Fluctuation and Noise Letters
Vol. 8, No. 2 (2008) L173–L185
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PECULIARITIES OF SCALING IN MODELS OF DUFFING OSCILLATOR UNDER ACTION OF KICKS WITH RANDOM MODULATION OF PARAMETERS

ALEXANDER P. KUZNETSOV, JULIA V. SEDOVA
Institute of Radio-Engineering and Electronics RAS, Saratov Branch,
Zelenaya 38, Saratov, 410019, Russia
alkuz@sgu.ru, sedovayv@rambler.ru

Received 2 January 2008
Revised 21 March 2008
Accepted 24 March 2008
Communicated by Alexander Neiman

We study noise influence on universal properties of period-doubling cascade in a well-defined model of periodically kicked nonlinear dissipative oscillator. Two approaches of noise adding are considered: (i) modulation of kick's amplitude and (ii) modulation of the interval between kicks. We then derive corresponding stochastic discrete one- and two-dimensional maps and provide a detailed study of the noise scaling properties of the Feigenbaum scenario and of the non-Feigenbaum tricritical case. Illustrations of bifurcations trees and Lyapunov exponent charts are given.

Keywords: Scaling; period-doublings; noise.

1. Introduction

Several basic scenarios of transition from regular behavior to chaos for nonlinear systems are known. All of them allow description with the help of a renormalization group method and therefore demonstrate property of universality with regard to concrete type of system [1]. One of consequences of an opportunity of a renormalization approach is the property of scaling which means reproducibility of dynamic characteristics on small scales in the vicinity of the critical point of transition to chaos. The scaling property can be observed for the structure of bifurcation tree, Lyapunov exponent diagram, phase portrait of attractor, etc. However, in real systems the noise is presented, which can greatly influence the picture of transition. For example in a case of period-doublings the noise washes away thin structure of bifurcation tree. As the result, only the limited number of period-doubling bifurcations can be observable. Thus, the known illustrations of scaling become unrealizable, because they assume asymptotically closed approach to a critical point. However, the renormalization group method can be generalized on a stochastic case (for example, see [2–7]). Due to this, it is possible to extend the scaling property on systems with the presence of noise. To demonstrate the scaling property we should renormalize noise amplitude by new universal constant at transition from one level of hierarchy to another. For a number of universality classes such constants were determined earlier [2, 5–8]. At the same time, it is important to give scaling illustrations not only for formal "canonical" models, but also for maps obtained from "the first principles" for physical systems, when the connection between parameters of model and parameters of initial physical system is clearly determined. Besides, it would be interesting to present illustrations of self-similarity for systems with period-doublings under action of noise on a plane of essential parameters. In this context, we study a periodically kicked Duffing oscillator. The noise is brought into system as random modulation of kicks amplitudes or intervals of time between the neighboring kicks. Such approach allows to obtain two-dimensional and one-dimensional maps analytically.

We consider two classes of universality associated with period-doublings. First one is a classical Feigenbaum type of behavior and it is characterized, besides the Feigenbaum scaling constants δ_F = 4.669201609… and α_F = –2.502907875… by noise factor μ_F = 6.61903... found out in works of Crutchfield et al. and Shraiman et al. [2, 8]. The second type of behavior assumes generalization of the Feigenbaum scenario at a case of two-parametrical maps. It is characterized by the universal self-similar structure of parameters plane in a vicinity of terminal points for Feigenbaum critical curves (so called tricritical points). Such organization of parameters plane for systems without noise was discussed in paper [9] with help of charts of dynamical regimes. We demonstrate the scaling regularities at investigating of two-dimensional chart of Lyapunov exponent both in systems without noise, and with noise. We specify the appropriate universal factors δ_T = 7.284686217… and α_T = –1.69030297… and noise constant μ_T = 8.2439...

In Section 2 we present a transition from initial differential equation of periodically kicked nonlinear dissipative oscillator to 2D Ikeda map and 1D cosine map. We obtain maps with presence of noise that is added both by amplitude modulation and by modulation of the duration of time intervals between kicks. In Section 3 we investigate scaling property in one-parameter case and give illustrations of self-similarity on bifurcation trees. In Section 4 we perform transition from cosine map to cubic map and show 2D scaling on Lyapunov exponent charts.

2. Transition from Differential System to Map. Addition of Noise

We shall consider dissipative oscillator with cubic nonlinearity (Duffing oscillator) under the action of periodic kicks. Let us assume that the action of kicks takes very short time, so during this time a coordinate of oscillator practically does not change, and the speed obtains addition determined by amplitude of kick. The behavior of such oscillator is described by the following differential equation:

$$\ddot{X} + \gamma\dot{X} + \omega_0^2 X + \beta X^3 = \sum_n C\delta(t - nT), \tag{1}$$

where X is the coordinate of oscillator, γ is the damping factor, ω_0 is eigen frequency of oscillator, β is the parameter of nonlinearity, T is the interval between kicks and C is amplitude of kick.

