORBIT FROM RENORMALIZATION-GROUP FUNCTIONAL EQUATION

Received February 24, 2005; accepted March 10, 2005 DOI: 10.1070/RD2005v010n01ABEH000298

A method is suggested for computation of the generalized dimensions for a fractal attractor associated with the quasiperiodic transition to chaos at the golden-mean rotation number. The approach is based on an eigenvalue problem formulated in terms of functional equations with coefficients expressed via the universal fixed-point function of Feigenbaum-Kadanoff-Shenker. The accuracy of the results is determined only by precision of representation of the universal function.

Multifractal or thermodynamic formalism is a commonly recognized tool for description of strange sets arising in dynamical systems in different contexts [1]–[7]; its basic ideas have been clearly formulated e.g. in the paper of Halsey et al. [2]. Some examples presented by these and other authors relate to fractal attractors that occur at the onset of chaos via period doubling and quasiperiodicity [3]–[7]. The multifractal analysis reveals global scaling properties of these attractors, such as the generalized dimensions and $f(\alpha)$ spectra. They are of principal interest because of their universality for systems of different nature. Moreover, they allow a measurement in physical experiments [7].

One of the well-studied multifractal objects is the golden-mean quasiperiodic motion for the critical circle map with a cubic inflection point [8], [9], [10]. In fact, all essential quantitative characteristics of this object relate to a universality class including many systems of different nature [7]. It should be noted that *global* description of scaling properties in the multifractal formalism is in some contrast to *local* description in terms of the renormalization group approach [9], [10]. The last is based on solution of the functional fixed-point equation and associated with scaling relations for the evolution operators in a narrow neighborhood of the inflection point of the considered map.

In this note I present a method for precise computation of the multifractal characteristics similar to that developed earlier in cooperation with A. Osbaldestin in the context of Feigenbaum's period-doubling transition to chaos [11]. It will be shown that the problem allows formulation in terms of the Feigenbaum–Kadanoff–Shenker renormalization transformation with an extension including an additional linear equation for some auxiliary function. The desirable quantities, such as generalized dimensions and $f(\alpha)$ spectra, can be extracted from an eigenvalue problem for a set of linear functional equations, whose coefficients involve the universal fixed-point function of Feigenbaum–Kadanoff–Shenker. A particular case of these equations associated with one special generalized dimension is linked with the problem of the effect of noise on the golden-mean quasiperiodic motion at the onset of chaos [12].

Mathematics Subject Classification: 37C05, 37F25, 37C45, 37C55, 28A80
Key words and phrases: circle map, golden mean, renormalization, dimension, generalized dimensions

Using a polynomial representation of the Feigenbaum–Kadanoff–Shenker function valid up to approximately 14 decimal digits, I performed a numerical solution of the functional equations and obtained the generalized dimensions with high precision. In addition, I present an accurate estimate for the scaling factor responsible for the effect of noise on the golden mean quasiperiodic critical orbit.

As known [8], [9], [10], [13], [14], in the critical circle map

$$x_{n+1} = \phi(x_n) = x_n + r - (1/2\pi)\sin 2\pi x_n, \tag{1}$$

the dynamics with the golden-mean rotation number takes place at

$$r = r_{GM} = 0.606\ 661\ 063\ 470\ 112\ 017\ldots$$

In a standard approach developed by Halsey et al., to estimate numerically the multifractal characteristics for the golden-mean quasiperiodic motion they introduce a partition function $\Gamma_k(q,\tau) = \sum_{i=1}^{F_k} p_i^q/l_i^\tau$. Here F_k is a Fibonacci number, q and τ are some real parameters. The values $l_i = \left|x_i - x_{i+2^k}\right|$ are the minimal distances for pairs of points of the orbit of the critical circle map starting from the inflection point, $x_0 = 0$. They define natural scales for the partition with measures $p_i = F_k^{-1}$ attributed to each interval l_i. Obviously, $\Gamma_k(q,\tau) = F_k^{-q} S_k(\tau)$, where $S_k(\tau) = \sum_{i=1}^{F_k} l_i^{-\tau}$. For each given τ an appropriate $q = q(\tau)$ exists that ensures an asymptotic equality $\Gamma_{k+1}(q,\tau) = \Gamma_k(q,\tau)$ as $k \to \infty$, namely,

$$q = \lim_{k\to\infty} \frac{\log(S_{k+1}(\tau)/S_k(\tau))}{\log(F_{k+1}/F_k)} = \lim_{k\to\infty} \log_W(S_{k+1}(\tau)/S_k(\tau)), \tag{2}$$

where $W = (\sqrt{5}+1)/2$. Vise versa, for a given q we can select a respective value of $\tau = \tau(q)$. This relation of q and τ is used then to obtain the generalized dimensions $D_q = \tau/(1-q)$ and the singularity spectrum $f(\alpha)$ via relations $f = q\alpha - \tau$, $\alpha = d\tau/dq$.

