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RENORMALIZATION GROUP APPROACH TO THE EFFECT OF NOISE AT THE BIRTH OF A STRANGE NONCHAOTIC ATTRACTOR

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Scaling regularities associated with additive noise are examined in a model of the pitch-fork bifurcation map with multiplicative quasiperiodic driving (Grebogi et al., Physica **D13**, 261) with the golden-mean frequency ratio at the birth of a strange nonchaotic attractor (SNA). This case of the onset of SNA termed as the blowout bifurcation route was discussed in the context of realistic systems governed by non-autonomous differential equations (Yalçynkaya and Lai, Phys. Rev. Lett., **77**, 5039). Our method taking into the account of noise is based on renormalization group (RG) analysis of the birth of SNA (Kuznetsov et al., Phys. Rev. **E51**, 1629) with application of an appropriate generalization of the approach of Crutchfield et al. (Phys. Rev. Lett., **46**, 933) and Shraiman et al. (Phys. Rev. Lett., **46**, 935) originally developed for the period doubling transition to chaos. A constant γ=7.4246 is evaluated that determines the scaling law regarding the intensity of noise: A decrease of the noise amplitude by this factor allows resolving one more level of the fractal-like structure associated with the characteristic time scale which is increased by a factor of $[(\sqrt{5}+1)/2]^3$. Numeric results demonstrating evidence of the expected regularities are presented, e.g. portraits of the noisy attractors in different scales.

Keywords: Noise; strange non-chaotic attractor; renormalization group.

The renormalization group (RG) approach in nonlinear dynamics was introduced by Feigenbaum [1] and then applied for different types of transitions to chaos, e.g. via period doubling [2,3], intermittency [4], quasiperiodicity [5]. This is an effective and powerful theoretical tool revealing deep and fundamental features of dynamics between order and chaos, like quantitative universality and scale invariance (scaling).

In quasiperiodically forced dissipative systems, a natural attribute of dynamics between order and chaos is a strange nonchaotic attractor (SNA) [6,7]. The word "nonchaotic" means absence of the exponential sensitivity of the orbits in respect to variance of initial conditions. The adjective "strange" reflects a subtle fractal-like structure intrinsic to the attractor. The RG analysis is relevant for understanding the birth of SNAs and associated quantitative regularities. This idea was applied to several types of critical behavior of quasiperiodically driven maps with the golden-mean frequency ratio [7–12].

In real physical systems, one must take into account the effect of noise, which obliterates subtle details of the fractal-like structure of SNA. Thus, understanding the effect of noise is a problem of significance e.g. for analysis and interpretation of the experiments on observation of these attractors and bifurcation routes of their birth.

A theoretical approach based on the RG analysis to explain the noise effects was suggested by Crutchfield et al. and Shraiman *et al.* [13, 14] in the context of the period-doubling in dissipative systems. They obtained a universal factor $\gamma = 6.619036\ldots$ responsible for the scaling properties at the onset of chaos regarding the noise. Namely, a decrease of the noise amplitude with that factor allows observing one more level of the period doubling. Analogous approaches were developed for some other types of behavior associated with the appearance of chaos [15–18].

This Letter is devoted to the effect of additive noise at the birth of SNA in the model originally introduced in Ref. [6] and modified in Ref. [8]. This is the pitch-fork bifurcation map with multiplicative quasiperiodic driving. Without noise the equations read

$$x_{n+1} = 2\lambda F(x_n)\sin 2\pi u_n, \quad u_{n+1} = u_n + w \text{ (mod 1)}, \tag{1}$$

where $F(x) = x/\sqrt{1+x^2}$, λ is a parameter, and $w = (\sqrt{5}-1)/2$ is the inverse golden-mean constant. At $\lambda < 1$ the only attractor is a trivial invariant set $x \equiv 0$. For $\lambda > 1$ the invariant set becomes unstable. This transition accompanied by the appearance of the SNA was interpreted later as the blow-out bifurcation route and discussed in the context of realistic systems governed by non-autonomous differential equations [18].

Portraits of the SNA for the map (1) are shown in Fig.1 in the left column. As stated, in a neighborhood of the transition the attractors obey some scaling properties [8,7]. E.g. with decreasing the distance from the bifurcation point by a factor $\delta = w^{-3} = 4.2360...$ the local structure of the attractor close to the point x=0, u=0 reproduces itself in smaller scales; the factors of scaling are $a = 1.3239$ and $b = -w^3 = -4.2360...$ along the axes x and u, respectively.

