Available online at www.sciencedirect.com
SCIENCE DIRECT®
Physics Letters A 350 (2006) 110–116
PHYSICS LETTERS A
www.elsevier.com/locate/pla

A new type of period-doubling scaling behavior in two-dimensional area-preserving map

S.P. Kuznetsov *, I.R. Sataev

Institute of Radio-Engineering and Electronics of RAS, Saratov Branch, Zelenaya 38, 410019, Saratov, Russian Federation

Received 21 October 2005; accepted 15 November 2005

Available online 22 November 2005

Communicated by A.B. Bishop

Abstract

A novel type of period-doubling scaling behavior in two-dimensional area-preserving maps is reported, a conservative analog of the critical behavior in period-doubling one-dimensional maps with quartic extremum. We present data of numerical solution of the two-dimensional version of the Feigenbaum–Cvitanović RG equation and accurate estimates for the universal constants. Illustrations are given for self-similarity in the phase space and in the parameter space of the model map.

PACS: 05.45.-a; 05.10.Cc

Keywords: Iterative map; Area-preserving map; Scaling; Bifurcation; Period doubling

Soon after discovery of quantitative universality in the period-doubling onset of chaos and development of the renormalization group (RG) analysis in this context [1–3], two generalizations were suggested in the literature.

One concerns a family of unimodal 1D maps of arbitrary real degree of the extremum, $x_{n+1} = 1 - \lambda|x_n|^r$, $r > 1$ [4–8]. It appears that the period-doubling bifurcation cascade in a course of increase of parameter λ is characterized by scaling constants, which depend on the exponent r: $\delta(r)$ (the convergence rate of the period doubling bifurcations) and $\alpha(r)$ (the orbital scaling factor). The original Feigenbaum results relate to a case of quadratic extremum, with $\delta_F = \delta(2) \approx 4.669$ and $\alpha_F = \alpha(2) \approx -2.503$. Other even integer degrees are of a certain interest too. Say, $r = 4$ corresponds to the so-called tricritical behavior [9,10]. As found, for this case $\delta_T = \delta(4) \approx 7.284$ and $\alpha_T = \alpha(4) \approx -1.690$.

Another generalization relates to a class of two-dimensional area-preserving maps associated with conservative (Hamiltonian) dynamics [11–17]. In this case, the period-doubling cascade is governed by the convergence factor $\delta_H \approx 8.721$, and there are two constants responsible for the phase space (orbital) scaling, $\alpha_H \approx -4.018$ and $\beta_H \approx -16.36$. An example is delivered by the map $x_{n+1} = 1 - \lambda x_n^2 - y_n,\ y_{n+1} = x_n$.

In Ref. [18], discussing a family of Hénon-like maps

$$x_{n+1} = 1 - \lambda|x_n|^r - Dy_n, \qquad y_{n+1} = x_n, \tag{1}$$

the authors argued in favor of dependence of the period-doubling scaling constants on the exponent r in the conservative case $D = 1$. It seems that these results are not correct. In particular, as noted by Briggs et al. [19] (with a reference to an unpublished work of Roberts), in the area-preserving case for this family the scaling constants are common for $r = 2$ and for values distinct from 2. Our computations support this conclusion too.

So, the question remains unsolved yet: either one can find a family of the scaling behaviors for the area-preserving maps analogous to those with different degrees of extremum for the 1D maps?

In this Letter, we report a particular result in this respect and present a new class of period-doubling scaling behavior, an area-preserving analog of the tricritical situation in 1D maps. It can occur generically in three-parameter analysis of the area-

* Corresponding author.
E-mail address: spkuz@rambler.ru (S.P. Kuznetsov).

0375-9601/$ – see front matter
doi:10.1016/j.physleta.2005.11.037

preserving maps at some special critical point HT in the parameter space. We suggest a model map manifesting this type of criticality and present data of numerical solution of the RG equation (the 2D version of the equation of Feigenbaum and Cvitanović). Also, we obtain accurate estimates for the associated universal constants, and illustrate self-similar formations in the phase space and in the parameter space of the model map at the critical point.