Now let us turn to derivation of the basic equation. For a large k the lengths of the intervals l_i are small and can be expressed via the derivatives as

$$l_i \cong |\partial x_i/\partial x_1|\, l_1. \tag{3}$$

Then, we can compute them step by step together with the sums S_k via simultaneous iterations of the equations

$$x_{i+1} = \phi(x_i),\quad l_{i+1} = \left|\phi'(x_i)\right| l_i,\quad S_{i+1} = S_i + l_{i+1}^{-\tau}\psi(x_i). \tag{4}$$

For now, the auxiliary function $\psi(x)$ is supposed to be identically equal to 1.

Now, let us write two sets of relations (similar to (4)) for F_k and F_{k+1} iterations of the original map:

$$x_{i+F_k} = \phi_k(x_i),\quad l_{i+F_k} = \left|\phi_k'(x_i)\right| l_i,\quad S_{i+F_k} = S_i + l_{i+F_k}^{-\tau}\psi_k(x_i) \tag{5}$$

and

$$x_{i+F_{k+1}} = \phi_{k+1}(x_i),\quad l_{i+F_{k+1}} = \left|\phi_{k+1}'(x_i)\right| l_i,\quad S_{i+F_{k+1}} = S_i + l_{i+F_{k+1}}^{-\tau}\psi_{k+1}(x_i). \tag{6}$$

From these formulas, we obtain the following relations for evolution over F_{k+2} steps:

$$\begin{aligned} &x_{i+F_{k+2}} = \phi_k(\phi_{k+1}(x_i)),\quad l_{i+F_{k+2}} = |\phi_k'(\phi_{k+1}(x_i))\phi_k'(x_i)|\, l_i,\\ &S_{i+F_{k+2}} = S_i + l_{i+F_{k+2}}^{-\tau}\left[|\phi_k'(\phi_{k+1}(x_i))|^\tau \psi_{k+1}(x_i) + \psi_k(\phi_{k+1}(x_i))\right]. \end{aligned} \tag{7}$$

In accordance with the basic idea of the renormalization approach, let us perform the scale change $x \mapsto x/\alpha^k$, $l \mapsto l/|\alpha|^k$, where $\alpha = -1.288\ 574\ 553\ 954\ 368\ldots$ is a universal constant, the rescaling factor for the critical golden-mean dynamics [8], [9], [10], [13]. Then, in terms of the rescaled functions

$$\begin{aligned} &g_k(x) = \alpha^k\phi_k(\alpha^{-k}x),\quad f_k(x) = \alpha^k\phi_{k+1}(\alpha^{-k}x),\\ &\Phi_k(x) = \psi_k(\alpha^{-k}x),\quad \Psi_k(x) = \psi_{k+1}(\alpha^{-k}x), \end{aligned} \tag{8}$$

the equations (6), (7) imply that

$$\begin{aligned} g_{k+1}(x) &= \alpha f_k(x/\alpha), \\ f_{k+1}(x) &= \alpha g_k(f_k(x/\alpha)), \\ \Phi_{k+1}(x) &= |\alpha|^\tau\, \Psi_k(x/\alpha), \\ \Psi_{k+1}(x) &= |\alpha|^\tau \left[|g'_k(f_k(x/\alpha))|^\tau\, \Psi_k(x/\alpha) + \Phi(f_k(x/\alpha)) \right]. \end{aligned} \tag{9}$$

These relations define an RG transformation for a set of functions $\{g,\ f,\ \Phi,\ \Psi\}$. The procedure may be repeated again and again to get the functions for larger and larger k.

