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SHORT COMMUNICATIONS

Fourier Spectrum of the Signal Generated at the Point of Period Tripling Bifurcation Accumulation

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Abstract—The object of consideration is universal rules observed in the Fourier spectrum of the signal generated at the point where bifurcations tripling the complex analytical map period are accumulated. The difference in the intensities of subharmonics at frequencies corresponding to neighboring levels in the hierarchical levels in the spectrum is characterized by constant $\gamma = 21.9$ dB, which is an analogue of constant $\gamma_F = 13.4$ dB for the Feigenbaum critical point. Physical observations of the spectrum at the point of period tripling bifurcation accumulation are presented.

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A simple object with nontrivial dynamics is known to be a discrete-time system specified by a quadratic map. In particular, this system demonstrates transition to chaos through a cascade of period doubling bifurcations, which is characterized by the Feigenbaum universal properties [1]. When generalized for the complex variable, quadratic map,

$$z_{n+1} = \lambda - z_n^2, \quad \lambda, z \in C \tag{1}$$

forms a fractal pattern on the parameter plane known as the Mandelbrot set (Fig. 1) [2]. This set comprises leaves where periodic or chaotic dynamics is restricted (iterations do not go to infinity). It is known [3, 4] that, when mapping paths on the complex parameter plane cross the leaves of the Mandelbrot set in an appropriate way, other cascades of period multiplying bifurcations (e.g., period tripling bifurcations) may be observed. In [4], Gol'berg, Sinaĭ, and Khanin introduced into consideration the critical point of period tripling bifurcation accumulation (GSKh point),

$$\lambda_c = 0.0236411685377 + 0.7836606508052i. \tag{2}$$

Separating out the real and imaginary parts of a 1D complex map, one will be able of equivalently describing the dynamics by 2D mapping onto a set of pairs of real numbers. Note that such a map is rather specific, since the respective functions must meet the Cauchy–Riemann conditions (conditions of analyticity). It was shown [5, 6] that even a small analytical perturbation of the map radically changes the dynamics, specifically, breaks the universal rules typical of period-tripling and other period-multiplying cascades. Therefore, of basic significance would be the possibility of observing the complicated analytical dynamics in real physical systems (see, e.g., [7]). In [8], the author considered the Mandelbrot set arising on the parameter plane of coupled systems, with the coupling providing a certain

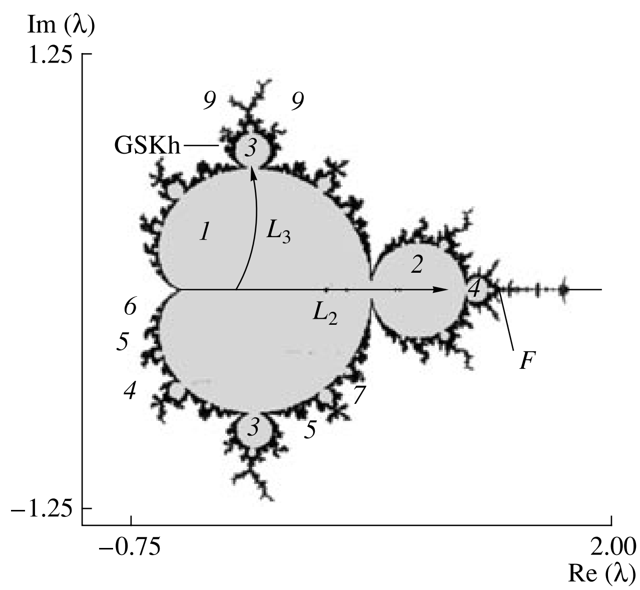

Fig. 1. Mandelbrot set for complex map (1). The regions with periodic dynamics are colored gray (the figures are the periods of the respective cycles), the black fractal fringe shows chaotic dynamics restricted in the phase space, and white regions correspond to orbits running away to infinity. Lines L_2 and L_3 are paths toward period doubling and period tripling, respectively. The arrows indicate the bifurcation accumulation points: Feigenbaum point F and GSKh point.