Let us consider the following Hénon-like 2D map:

$$x_{n+1} = f(x_n) - Dy_n, \qquad y_{n+1} = x_n, \tag{2}$$

where

$$f(x) = 1 + Ax + Bx^2 + Cx^4. \tag{3}$$

This model contains four parameters A, B, C, D. At $D = 0$ the dynamics is of one-dimensional nature; it is governed by the map $x_{n+1} = f(x_n)$. At $A = 0$ and $B = 0$ the function possesses a quartic extremum, and the limit of the period-doubling cascade observed with increase of C is a tricritical point: $C_T = 1.594901356\ldots$. Starting at this point, we can gradually increase D and tune simultaneously A, B, and C to preserve the tricriticality. In computations this procedure was realized in an earlier work of one of the authors [20]. As noted there, in the course of increase of D it is necessary to control exactly three other parameters to compensate three relevant unstable modes arising due to a perturbation of the fixed point of the RG equation associated with the tricriticality. The condition to be fulfilled consists in selection of these three parameters to keep constant the main multipliers (the larger of two Floquet eigenvalues) for three cycles of sufficiently long periods 2^k, 2^{k+1}, 2^{k+2}. (Asymptotically, these multipliers should be equal to a universal number associated with the tricriticality, $\mu_T = -2.0509404$.) Fig. 1 shows a graphical representation of the results of the computations, namely, the tricritical curve in the parameter space (A, B, C). The limit of this curve at $D \to 1$, where the map becomes area preserving, yields a new critical point, which is the main matter of our interest here. We designate it as HT-point (that stands for "Hamiltonian + Tricritical").

For D close to one, the accurate computations become difficult. Indeed, the regularities intrinsic to the tricriticality are valid only in the strong-dissipation limit. For D approaching unity, it becomes necessary to deal with orbits of larger and larger period 2^k with $k \to \infty$ to get notable dissipation over the period. Nevertheless, performing the procedure with subsequently increasing k we could approach sufficiently close to $D = 1$ and were able to estimate parameters corresponding to the HT-point. Our best result (with improvements with a use of universal constant for Green's residue, as explained later) is the following:

$$A_{\mathrm{HT}} = -0.1956089759639,$$
$$B_{\mathrm{HT}} = -0.064248541437467,$$
$$C_{\mathrm{HT}} = -17.953222255456, \qquad D_{\mathrm{HT}} = 1. \tag{4}$$

To analyze behavior at the critical point HT on a basis of the RG analysis it is convenient to rewrite the map in the equivalent

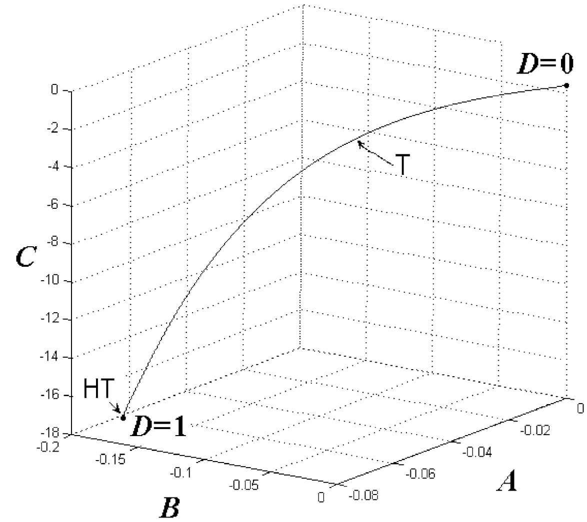

Fig. 1. The tricritical curve of the map (2) in the parameter space (A, B, C) obtained in computations for D varied from 0 to 1. The limit point at $D \to 1$ corresponds to the new type of criticality designated as HT.

form (by analogy with [11,12,14–16])

$$x_{n+1} = Y_n + \frac{1}{2} f(x_n), \qquad Y_{n+1} = -x_n + \frac{1}{2} f(x_{n+1}), \tag{5}$$

where $Y = \frac{1}{2} f(x) - y$. Next, it is convenient to shift origin in x and set $X = x - x_c$, where x_c is the "scaling center", the limit point for a sequence of elements of the period-2^k cycles at the critical point. For our case, as computed, $x_c = 0.00032689$. The new variables (X, Y) represent a canonically conjugate pair and in respect to the formulation of the RG equation have an advantage that the scaling transformation in these variables will be diagonal.

Let us assume that the evolution operator of dynamics at the HT point over 2^k units of discrete time is defined by a pair of functions $\{g_k(X, Y), f_k(X, Y)\}$ and normalized in such way that $g(0, 0) = 1$, $f(0, 0) = 1$. By two-fold application of this operator and after variable change $X \to X/\alpha_k$, $Y \to Y/\beta_k$, where $\alpha_k = 1/g_k(1, 1)$ and $\beta_k = 1/f_k(1, 1)$, we get the renormalized evolution operator for 2^{k+1} units of time:

$$g_{k+1}(X, Y) = \alpha_k g_k\big(g_k(X/\alpha_k, Y/\beta_k), f_k(X/\alpha_k, Y/\beta_k)\big),$$
$$f_{k+1}(X, Y) = \beta_k f_k\big(g_k(X/\alpha_k, Y/\beta_k), f_k(X/\alpha_k, Y/\beta_k)\big). \tag{6}$$

One can apply this RG transformation repeatedly to obtain a sequence of the evolution operators for larger and larger time scales.