Let us introduce a sequence ξ_n that represents a discrete-time white noise (the terms of the sequence are statistically independent). The maximal magnitude of ξ_n is supposed to be bounded, the average is assumed to be zero: $\langle \xi_n \rangle = 0$, and the standard deviation is constant, $\sigma = \sqrt{<\xi_n^2>}$. Now we consider the stochastic map

$$x_{n+1} = 2\lambda F(x_n)\sin 2\pi u_n + \kappa\xi_n, \quad u_{n+1} = u_n + w \text{ (mod 1)}, \tag{2}$$

where κ characterizes the intensity of the additive noise source. If the amplitude of noise is small enough, the concrete form of the probability distribution for ξ_n will be not essential. The behavior of the noisy system will be of a universal nature (like in other critical situations allowing analysis in terms of the RG method, cf. [19]). In derivation of the RG equations we will assume the noise to be Gaussian, but in computations we define ξ_n as a random variable uniformly distributed over an interval [–0.5, 0.5]; hence, $\sigma^2 = 1/12$.

The right column in Fig.1 shows noisy attractors at λ=1 (the threshold of the SNA birth in the noiseless model) in dependence of the noise amplitude κ. Observe qualitative similarity with the pictures without noise at supercritical values of λ.

In Fig. 2 the Lyapunov exponent of the stochastic map (2)

$$\Lambda \cong N^{-1}\sum \log|f'(x_n)\sin 2\pi u_n|, \tag{3}$$

is plotted versus parameters λ and κ. Here N is a number of iterations in the course of the computation, and x_n and u_n are values of the variables at each step of the iterations. As seen from the plot, the Lyapunov exponent is negative except the point $\lambda = 1$, $\varepsilon = 0$, at which it vanishes. The main result of the effect of noise is rather obvious: it smoothes over the sharp form of the Lyapunov exponent dependence on the control parameter λ.

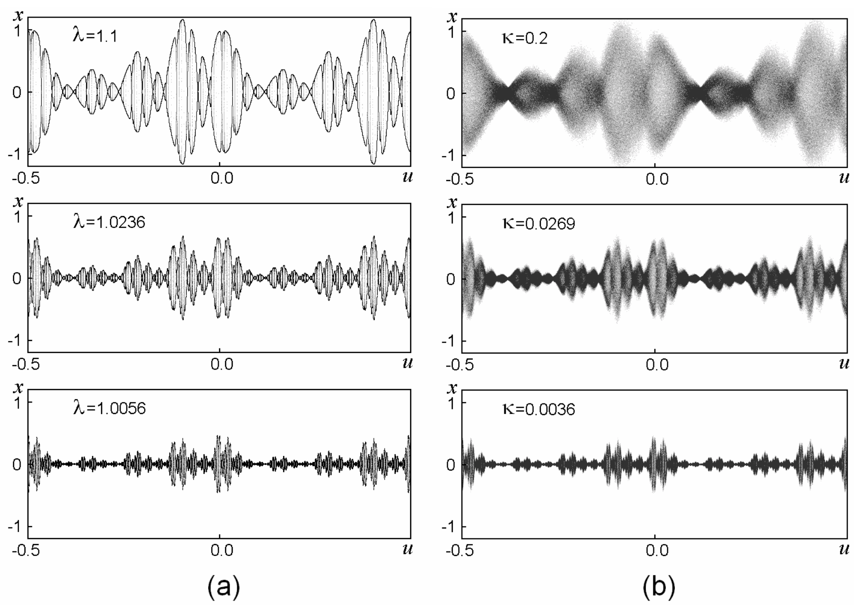

Fig. 1. Portraits of SNA for the model (1) on the plane of variables u and x at several values of λ (a) and portraits of noisy attractors of the stochastic map (2) at different amplitudes of noise at $\lambda = 1$ (the threshold value of the parameter corresponding to the birth of SNA in the noiseless model).

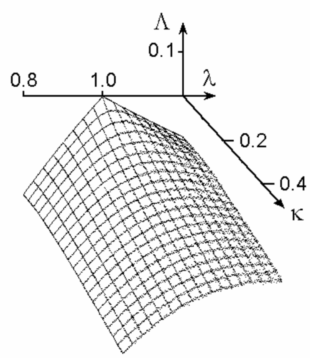

Fig. 2. Lyapunov exponent versus the control parameter λ and amplitude of noise κ for the stochastic map (2).