Asymptotically, the functions $g_k(x)$, $f_k(x)$ converge to the fixed-point solution, which satisfies $g(x) = \alpha f(x/\alpha)$, $f(x) = \alpha g(f(x/\alpha))$, or $g(x) = \alpha^2 g(\alpha^{-1} g(x/\alpha))$. This result was obtained first by Feigenbaum, Kadanoff, Shenker [9] and Rand, Ostlund, Sethna and Siggia [10], and reproduced latter by many authors [13], [15], [16], [17], [18]. Numerical data for polynomial expansion of the universal function $g(x)$ may be found e.g. in Ref. [18].

Convergence of the functions g and f to the fixed point of the renormalization transformation implies that the recursive linear functional equations for the functional pairs $\{\Phi_k(x),\ \Psi_k(x)\}$ have coefficients asymptotically independent of k. Hence, as $k \to \infty$, the solution will tend to the eigenvector associated with the largest eigenvalue for the matrix functional equation

$$\nu(\tau) \begin{pmatrix} \Phi(x) \\ \Psi(x) \end{pmatrix} = |\alpha|^\tau \begin{pmatrix} 0 & 1 \\ |g'(f(x/\alpha))|^\tau & 1 \end{pmatrix} \begin{pmatrix} \Phi(f(x/\alpha)) \\ \Psi(x/\alpha) \end{pmatrix}. \tag{10}$$

One special case of this equation ($\tau = 2$) appears in the theory of noise effect on the golden-mean quasiperiodic transition to chaos, see Ref. [12].

By the construction, the eigenvalue $\nu(\tau)$ indicates a rate of growth or decrease of sums S:

$$S_k(\tau) \propto \nu^k(\tau). \tag{11}$$

To have $\Gamma_k \to \text{const}$ as $k \to \infty$ we must set

$$\nu(\tau) = F_k^q \propto W^q \text{ or } q = \log_W \nu(\tau). \tag{12}$$

Then, in accordance with the known equations of the multifractal formalism, we can obtain the generalized dimensions as

$$D_q = \frac{\tau}{1-q}, \tag{13}$$

and $f(\alpha)$ spectrum as an implicitly defined relation between the variables

$$\alpha = \frac{d\tau}{dq} \text{ and } f = q\frac{d\tau}{dq} - \tau. \tag{14}$$

Although my argumentation starts from the approximate relation (3), apparently the final Eq.(10) is exact. Indeed, in the asymptotic of $k \to \infty$ the approximate nature of (3) becomes inessential. It may be thought that the corresponding rigorous proof can be found. (See an analogous approach in Ref. [11], where the data of numerical computations for period-doubling critical attractors manifest precise agreement with the best known numerical results, up to all reliable digits.)

With the known polynomial representation of $g(x)$ and $f(x)$ and the scaling constant α, I constructed the functional transformation defined by the right-hand part of Eq.(10) as a computer program. The unknown functions $\Phi(x)$ and $\Psi(x)$ are represented by tables of their values at the nodes of a one-dimension grid on the interval $[-1.2, 1.2]$ and by an interpolation scheme of the fourth order between the nodes. With the tables for $\Phi(x)$ and $\Psi(x)$ as an input, the program yields analogous tables as output. In principle, the achievable precision of the results is determined only by the accuracy of representation for the universal functions and the interpolation scheme.

Suppose we fix τ and wish to find q. Let us define an initial condition arbitrarily, as $\Phi(x) = \Psi(x) \equiv 1$, perform the functional transformation, and normalize the resulting functions as $\Phi^0(x) = \Phi(x)/\Phi(x_0)$, $\Psi^0(x) = \Psi(x)/\Phi(x_0)$. (Here x_0 is an arbitrarily chosen coordinate inside the function definition domain.) Then, the new pair of functions is taken as the initial condition and so on. This operation is repeated many times, until the form of the functions $\Phi(x)$ and $\Psi(x)$ stabilizes. Then, the value of $\Phi(x_0)$ before the normalization becomes equal to ν and we get $q(\tau) = \log_W \nu$.

To find τ for a given q the above procedure was supplemented with a simple iteration scheme for numerical solution of the algebraic equation $q(\tau) = q$. Then, it becomes possible to find $D_q = \tau/(q-1)$ at $q \neq 1$. In particular, D_0 is the Hausdorff dimension, and D_2 is the correlation dimension.