Numerically, the functions $\{g_k(X, Y), f_k(X, Y)\}$ may be obtained from 2^k-fold iteration of the map (5) at the critical point (4) by means of appropriate rescaling of the dynamical variables. These computations indicate that the sequence of the functional pairs $\{g_k(X, Y), f_k(X, Y)\}$ tend to a definite limit, which corresponds to a fixed point of the set of functional equations:

$$g(X, Y) = \alpha g\big(g(X/\alpha, Y/\beta), f(X/\alpha, Y/\beta)\big),$$
$$f(X, Y) = \beta f\big(g(X/\alpha, Y/\beta), f(X/\alpha, Y/\beta)\big). \tag{7}$$

Here $\alpha = \lim \alpha_k = 1/g(1,1)$, $\beta = \lim \beta_k = 1/f(1,1)$, $g(0,0) = f(0,0) = 1$. (The same relations were derived by several authors in application to the usual conservative period-doubling criticality [11–16] and to some other situations [17], but here we deal with a distinct solution of the equations.)

The existence of the fixed point of the RG transformation means that the rescaled long-time evolution operators at the criticality will be of a universal form, up to a characteristic scale. The renormalized evolution operator $\{g(X,Y), f(X,Y)\}$ can be recovered (say, numerically) from the functional fixed-point equations (7). In other words, without any reference to a concrete system under examination, it is determined entirely by structure of the RG scheme. Therefore, like in other cases of applicability of the RG method, a fixed-point solution of RG equation gives rise to a universality class, which may include systems of very different mathematical nature (e.g., iterative maps, ordinary differential equations, etc.).

For the numerical solution of the functional equations (7), we represent the functions $\{g(X,Y), f(X,Y)\}$ by finite polynomial expansions inside a rectangle $(-1.1 \leqslant X \leqslant 1.1, -2.4 \leqslant Y \leqslant 2.4)$, and organize the procedure of RG transformation as a computer program operating with coefficients of the polynomials. In this way, the equations (7) reduce to a finite set of nonlinear algebraic equations for the coefficients, which may be solved by the multidimensional Newton method. (As the initial approximation, we use functions obtained from direct iterations of the map (5) at the HT point.) The results of the computations are summarized in Table 1. (The accuracy of representation of the universal functions with the data of the Table 1 is about six decimal digits, but we have obtained higher precision data as well.)

In the course of the procedure, we get also accurate estimates for the rescaling constants

$$\alpha = -2.059370935, \qquad \beta = 17.991734478. \tag{8}$$

From the polynomial representation of the functions, one can find out that the map $(X,Y) \mapsto (g(X,Y), f(X,Y))$ has a fixed point, namely, $X_* = 0.680386\ldots$, $Y_* = 0.000100\ldots$. As the functional pair $\{g(X,Y), f(X,Y)\}$ corresponds to the rescaled evolution operator at the HT point for an asymptotically large number of iterations 2^k, we conclude that periodic orbits of arbitrarily large periods 2^k coexist at this point. All of them are unstable, and asymptotical values of the multipliers (Floquet eigenvalues) may be estimated as eigenvalues of the Jacobian matrix $\partial(g(X,Y))/\partial(X,Y)$ at the point (X_*, Y_*). From the numerical data we get

$$\mu_1 = -2.795180794, \qquad \mu_2 = \frac{1}{\mu_1} = -0.357758265. \tag{9}$$

Also, we obtain a universal number for Green's residue (see [11]), a characteristic value for the long-period orbits at the HT point

$$R_{\mathrm{HT}} = \frac{2 - \mu_1 - \mu_2}{4} = 1.288234851. \tag{10}$$

We used the constant R_{HT} to improve effectively the estimates for coordinates of the HT critical point in the original map (5) (or (2)). Starting with a crude estimate, we compute three first period-2^k cycles of the map, which have residues close to R_{HT}. By tuning the control parameters (A, B, C) we try to reach equality of the residues to the universal number (10). Then, we increase k and repeat the tuning for the triplet of longer-period cycles. For k about 10 we get a very accurate estimate, see (4).

From the results of the RG analysis, we can derive scaling properties of the dynamics at the critical point HT. Namely, let us suppose that we consider an orbit started near the "scaling center" $X = 0$, $Y = 0$ at some point (X_0, Y_0). (We remind the link of the redefined variables with the original ones: $X = x - x_c$, $Y = \frac{1}{2} f(x) - y$.) Then, for the trajectory launched at $(X_0/\alpha, Y_0/\beta)$ we shall observe the similar behavior but with doubled characteristic time scale. Formally speaking, this property is asymptotic: smaller the vicinity of origin, larger the characteristic time scale, and better the precision the scaling holds.