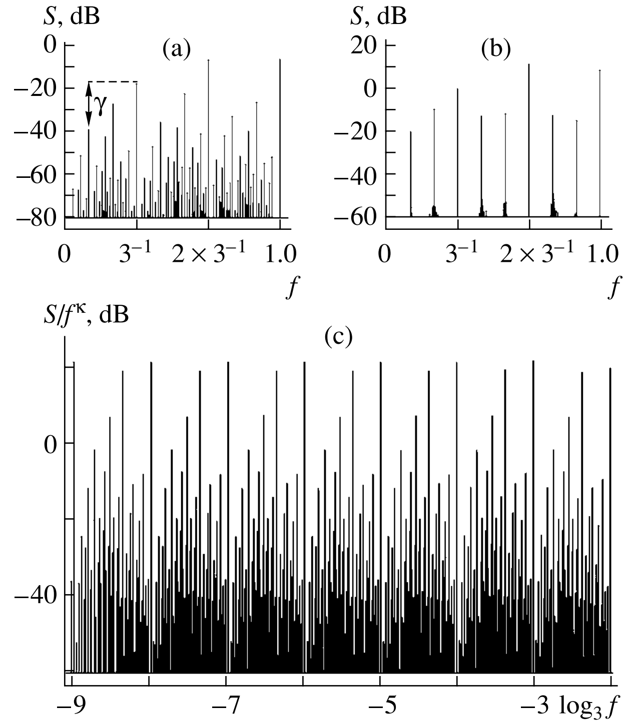

Fig. 2. Fourier spectrum for the (a) signal generated by complex map (1) at the critical point, (b) experimental signal corresponding to the cycle of period 9, and (c) signal at the critical point plotted in the log–log scale.

symmetry that is necessary for the conditions of analyticity to be met in the system. Based on this idea, an electronic analog device was suggested [9] that simulates the dynamics of coupled logistic maps. This device has made it possible to observe the Mandelbrot set in a physical experiment for the first time. In addition, using this device, one can observe bifurcations typical of complex maps, specifically, a period tripling cascade.

Dynamic regimes observed in various systems are convenient to analyze using the spectral characteristics of the signal generated. With this in mind, let us consider the Fourier spectrum of the signal generated by the quadratic complex mapping at the GSKh critical point of period tripling bifurcation accumulation. As follows from the renormalization group analysis [3, 4], an infinite number of unstable cycles with periods $N = 3^k$, where $k = 1, 2, 3, \ldots$, are bound to originate at the GSKh point, which, as N tends toward infinity, more and more accurately approximate the critical attractor. The direct and inverse discrete Fourier transformations are defined as

$$z_n = \sum_{m=0}^{N-1} c_m \exp\left(\frac{2\pi i}{N} mn\right),$$
$$c_m = \frac{1}{N}\sum_{m=0}^{N-1} z_n \exp\left(-\frac{2\pi i}{N} mn\right), \tag{3}$$

where z_n is the sequence of values of the dynamic variable at the GSKh point that starts from the map extremum $z_0 = 0$. Ratio $m/N = f$ is the frequency of component c_m. Below, the mth component amplitude squared is designated as $S(f) = S(m/N) = |c_m|^2$.

Figure 2a shows the spectrum of the signal generated at the GSKh point. It is represented as the intensity of harmonics in decibels (10log$S$) versus frequency f. The spectrum is seen to be of fractal character: it peaks at frequencies corresponding to the periods of the unstable cycles appearing at the GSKh point. In the regions between the peaks, the structure of the spectrum is the same. The difference between the intensities of the harmonics at frequencies $1/3^k$ and $1/3^{k+1}$ is $\gamma = 21.9$ dB on average. This constant is an analogue of the constant $\gamma_F = 13.4$ dB known for the Feigenbaum point [1].

In Fig. 2c, the same spectrum of map (1) is shown in the log–log scale. Here, the spectral intensities are normalized to the respective frequencies to the power $\kappa = 6.15$ (this exponent was found empirically). Such a representation highlights the self-similarity of the spectrum. Note that the structure of the spectrum (the heights of the peaks and their mutual arrangement) is periodic.

The values of constants γ and κ are universal, being the attributes of the GSKh critical point. With constant κ, the self-similar hierarchical structure of the spectrum at the GSKh point was demonstrated for the complexified Hénon map [10].

In addition, the universal attributes of the spectrum at the GSKh point were observed experimentally. Using the device described in [9], we managed to observe several initial period tripling bifurcations in a real physical experiment (the GSKh point itself is hard to reach because of experimental noise and errors). Figure 2b shows the experimental spectrum taken at control parameters corresponding to the cycle of period 9 (subsequent period triplings are hardly discernible under the experimental conditions). High peaks at the respective frequencies are seen. The difference between the intensities of harmonics at tripled frequencies in accordance with constant $\gamma = 21.9$ dB is thus confirmed.

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