The main idea of the RG analysis, like in other situations of the golden-mean quasiperiodicity [5, 7, 8–11], consists in examination of evolution operators for time intervals given by Fibonacci's numbers F_k: $F_0 = 0$, $F_1 = 1$, $F_{k+2} = F_{k+1} + F_k$.

Let the equations at the SNA birth for F_k and F_{k+1} steps of discrete time be

$$x_{i+F_k} = \phi_k(x_i, u_i) + \kappa\xi_i \psi_k(x_i, u_i), \quad u_{i+F_k} = u_i + wF_k = u_i - (-w)^k \ (\mathrm{mod}\, 1) \tag{4}$$

and

$$x_{i+F_k} = \phi_{k+1}(x_i, u_i) + \kappa\xi_i \psi_{k+1}(x_i, u_i), \ u_{i+F_{k+1}} = u_i + wF_{k+1} = u_i - (-w)^{k+1} \ (\mathrm{mod}\, 1). \tag{5}$$

Here ξ_i is the random sequence, the noise amplitude parameter κ is supposed to be small, $\psi_{k+1}(x, y)$ and $\psi_{k+1}(x, y)$ are some auxiliary functions. Obviously, the model (2) corresponds to a particular version of those relations; at $F_1 = F_2 = 1$ we have to set

$$\phi_1(x,u) = \phi_2(x,u) = 2F(x)\sin 2\pi u = \frac{2x\sin 2\pi u}{\sqrt{1+x^2}}, \ \psi_1(x,y) = \psi_2(x,y) \equiv 1. \tag{6}$$

By composition of (4) and (5), retaining terms up to the first order in κ, we obtain the following equation for evolution over F_{k+2} steps of discrete time:

$$\begin{aligned} x_{i+F_{k+2}} &= \phi_k(\phi_{k+1}(x_i,u_i), u_i - (-w)^{k+1}) \\ &+ \kappa[\xi_i \phi_k'(\phi_{k+1}(x_i,u_i), u_i - (-w)^{k+1})\psi_{k+1}(x_i,u_i) + \xi_{i+F_{k+1}}\psi_k(\phi_{k+1}(x_i,u_i), u_i - (-w)^{k+1})] \end{aligned} \tag{7}$$

In respect to the stochastic terms, the following remark is relevant. Let us suppose that at some moment an orbit starts at (x_i, u_i). Consider an ensemble of random numbers $\{\xi_i, \xi_{i+F_{k+1}}\}$ of zero mean and the mean square σ^2, and compose them with coefficients given by functions of (x_i, u_i). As $\{\xi_i, \xi_{i+F_{k+1}}\}$ are independent, the sum is again a random number of zero mean, and the mean square σ^2 is multiplied by a proper function:

$$\xi_i \phi_k'(\phi_{k+1}(x_i,u_i), u_{i+1})\psi_{k+1}(x_i,u_i) + \xi_{i+F_{k+1}}\psi_k(\phi_{k+1}(x_i,u_i), u_{i+1}) = \tilde{\xi}_i \psi_{k+2}(x_i,u_i) \tag{8}$$

Now, we set

$$\phi_{k+2}(x,u) = \phi_k(\phi_{k+1}(x,u), u - (-w)^{k+1}) \tag{9}$$

and rewrite Eq. (7) in the form analogous to (4) and (5), with redefined random variable and functions ϕ and ψ:

$$x_{i+F_{k+2}} = \phi_{k+2}(x_i,u_i) + \kappa\tilde{\xi}_i \psi_{k+2}(x_i,u_i). \tag{10}$$

To obtain closed functional equations, we square both parts of Eq. (10) and perform averaging over ensemble of realizations of the noise. As $<\tilde{\xi}_i^2> = <\xi_i^2> = \sigma^2$, and $<\xi_i \xi_{i+F_{k+1}}> = 0$, we come to the relation

$$[\psi_{k+2}(x,u)]^2 = [\phi_k'(\phi_{k+1}(x,u), u - (-w)^{k+1})\psi_{k+1}(x,u)]^2 + [\psi_k(\phi_{k+1}(x,u), u - (-w)^{k+1})]^2, \tag{11}$$

where the apostrophe designates a derivative of the function over the first argument. In accordance with the basic content of the RG approach, we implement a scale change $x \mapsto x/\alpha^k$, $u \mapsto (-w)^k u$, where α is some constant. Then, for the rescaled functions