Because of conservative nature of the dynamics, no attractors exist at the critical point. However, other phase space objects do obey the scaling property. In particular, it relates to a

Table 1
Coefficients of polynomial expansions for the functions representing the fixed-point of the RG transformation (7)

g	1	Y	Y^2	Y^3	Y^4
1	1.000000	−0.250023	0.003653	0.000017	−0.000001
X	−0.096847	−0.025497	−0.000260	0.000012	0
X^2	−0.010754	0.001956	−0.000055	−0.000004	0
X^3	−0.001888	0.000045	0.000001	−0.000002	0
X^4	−1.099936	0.032221	0.000246	−0.000016	0
X^5	−0.112234	−0.002378	0.000167	0	0
X^6	0.008549	−0.000870	−0.000081	0	0
X^7	0.000340	0.000233	−0.000021	0	0
X^8	0.071147	0.001866	−0.000081	0	0
X^9	−0.005514	0.000529	0	0	0
X^{10}	−0.002189	−0.000999	0	0	0
X^{11}	0.000666	0	0	0	0
X^{12}	0.003078	0	0	0	0
X^{13}	0.000703	0	0	0	0
X^{14}	−0.001362	0	0	0	0

Table 1 (*continued*)

f	1	Y	Y^2	Y^3	Y^4	Y^5	Y^6
1	1.000000	−4.601519	1.714849	−0.323223	0.029363	−0.001105	0.000015
X	2.217240	0.924041	−0.003203	−0.049555	0.007197	−0.000169	0
X^2	−0.132037	0.176217	−0.093645	0.011220	−0.000185	0.000025	0
X^3	0.029112	−0.005566	−0.007935	0.002499	−0.000122	0.000021	0
X^4	−20.237814	15.082968	−4.265691	0.516918	−0.023772	0.000324	0
X^5	4.065733	−0.028476	−0.652280	0.127062	−0.003710	−0.000214	0
X^6	0.777049	−0.821283	0.152199	−0.003733	−0.000939	0.000085	0
X^7	−0.037778	−0.072452	0.020591	−0.003307	0.000374	0.000169	0
X^8	33.139997	−18.779353	3.399545	−0.207428	0.005296	0	0
X^9	0.031313	−2.848674	0.893099	−0.030208	−0.002411	0	0
X^{10}	−1.638544	0.729382	−0.000334	−0.006680	0	0	0
X^{11}	−0.544947	0.027758	−0.159716	−0.000381	0.001516	0	0
X^{12}	−28.172982	9.830790	−0.938270	0.030969	0	0	0
X^{13}	−3.229990	2.781572	0.081899	−0.012033	0	0	0
X^{14}	2.595528	0.189221	−0.003871	0	0	0	0
X^{15}	−1.303847	−0.624572	−0.197516	0.007071	0	0	0
X^{16}	8.370378	−2.231363	0.096046	0	0	0	0
X^{17}	4.081385	0.354591	0.058735	0	0	0	0
X^{18}	2.725813	0.081417	0	0	0	0	0
X^{19}	−0.688120	−0.462671	0	0	0	0	0
X^{20}	−3.647391	0.149413	0	0	0	0	0
X^{21}	−0.192762	0.117978	0	0	0	0	0
X^{22}	0.713379	0	0	0	0	0	0

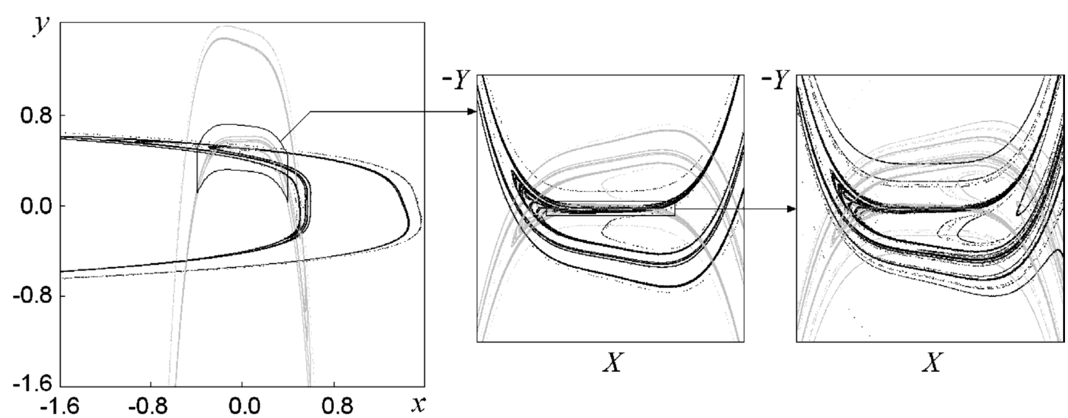

Fig. 2. Illustration of the scaling property for the stable and unstable manifolds of the trajectory starting at the "scaling center" $x = x_c$, $y = \frac{1}{2} f(x_c)$ at the HT critical point. The unstable manifold is shown in black, and stable one in gray. A small curvilinear quadrangle formed by coordinate curves $X = \pm 0.4$ and $Y = \pm 0.2$ is shown separately in coordinates (X, Y). The second inset presents the interior of the depicted rectangular under magnification by factors $\alpha = -2.0593\ldots$, $\beta = 17.9917\ldots$ along the horizontal and vertical axes, respectively.