Since in a right part of the equation (1) there is a δ-function, then during an interval between kicks the right part of the equation vanishes:

$$\ddot{X} + \gamma\dot{X} + \omega_0^2 X + \beta X^3 = 0. \tag{2}$$

In that case, it is possible to find an approximate analytical solution, using a method of slowly varying amplitudes. In other words, we can present coordinate X in the following form:

$$X = \frac{a}{2} e^{i\omega_0 t} + \frac{a^*}{2} e^{-i\omega_0 t}, \tag{3}$$

where $a = a(t)$ and $a^* = a^*(t)$ are complex and complex conjugate slowly varying amplitudes correspondingly. We shall take into account a traditional additional condition

$$\dot{a} e^{i\omega_0 t} + \dot{a}^* e^{-i\omega_0 t} = 0 . \tag{4}$$

Substituting relation (3) to (2), using condition (4) and performing averaging over time, we get well-known abridged equation for complex amplitude

$$\dot{a} = -\frac{\gamma}{2} a + \frac{3}{8} \frac{i\beta}{\omega_0} |a|^2 a . \tag{5}$$

Let us further introduce the real amplitude $R(t)$ and real phase $\varphi(t)$ by means of next expression:

$$a(t) = R(t) e^{i\varphi(t)} .$$

We substitute this expression in the abridged equation (5) and separate the real and imaginary parts. Then for the real amplitude $R(t)$ and the real phase $\varphi(t)$ we obtain the next differential equations:

$$\dot{R} = -\frac{\gamma}{2} R , \ \dot{\varphi} = \frac{3\beta}{8\omega_0} R^2 . \tag{6}$$

Solutions of these equations give dependences of real amplitude $R(t)$ and phase $\varphi(t)$ on time in an interval between kicks:

$$R(t) = R_n e^{-\gamma t/2}, \varphi(t) = \frac{3\beta}{8\omega_0} R_n^2 \frac{1 - e^{-\gamma t}}{\gamma} + \varphi_n . \tag{7}$$

Here R_n and φ_n are the initial amplitude and initial phase immediately after n-th kick.

From Eq. (3) it is possible to find expressions specifying dependences of coordinate $X(t)$ and speed $V(t)$ of oscillator on time:

$$X(t) = R(t) \cos[\omega_0 t + \varphi(t)] ,$$
$$V(t) = -\omega_0 R(t) \sin[\omega_0 t + \varphi(t)].$$

Substituting (7) in last expressions, we shall find dependences of coordinate and speed of oscillator on time in an interval between kicks:

$$\begin{aligned} X(t) &= R_n e^{-\gamma t/2} \cos\left(\omega_0 t + \frac{3\beta}{8\omega_0} |R_n|^2 \frac{1 - e^{-\gamma t}}{\gamma} + \varphi_n \right), \\ V(t) &= -\omega_0 R_n e^{-\gamma t/2} \sin\left(\omega_0 t + \frac{3\beta}{8\omega_0} |R_n|^2 \frac{1 - e^{-\gamma t}}{\gamma} + \varphi_n \right). \end{aligned} \tag{8}$$

To the moment of the beginning of $(n+1)$-th kick, the time equal to period of external influence T has passed. Hence the coordinate and speed of oscillator are equal to $X(T)$ and $V(T)$ correspondingly. Owing to δ-shaped character of external force, immediately after kick the coordinate does not change and speed gets the addition that is equal to amplitude of external influence C. Consequently for coordinate and speed immediately after $(n+1)$-th kick we obtain next expressions [10]:

$$\begin{aligned} X_{n+1} &= R_n e^{-\gamma T/2} \cos\left(\omega_0 T + \frac{3\beta}{8\omega_0} |R_n|^2 \frac{1-e^{-\gamma T}}{\gamma} + \varphi_n \right), \\ V_{n+1} &= -\omega_0 R_n e^{-\gamma T/2} \sin\left(\omega_0 T + \frac{3\beta}{8\omega_0} |R_n|^2 \frac{1-e^{-\gamma T}}{\gamma} + \varphi_n \right) + C. \end{aligned} \tag{9}$$

These expressions represent required two-dimensional map, which gives dependences of coordinate and speed of oscillator immediately after $(n+1)$-th kick on its coordinate and speed immediately after n-th kick. It is more convenient to introduce a new complex variable Z:

$$Z = \left(iX + \frac{V}{\omega_0} \right) \sqrt{\frac{3\beta}{8\omega_0} \frac{1-e^{-\gamma T}}{\gamma}} \,. \tag{10}$$

Using initial conditions

$$\begin{aligned} X(t=0) &= X_n = R_n \cos\varphi_n, \\ V(t=0) &= V_n = -\omega_0 R_n \sin\varphi_n, \end{aligned}$$

and also using expression for complex variable Z_n, we have got from (9):

$$Z_{n+1} = A + BZ_n \exp\left(i\left(|Z_n|^2 + \psi\right)\right). \tag{11}$$

Here new parameters A, B and ψ are defined via parameters of initial differential system as follows

$$\begin{aligned} A &= \frac{C}{\omega_0} \sqrt{\frac{3\beta}{8\omega_0} \frac{1-e^{-\gamma T}}{\gamma}}, \\ B &= e^{-\gamma T/2}, \\ \psi &= \omega_0 T. \end{aligned}$$

The map (11) is called Ikeda map [11]. It is necessary to give a well-known illustration of chart of dynamical regimes for Ikeda map (Fig. 1(a)). The chart of dynamical regimes is a diagram on the parameter plane where domains of qualitatively distinct regimes are indicated by colors. To depict such chart one needs to scan step by step an area on the parameter plane. At each point of area, the discrete map needs to be iterated. Then after transient process and arrival to attractor, the nature of regime is analyzed and the point is marked by an appropriate color.