To obtain the information dimension D_1 it is necessary to determine the limit at $q \to 1$, that is, at $\tau \to 0$. Formally, from L'Hospital rule, $D_1 = \lim\limits_{q\to 1} \dfrac{\tau(q)}{q-1} = \left(\dfrac{d\tau}{dq}\right)_{q=1} = \left(\dfrac{dq}{d\tau}\right)^{-1}_{\tau=0}$. To compute it without loss of accuracy let us write for $\tau \ll 1$

$$\Phi_k(x) = W^k |\alpha|^{k\tau} [1 + \tau\varphi_k(x)], \quad \Psi_k(x) = W^k |\alpha|^{k\tau} [1 + \tau\psi_k(x)] \tag{15}$$

and substitute this expression into Eq.(9). In the first order in ε we have

$$\begin{aligned} &\psi_{k+1}(x) = \varphi_k(x/\alpha), \\ &\varphi_{k+1}(x) = W^{-1}\left[\varphi_k(x/\alpha) + W^{-1}\psi_k(f(x/\alpha)) + \ln|g'(g(x/\alpha))|\right]. \end{aligned} \tag{16}$$

Numerically, representing $\varphi_k(x)$ and $\psi_k(x)$ by tables of their values at the grid nodes and performing a large number of steps of the transformation one can observe that $\varphi_{k+1}(x) - \varphi_k(x) \underset{k\to\infty}{\to} \theta = \text{const}$, and the same is true for the component ψ. It means that Φ_k and $\Psi_k \propto |\alpha|^{k\tau} W^k e^{k\gamma\tau} = W^{k(q+\tau dq/d\tau)}$. As follows from this relation, the information dimension is

$$D_1 = \left(\frac{dq}{d\tau}\right)^{-1}_{\tau=0} = \frac{\log W}{\log|\alpha| + \theta}. \tag{17}$$

In computations I used polynomial representations of $g(x)$ and $f(x) = g(\alpha x)/\alpha$ containing 32 terms of expansion in powers of x^3, which were valid, as checked, up to 14 decimal digits. Some loss of precision occurs in computation of derivatives via analytical expressions from the finite power expansion for $g(x)$, so the resulting precision is about 12 digits. The data for D_q are presented in the first two columns of Table 1. They are in good correspondence with results from the literature (e.g. [2]), although I have managed to find only graphical (not numerical) data. The only exception is the information dimension, which may be compared with Ref. [19] and coincides with it up to all true digits of that estimate.

As an alternative to the traditional definition of the generalized dimensions D_q one might consider a family of dimensions indexed by τ. Let us designate them as $D^{(\tau)}$: $D^{(\tau)} = D_{q(\tau)} = \tau/[q(\tau) - 1]$. Note that $D^{(-1)} = D_0$ and $D^{(0)} = D_1$. The numerical results are given in the last two columns of Table 1.

As mentioned above, for τ=2 the equation (6) is of the form studied in the theory of noise effect on the critical golden-mean quasiperiodic dynamics [12]. The scaling constant $\gamma = \sqrt{\nu(2)}$ determines a factor by which the noise amplitude must be reduced to reveal one more level of the fractal structure. Hence, the dimension $D^{(2)}$ is linked with the effect of noise. The scaling factor γ is expressed via $D^{(2)}$ as $\gamma = W^{1/D^{(2)}+1/2}$. As follows from the computations,

$$\gamma = 2.3061852653..., \tag{18}$$

which improves significantly the previously known result [12].

The method of calculation of the generalized dimensions developed here is accurate and outlines a link between global and local description of the scaling regularities. Moreover, it seems promising

Table 1. Generalized dimensions for the golden-mean critical quasiperiodic orbit

q	D_q	τ	$D^{(\tau)}$
0	1.000000000000	1	0.858785542316
1	0.921578263514	2	0.808774530475
2	0.866393010548	3	0.770186425804
3	0.824931598145	4	0.741619983463
4	0.792879513565	5	0.720982145614
5	0.767837633226	6	0.706068210093

for further generalizations dealing with a variety of critical behavior scenarios at the onset of chaos or strange non-chaotic behavior.

The author acknowledges support from Max Planck Society and from RFBR (grant No 03-02-16074).

References

[1] *E. B. Vul, Y. G. Sinai, K. M. Khanin.* Feigenbaum universality and thermodynamic formalism. Russ. Math. Surv. 1984. V. 39. №3. P. 1–40. Russian original: Usp. Mat. Nauk. 1984. V. 39. №3. P. 3–37.

[2] *T. S. Halsey, M. H. Jensen, L. P. Kadanoff, I. Procaccia, B. I. Shraiman.* Fractal measures and their singularities: the characterization of strange sets. Phys. Rev. 1986. V. A33. P. 1141–1151.