family of unstable periodic orbits of period 2^k mentioned above and having asymptotically the universal multipliers (9). Fig. 2 illustrates the scaling property for the stable and unstable manifolds of the trajectory starting at the "scaling center" $X = 0$, $Y = 0$, i.e., $x = x_c$, $y = \frac{1}{2} f(x_c)$. The unstable manifold shown in black is generated from iterations of an ensemble of random initial conditions very close to the origin (distances less than a pixel of the graphical presentation). After sufficiently large number of iterations, the depicted points take up positions along the unstable manifold. The stable manifold is generated in the same way, but from iterations in backward time. In the main panel, the manifolds are plotted in coordinates (x, y). A selected curvilinear quadrangle is shown separately in coordinates (X, Y). On a plane of these variables magnification by factors α and β along the horizontal and vertical axes, respectively, reveals the similar structure of the manifolds as seen from comparison of the first and the second inset.

The next step in the RG analysis consists in consideration of small perturbations of the fixed-point solution of the functional equations:

$$g(X,Y) \mapsto g(X,Y) + \varepsilon u(X,Y),$$
$$f(X,Y) \mapsto f(X,Y) + \varepsilon v(X,Y), \quad \varepsilon \ll 1. \tag{11}$$

On this stage it is convenient to use slightly reformulated RG transformation: now we regard the scaling factors α and β as constants equal to the universal numbers (8), not depending on the level of the construction. Then, by linearization of the RG transformation at the fixed point $\{g(X,Y), f(X,Y)\}$ we come to the eigenvalue problem

$$\begin{aligned}
\delta u(X,Y) = \alpha\big[&g_1'\big(g(X/\alpha,Y/\beta), f(X/\alpha,Y/\beta)\big)u(X/\alpha,Y/\beta)\\
&+ g_2'\big(g(X/\alpha,Y/\beta), f(X/\alpha,Y/\beta)\big)v(X/\alpha,Y/\beta)\\
&+ u\big(g(X/\alpha,Y/\beta), f(X/\alpha,Y/\beta)\big)\big],\\
\delta v(X,Y) = \beta\big[&f_1'\big(g(X/\alpha,Y/\beta), f(X/\alpha,Y/\beta)\big)u(X/\alpha,Y/\beta)\\
&+ f_2'\big(g(X/\alpha,Y/\beta), f(X/\alpha,Y/\beta)\big)v(X/\alpha,Y/\beta)\\
&+ v\big(g(X/\alpha,Y/\beta), f(X/\alpha,Y/\beta)\big)\big]. \qquad (12)
\end{aligned}$$

Here δ is an eigenvalue, and the indices 1 and 2 designate derivatives of the functions in respect to the first and the second argument.

Among the eigenmodes we have to collect the relevant ones, which have $|\nu| > 1$ and response for the asymptotic behavior of the solution under subsequent repetition of the RG transformation. It is essential to exclude modes, which are associated with infinitesimal variable changes (e.g., shifts along X and Y and rotation of the coordinate system, cf. Refs. [1–3,9–17]). Then, from the numerical solution based on representation of the functions by finite polynomial expansions, we extract four relevant eigenvalues

$$\begin{aligned}
&\delta_1 = 14.2808753, \qquad \delta_2 = -8.5311613,\\
&\delta_3 = 4.2018184, \qquad \delta_4 = 2. \qquad (13)
\end{aligned}$$

As checked, the first three of them are associated with perturbations, which do not violate the area-preserving nature of the map. The last one, δ_4 corresponds to implementing infinitesimal dissipation. Note that $\delta_4 = 2$ precisely (cf. [21–26]). (Indeed, if we have a perturbation corresponding to a slight deflection of the Jacobian determinant from 1, then, for the doubled number of iterations it becomes twice larger, as follows from the trivial relation for the product $(1-\varepsilon)(1-\varepsilon) \approx 1 - 2\varepsilon$.)

Let us consider a vicinity of the HT point in the parameter space and state the intrinsic scaling regularities assuming the conservative nature of the dynamics. In this case we set $D = 1$ and deal with three-dimensional parameter space (A, B, C).