$$\begin{aligned} g_k(x,u) &= \alpha^k \phi_k(\alpha^{-k}x, (-w)^k u), \qquad f_k(x,u) = \alpha^k \phi_{k+1}(\alpha^{-k}x, (-w)^k u), \\ \Phi_k(x,u) &= [\psi_k(\alpha^{-k}x, (-w)^k u)]^2, \ \ \Psi_k(x) = \alpha^k[\psi_{k+1}(\alpha^{-k}x, (-w)^k u)]^2, \end{aligned} \tag{12}$$

the above equations imply that

$$g_{k+1}(x,u)=\alpha f_k(x/\alpha,-wu),\ \ f_{k+1}(x,u)=\alpha g_k(f_k(x/\alpha,-wu),-wu+w), \tag{13}$$

and

$$\begin{aligned}\Phi_{k+1}(x,u)&=\alpha^2\Psi_k(x/\alpha,-wu),\\ \Psi_{k+1}(x,u)&=\alpha^2\{[g_k'(f_k(x/\alpha,-wu),-wu+w)]^2\Psi_k(x/\alpha,-wu)\\ &\quad+\Phi_k(f_k(x/\alpha,-wu),-wu+w)\}.\end{aligned} \tag{14}$$

These relations define the RG transformation for a set of functions $\{g_k, f_k, \Phi_k, \Psi_k\}$. The procedure may be applied repeatedly to get the functions for larger and larger k, i.e. to determine the renormalized evolution operators for Fibonacci's numbers of steps F_k.

A reason for preference of the nonlinear function $F(x)=x/\sqrt{1+x^2}$ in the model (1) [7,8] is that a composition of such functions generates again a function of the same class. Indeed, if $f_{1,2}(x)=p_{1,2}\,x\big/\sqrt{1+s_{1,2}x^2}$, then $f_1(f_2(x))=f_3(x)=p_3x\big/\sqrt{1+s_3x^2}$, where $p_3=p_1p_2$, $s_3=s_2+s_1p_2^2$. In particular, it relates to Fibonacci's numbers of steps, and we set

$$g_k(x,u)=P_k(u)x\big/\sqrt{1+R_k(u)x^2}\,,\ \ f_k(x,u)=Q_k(u)x\big/\sqrt{1+S_k(u)x^2}\,, \tag{15}$$

Then, substitution in Eqs. (13) yields

$$P_{k+1}(u)=Q_k(-wu),\ \ Q_{k+1}(u)=Q_k(-wu)P_k(-wu+w)\ , \tag{16}$$

$$R_{k+1}(u)=\alpha^{-2}S_k(-wu),\ \ S_{k+1}(u)=\alpha^{-2}\left[S_k(-wu)+Q_k^2(-wu)R_k(-wu+w)\right]. \tag{17}$$

As seen from computations [7,8], the iterative functional equations (16) with initial conditions (6) generate asymptoptically a periodic orbit of period 6, and one half of this period corresponds to inverse of signs of P and Q. The computed solution was expressed in terms of high-precision polynomial approximation valid on an interval [-w,1]. Next, we notice that the equations (17) are linear in respect to the functions R and S and contain only one k-dependent coefficient Q^2 of period 3. So, the third iteration of these equations gives rise to an eigenvalue problem. A condition of periodic repetition of R and S after three iterations requires to assign

$$\alpha=1.098041568241\ldots \tag{18}$$

Now we have to substitute the periodic solution of the RG equations (13) into equations (14) taking into account its concrete form (15). As the relations (14) arise from squared original equations, the change of sign is not essential, and we may operate as if we would have the period-3 orbit of the RG equation. The recursive linear functional equations for the functional pairs $\{\Phi_k(x,y),\Psi_k(x,y)\}$ asymptotically is determined by the eigenvector associated with the largest eigenvalue Ω for the equation

$$\Omega\begin{pmatrix}\Phi\\ \Psi\end{pmatrix}=\hat{R}_3\hat{R}_2\hat{R}_1\begin{pmatrix}\Phi\\ \Psi\end{pmatrix}, \tag{19}$$

where $\hat{R}_k$ are linear operators

$$\hat{R}_k\begin{pmatrix}\Phi\\ \Psi\end{pmatrix}=\begin{pmatrix}\alpha^2\Psi(x/\alpha,-wu)\\ \alpha^2\{[g'_k(f_k(x/\alpha,-wu),-wu+w)]^2\Psi(x/\alpha,-wu)+\Phi(f_k(x/\alpha,-wu),-wu+w)\}\end{pmatrix}$$