[3] *D. Bensimon, M. H. Jensen, L. P. Kadanoff.* Renormalization-group analysis of the period-doubling attractor. Phys. Rev. 1986. V. A33. P. 3622–3624.

[4] *Z. Kovács.* Universal $f(\alpha)$ spectrum as an eigenvalue. J. Phys. A: Math. Gen. 1989. V. 22. P. 5161–5165.

[5] *F. Christiansen, P. Cvitanović, H. H. Rugh.* The spectrum of the period-doubling operator in terms of cycles. J. Phys. A: Math. Gen. 1990. V. 23. P. L713–L717.

[6] *A. H. Osbaldestim.* Siegel disk singularity spectra. J. Phys. A: Math. And Gen. 1992. V. 25. P. 1169–1175. *A. D. Burbanks, A. H. Osbaldestin, A. Stirnemann.* Fractal dimension of Siegel disc boundaries. European Phys. J. 1998. V. B4. P. 263–265.

[7] *J. A. Glazier, M. H. Jensen, A. Libchaber, J. Stavans.* Structure of Arnold tongues and the f(α) spectrum for period doubling – Experimental results. Phys. Rev. 1986. V. A34. P. 1621–1624. *Z. Su, R. W. Rollins, E. R. Hunt.* Measurements of f(α) spectrum in driven diode resonator systems. Phys. Rev. 1987. V. A36. P. 3515–3517. *J. A. Glazier, G. Gunaratne, A. Libchaber.* $F(\alpha)$ curves – experimental results. Phys. Rev. 1988. V. A37. P. 523–530. *R. E. Ecke, R. Mainieri, T. S. Sullivan.* Universality in quasi-periodic Rauleigh–Benard convection. Phys. Rev. 1991. V. A44. P. 8103–8118.

[8] *S. J. Shenker.* Scaling behavior in a map of a circle onto itself: Empirical results. Physica. 1982. V. D5. P. 405–411.

[9] *M. J. Feigenbaum, L. P. Kadanoff, S. J. Shenker.* Quasiperiodicity in dissipative systems: A renormalization group analysis. Physica. 1982. V. D5. P. 370–386.

[10] *S. Ostlund, D. Rand, J. Sethna, E. D. Siggia.* Universal properties of the transition from quasi-periodicity to chaos in dissipative systems. Physica. 1983. V. D8. №3. P. 303–342.

[11] *S. P. Kuznetsov, A. H. Osbaldestin.* Generalized dimensions of Feigenbaum's attractor from renormalization-group functional equations. Reg. & Chaot. Dyn. 2002. V. 7. №3. P. 325–330.

[12] *A. Hamm, R. Graham.* Scaling for small random perturbations of golden critical circle maps. Phys. Rev. 1992. V. A46. P. 6323–6333.

[13] *T. W. Dixon, T. Gherghetta, and B. G. Kenny.* Universality in the quasiperiodic route to chaos. Chaos. 1996. V. 6. P. 32–42.

[14] *R. de la Llave, N. P. Petrov.* Regularity of Conjugacies between Critical Circle Maps: An Experimental Study. Experimental Mathematics. 2002. V. 11. №2. P. 219–241.

[15] *O. E. Lanford III.* Renormalization group methods for circle mappings. In: Statistical Mechanics and Field Theory: Mathematical Aspects (Groningen, 1985).

P. 176-189. Lecture Notes in Phys. V. 257. Springer, Berlin. 1986.

[16] *T. W. Dixon, B. G. Kenny.* Transition to criticality in circle maps at the golden mean. J. Math. Phys. 1998. V. 39. №11. P. 5952–5963.

[17] *B. Fourcade, A.-M. S. Tremblay.* Universal multifractal properties of circle maps from the point of view of view of critical phenomena. I. Phenomenology. II. Analytical results. J. Stat. Phys. 1990. V. 61. №3–4. P. 607–665.

[18] *N. Yu. Ivankov, S. P. Kuznetsov.* Complex periodic orbits, renormalization and scaling for quasiperiodic golden-mean transition to chaos. Phys. Rev. V. E63. 2001. №4. P. 146–210.

[19] *S. K. Sarkar.* Information dimension for quasiperiodic trajectories with quadratically irrational winding number. Phys. Lett. V. A106. 1984. №3. P. 95–98.