To formulate the scaling properties, we have to define a special local coordinate system in the parameter space ("scaling coordinates"). It is natural to take the critical point HT as the origin. The coordinate axes must be directed in such way that a shift from the critical point along each axis has to produce a perturbation associated with one certain relevant eigenvalue δ_1, δ_2, or δ_3 of the linearized RG equation.

The numerical method of determining the scaling coordinates consists in the following. For the area-preserving case $D = 1$ we rewrite the map (2) as

$$\begin{aligned}
x_{n+1} &= 1 + A_{\mathrm{HT}}x_n + B_{\mathrm{HT}}x_n^2 + C_{\mathrm{HT}}x_n^4\\
&\quad + \varepsilon\big(px_n + qx_n^2 + sx_n^4\big) - y_n,\\
y_{n+1} &= x_n. \qquad (14)
\end{aligned}$$

Then, we compose a program that finds out numerically the unstable orbits of period 2^k and computes residues for these orbits R_k and derivatives of R_k with respect to ε at $\varepsilon = 0$. Under arbitrary generic selection of the vector of additional parameters $\mathbf{u} = (p, q, s)$, say, $\mathbf{u}_1 = (0, 1, 0)$, we observe that the sequence of the derivatives follows asymptotically the power law, $\partial R_k/\partial\varepsilon \propto \delta_1^k$. Fixing two of the parameters, with accurate appropriate selection of the third one, we are able to force the sequence to behave as δ_2^k. In particular, it takes place at $\mathbf{u}_2 = (1, -0.9076928, 0)$ and $\mathbf{u}_2' = (0, -0.19517581, 1)$. Next, taking a linear combination of the last two vectors, $\mathbf{u}_3 = \kappa\mathbf{u}_2 + \mathbf{u}_2'$ we select numerically the coefficient $\kappa = -0.0042178$ to get the sequence growing as $\partial R_k/\partial\varepsilon \propto \delta_3^k$. Finally, we use the vectors $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$ as the basis for the new local coordinate system and set

$$(A - A_{\mathrm{HT}}, B - B_{\mathrm{HT}}, C - C_{\mathrm{HT}}) = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3, \qquad (15)$$

or

$$\begin{aligned}
A &= A_{\mathrm{HT}} + c_2 - 0.0042178c_3,\\
B &= B_{\mathrm{HT}} + c_1 - 0.9076928c_2 - 0.1913459c_3,\\
C &= C_{\mathrm{HT}} + c_3. \qquad (16)
\end{aligned}$$

Now, let us suppose that some certain dynamical behavior occurs at a point in the parameter space close to the critical point HT, with a small parameter shift characterized by certain values of c_1, c_2, c_3, for an orbit started at a point (X_0, Y_0) near the origin. (We remind the link of the redefined variables with the original ones: $X = x - x_c$, $Y = \frac{1}{2}f(x) - y$.) Then, at the parameters corresponding to c_1/δ_1, c_2/δ_2, c_3/δ_3 we will observe a similar dynamics for the trajectory starting at $(X_0/\alpha, Y_0/\beta)$, but with doubled characteristic time scale. Hence, $\delta_1, \delta_2, \delta_3$ play a role of scaling factors in the parameter space of the area-preserving map: under magnification with these factors along three respective coordinate axes one will observe repetition of the parameter space structure in smaller and smaller vicinities of the critical point. Moreover, in scaling coordinates, the local topography of the parameter space must be regarded as an attribute of the given type of criticality, universal in this sense.

In Fig. 3 we show charts for two cross-sections of the parameter space by coordinate surfaces of the scaling coordinate system, namely, $c_3 = 0$ (a) and $c_2 = 0$ (b). They depict in gray tones the regions of stability for cycles of period 2^k participating in the period doubling associated with appearance of the critical point. In the first diagram in a row, we select a small rectangular in a vicinity of the critical point and show it separately in the second diagram with magnification. Then, again we select a small rectangular and show it in the third diagram. Factors of enlargement are given by the pairs of eigenvalues, (δ_1, δ_2) for the panels (a) and (δ_1, δ_2) for the panel (b). Observe approximate similarity of the structures. (As expected, it will be better for further subsequent levels of the magnification.)