Using the universal functions $g_k(x,u)$ expressed through $P_k(u)$, $Q_k(u)$, $R_k(u)$, $S_k(u)$ represented by finite power expansions in u, we have constructed the functional transformation of the right-hand part of Eq. (19) as a computer program. The unknown functions $\Phi(x,y)$, $\Psi(x,y)$ were represented by their values at nodes of a grid in a rectangular $\{-2<x<2,\ \ -w<u<1\}$, and by an interpolation scheme between them. Taking random initial conditions for $\Phi(x,y)$, $\Psi(x,y)$, the program performed the functional transformation many times and normalized the resulting functions at each step as $\Phi^0(x,y)=\Phi(x,y)/\Phi(0,0)$, $\Psi^0(x,y)=\Psi(x,y)/\Phi(0,0)$, until the form of the functions stabilized. The value $\Phi(0,0)$ (before the normalization) converges to the eigenvalue

$$\Omega = 55.125332... \tag{20}$$

Now, in linear approximation with respect to the noise amplitude, the stochastic map for the evolution over F_{3k+q} and F_{3k+q+1} steps at the critical point may be written in terms of the renormalized variables as

$$\begin{aligned} x_{i+F_{3k+q}} &= g_q(x_i,y_i)+\kappa\gamma^k\xi_i\varphi_q(x_i,y_i),\quad y_{i+F_{3k+q}}=y_i-1\\ x_{i+F_{3k+q+1}} &= f_q(x_i,y_i)+\kappa\gamma^k\xi_i\upsilon_q(x_i,y_i),\quad y_{i+F_{3k+q+1}}=y_i+w \end{aligned} \tag{21}$$

where q=1, 2, 3, and

$$\varphi_q(x,y)=\sqrt{\Phi_q^0(x,y)},\ \ \upsilon_q(x,y)=\sqrt{\Psi_q^0(x,y)},\ \ \gamma=\sqrt{\Omega}=7.4246436\ldots \tag{22}$$

Next, if we consider a small shift of parameters λ from the critical point, some additional perturbation term will appear in the equations, which corresponds to an eigenmode of the RG equation linearized at the periodic RG equation solution with eigenvalue $\delta=W^3=4.236068...$ (see [7,8]). With account of it together with the noise we have

$$x_{i+F_{3k+q}} = g_q(x_i,y_i)+C\delta^k h_q(x_i,y_i)+\kappa\gamma^k\xi_i\varphi_q(x_i,y_i),\ \ q=1,2,3\,, \tag{23}$$

where $\{h_1(x,y),\ h_2(x,y),\ h_3(x,y)\}$ is the respective egenfunvector. A coefficient C depends on parameter λ and vanishes at the critical point λ_c=1. In a close neighborhood of this point it is sufficient to account only the leading, linear term of the expansion in respect to the original parameter, proportional to $\lambda-\lambda_c$.

If we decrease a parameter shift from the critical point to have the coefficient C reduced by factor δ, and decrease the noise amplitude κ by factor γ, then the form of the stochastic map (23) remains unchanged. Thus, with the new parameters, the noisy system will demonstrate statistically similar behavior as with the old ones, but with characteristic time scale increased by factor $F_{k+3}/F_k\cong W^3=[(\sqrt{5}+1)/2]^3$, scale along the axis u decreased by $b\cong -W^3=-[(\sqrt{5}+1)/2]^3$ (the minus sign of the value b corresponds to inversion of the direction of the coordinate axis u accompanying the scale change), and scale along the axis x decreased by $a=\alpha^3=1.3239035422...$.

The last is true locally, near the origin u=0. At other points on the axis u the scaling properties are, in general, different. Particularly, with one iteration of the original map, a point (x, u) is mapped from a vicinity of u=0 to a neighborhood of u=w with multiplication of x by $\sin 2\pi u \propto u$. Thus, the scaling factor for x near this new point will be a product of factors for x and u at the old point, namely, $a' = ab = \alpha^3/w^3 = 5.60814540043...$

Figure 3 illustrates the local scaling properties for the noisy attractors with $\lambda = \lambda_c = 1$ at $u = 0$ (a-c) and $u = w$ (d-f). Diagrams (a) and (d) correspond to the noise intensity $\kappa = 0.0269$. Then we decrease the noise amplitude and set $\kappa = 0.0269/\gamma \approx 0.0036$ to plot the portraits shown in panels (b), (e). From those diagrams we select the rectangular fragments and reproduce them separately with magnification by factor b along the u axis and by factors a and a', respectively, along the x axis (c, f). Observe similarity of the pictures in the panels (a) and (c), (d) and (f).