Finally, let us discuss shortly a role of dissipation. If we allow the parameter D to take values slightly less than 1, one more relevant eigenvalue $\delta_4 = 2$ comes into play. Now, the HT criticality must be regarded as a phenomenon of codimension 4. If we move a little bit in the parameter space from the HT point along the tricritical curve (see Fig. 1), then the crossover from HT to tricritical behavior occurs. Namely, at the first levels of period doubling we will see yet the scaling regularities intrinsic to HT universality class, but for high levels the tricritical regularities will hold. (Analogous passage from period-doubling

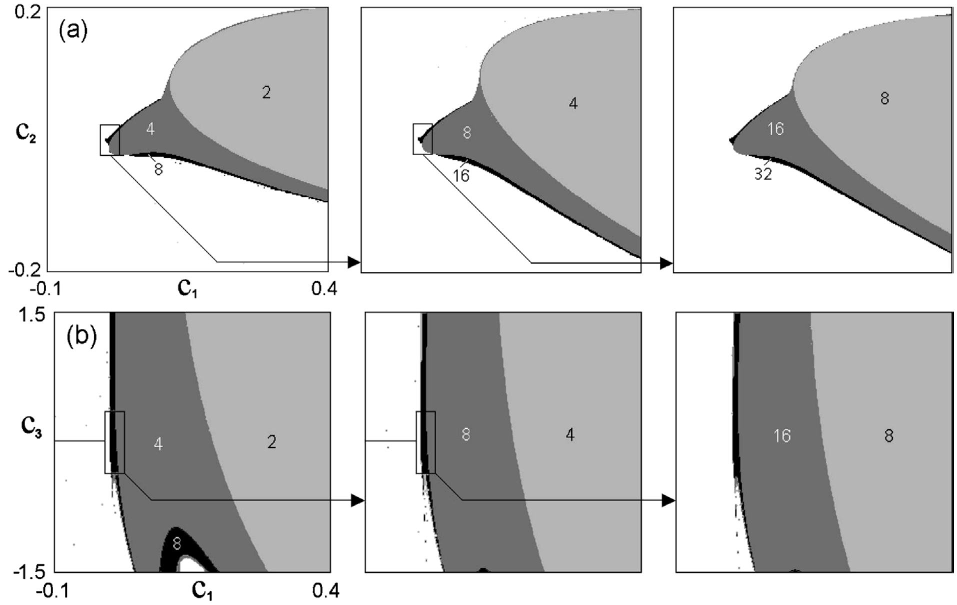

Fig. 3. Charts for two cross-sections of the parameter space by coordinate surfaces of the scaling coordinate system, (c_1, c_2) (a) and (c_1, c_3) (b). The regions of stability for periodic orbits participating in the period doubling associated with appearance of the critical point are shown by gray tones. Periods of the cycles are indicated by numbers. Magnification factors for each subsequent picture in a row are $\delta_1 = 14.2808\ldots$ and $\delta_2 = -8.5311\ldots$ for horizontal and vertical axes, respectively, for the panel (a), and $\delta_1 = 14.2808\ldots$ and $\delta_3 = 4.2018\ldots$ for the panel (b).

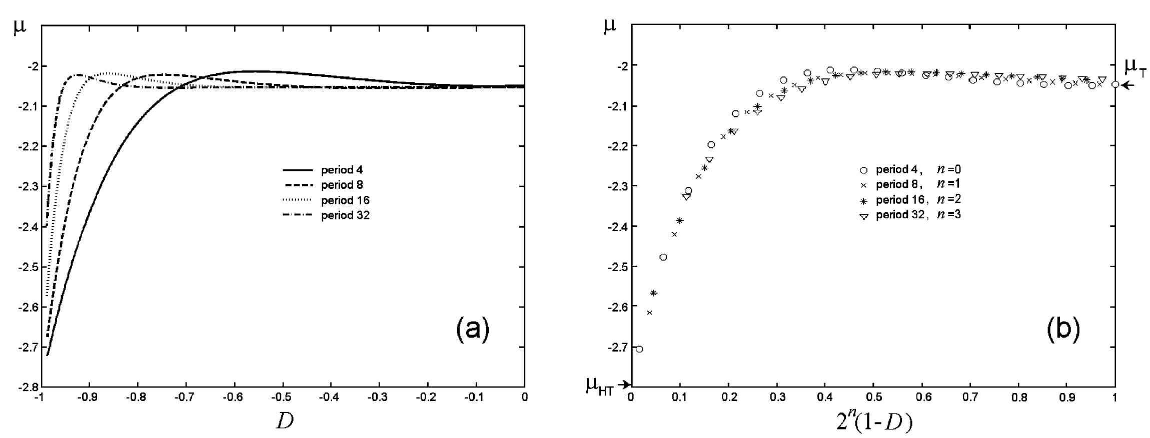

Fig. 4. Plots illustrating crossover between HT and tricritical behavior and scaling properties near the HT point associated with dissipation. The larger multipliers are shown for periodic orbits of periods 4, 8, 16, 32 versus parameter D along the tricritical curve (a) and the same data with rescaling along the abscissa axis (b).

criticality of usual type to the Feigenbaum regularities corresponding to 1D maps was widely discussed in the literature, see, e.g., [21–26].) This crossover phenomenon is illustrated in Fig. 4.