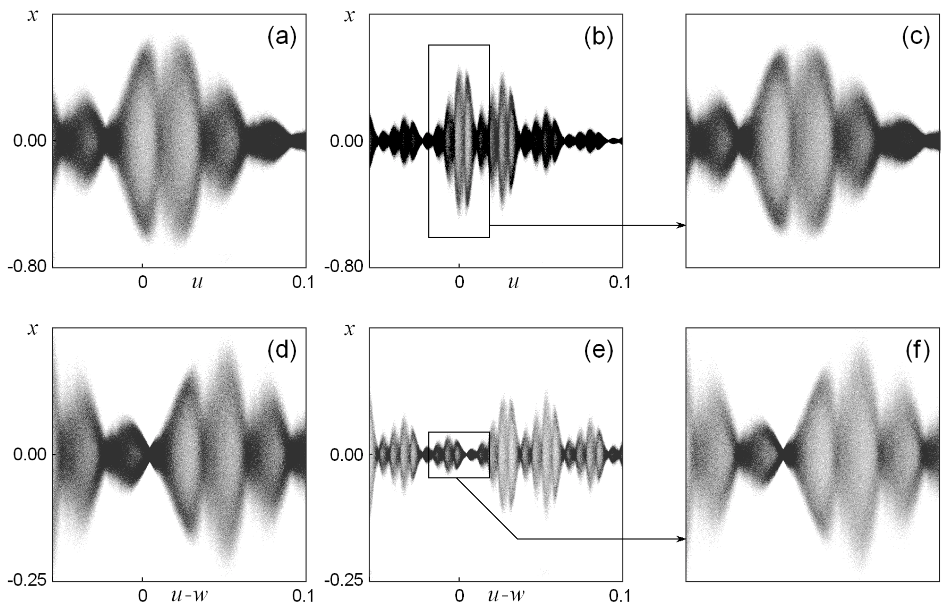

Fig. 3. Portraits of noisy attractors of the model (2) at the critical point $\lambda = \lambda_c = 1$. u=0 (a-c) and $u = w$ (d-f). Diagrams (a), (d) correspond to the noise intensity $\kappa = 0.0269$, and diagrams (b), (c) to $\kappa = 0.0269/\gamma \approx 0.0036$. The rectangular fragments are reproduced separately with magnification by factor $b = -4.2360\ldots$ along the u axis and by factors a and a', respectively, along the x axis (c, f). Observe similarity of the pictures (a) and (c), (d) and (f).

In accordance with the RG analysis, at λ=1 the model will demonstrate similar behaviors for the noise intensities κ and κ/γ, but with the characteristic time scale multiplied by $W^3 = w^{-3} = 4.2360\ldots$ in the second case. Accounting that decrease of κ by γ is accompanied with a decrease of the magnitude of the Lyapunov exponent by W^3, it must behave as

$$\Lambda \propto \kappa^{\eta} \tag{24}$$

with $\eta = 3\log_{\gamma} W = 0.7201$. Figure 4 shows the Lyapunov exponent Λ versus the noise amplitude κ in double logarithmic scale. The points obtained in the numerical computations are clearly placed along the straight line of slope η. It is worth noticing a kind of

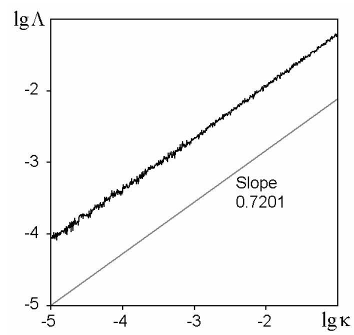

Fig. 4. Plot of the Lyapunov exponent versus noise intensity at $\lambda = 1$ in double logarithmic scale. The straight line corresponds to the relation (24).

noisy stabilization at the birth of SNA. Indeed, the effect of noise promotes a decrease of the Lyapunov exponent, i.e. a decrease of sensitivity in respect to the initial conditions.

To conclude, in this Letter we established scaling regularities associated with the effect of additive noise in a model system near the birth of a strange nonchaotic attractor in the situation of the golden-mean quasiperiodic force. A renormalization group analysis of the effect of noise was developed, and the respective scaling constant was computed. We also presented computer graphical illustrations for the scaling regularities. We considered a particular representative of the universality class (forced 1D pitchfork map). Nevertheless, on a basis of the RG argumentation it may be conjectured that the same regularities will be intrinsic to a class of dissipative quasiperiodically forced systems demonstrating the blowout birth of SNA [7, 18].

Acknowledgements

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