The left diagram shows the larger multipliers of periodic orbits of periods 4, 8, 16, 32 versus parameter D along the tricritical curve of Fig. 1. The right diagram presents the same data with rescaling along the horizontal axis. The observation that points designated by different symbols and relating to the cycles of all periods approximately fit a unique common curve illustrates the scaling property near the HT point associated with dissipation: indeed, to observe similar dynamics on a doubled time scale one has to make twice less the deflection of parameter D from 1. Observe that on the left end of the diagram all the multipliers become approximately equal $-2.795\ldots$, which is the universal value for the HT critical point, and in the right-hand part they tend to $-2.050\ldots$, which is the universal value for the tricriticality.

To conclude, we have reported on a novel type of critical behavior associated with period-doubling bifurcation cascade in a model area-preserving map. We have presented data of numerical solution of the RG equation and obtained accurate estimates for the associated universal constants. Note that all the scaling constants are distinct from those known in literature for the period doubling in conservative systems [11–16]. This new type of critical behavior has to be regarded as a conservative analog of the tricritical behavior observed in period-doubling for 1D maps with quartic extremum. It follows both from the procedure used to find out the new critical point and from crossover properties observed in presence of weak dissipation. In accordance with usual argumentation associated with RG approach and concepts of universality and scaling, our new type of critical behavior may appear generically in the three parameter analysis of period-doubling dynamics in various conservative systems reducible (with the Poincare section technique) to two-dimensional area-preserving maps.

From theoretical point of view, our results fill up a gap in the existing picture of the period-doubling phenomena: Up to now, only one type of the universal period-doubling behavior in the conservative area-preserving maps was known in contrast to a family of the universality classes for 1D map depending on the degree of the extremum.

Acknowledgement

This research was supported by RFBR grant No. 03-02-16074.

References

[1] M.J. Feigenbaum, J. Stat. Phys. 19 (1978) 25.
[2] M.J. Feigenbaum, J. Stat. Phys. 21 (1979) 669.
[3] B. Derrida, A. Gervois, Y. Pomeau, J. Phys. A: Math. Gen. 12 (1979) 269.
[4] B. Hu, J.M. Mao, Phys. Rev. A 25 (1982) 3259.
[5] B. Hu, I.I. Satija, Phys. Lett. A 98 (1983) 143.
[6] P.R. Hauser, C. Tsallis, M.F. Curado, Phys. Rev. A 30 (1984) 2074.
[7] J.-P. Eckmann, H. Epstein, Commun. Math. Phys. 128 (1990) 427.
[8] K.M. Briggs, T.W. Dixon, G. Szekeres, Int. J. Bifur. Chaos 8 (1998) 347.
[9] S.J. Chang, M. Wortis, J.A. Wright, Phys. Rev. A 24 (1981) 2669.
[10] S. Fraser, R. Kapral, Phys. Rev. A 30 (1984) 1017.
[11] J.M. Greene, R.S. MacKay, F. Vivaldi, M.J. Feigenbaum, Physica D 3 (1981) 468.
[12] R.H.G. Helleman, in: E.G.D. Cohen (Ed.), Fundamental Problems in Statistical Mechanics, vol. 5, North-Holland, Amsterdam, 1981, p. 165.
[13] P. Collet, J.-P. Eckmann, H. Koch, Physica D 3 (1981) 457.
[14] M. Widom, L.P. Kadanoff, Physica D 5 (1982) 287.
[15] J.-P. Eckmann, H. Koch, P. Wittwer, Phys. Rev. A 26 (1982) 720.
[16] R.S. MacKay, in: W. Horton, L.E. Reichl, V. Szebehely (Eds.), Long Time Prediction in Dynamics, J. Wiley and Sons, New York, 1983, p. 127.
[17] A.P. Kuznetsov, S.P. Kuznetsov, I.R. Sataev, Physica D 109 (1997) 91.
[18] P.R. Hauser, E.M.F. Curado, C. Tsallis, Phys. Lett. A 108 (1985) 308.
[19] K.M. Briggs, G.R.W. Quispel, C.J. Thompson, J. Phys. A: Math. Gen. 24 (1991) 3363.
[20] S.P. Kuznetsov, Phys. Lett. A 169 (1992) 438.
[21] C. Chen, G. Gyorgyi, G. Schmidt, Phys. Rev. A 36 (1987) 5502.
[22] C. Chen, G. Gyorgyi, G. Schmidt, Phys. Rev. A 34 (1986) 2568.
[23] C. Chen, G. Gyorgyi, G. Schmidt, Phys. Rev. A 35 (1987) 2660.
[24] K. Flensberg, H. Svensmark, Phys. Rev. E 47 (1993) 2190.
[25] C. Reick, Phys. Rev. A 45 (1992) 777.
[26] G. Reinout, W. Quispel, Phys. Rev. A 31 (1985) 3924.