Another graphic representation of complex dynamics of nonlinear multi-parameter maps is the chart of Lyapunov exponent [12–15]. For construction of such chart at each point of parameters space the value of Lyapunov exponent Λ is calculated and is coded by gray shadings. The white color corresponds to values of Λ that are close to zero.

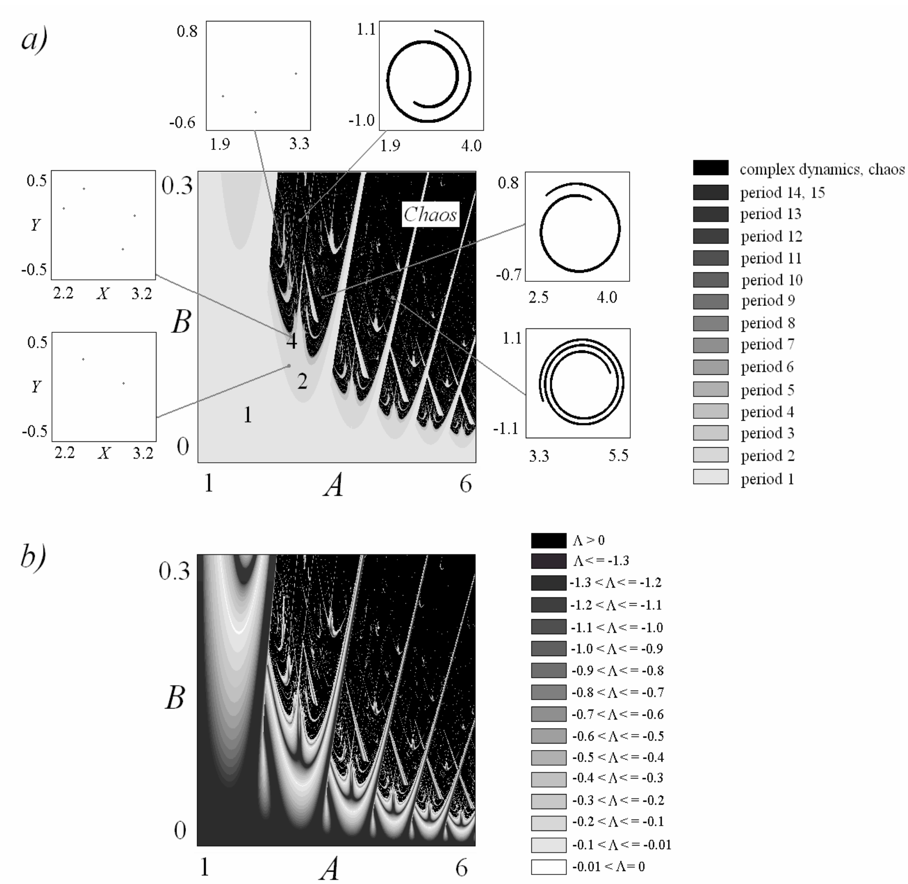

Fig. 1. A chart of dynamical regimes with some phase portraits of attractor in indicated points on parameters plane (a) and chart of Lyapunov exponent Λ (b) for Ikeda map (11). With help of figures on chart of dynamical regimes we mark the regions of existence of cycles with some basic periods. Separately markings of colors are presented.

Negative values of Λ are associated with gray nuance the darker the more $|\Lambda|$. The black color corresponds to positive values of Lyapunov exponent. Also white color designates points at which the iterative process diverges. The Lyapunov exponent chart for Ikeda map is presented on Fig. 1(b).

Let now assume that amplitudes of kicks are modulated in a random way and ΔC_n is random addition to amplitude of n-th kick:

$$\ddot{X} + \gamma\dot{X} + \omega_0^2 X + \beta X^3 = \sum_n \left(C + \Delta C_n\right)\delta\left(t - nT\right). \tag{12}$$

For Eq. (12) by analogy with (11) the following map may be obtained:

$$Z_{n+1} = A\left(1 + \frac{\Delta C_n}{C}\right) + BZ_n \exp\left(i\left(\left|Z_n\right|^2 + \psi\right)\right). \tag{13}$$

It is possible to consider another way of insertion of fluctuations in investigated system. Let us suppose, that the duration of time intervals between kicks slightly changes about the average period T so, that the duration of these intervals is $T + \Delta T_n$, where ΔT_n is small random value. Let us consider the case (that is usual for oscillators) when parameter of damping γ is much less than eigen frequency ω_0. Then it is easy to see from (8), that as time $T + \Delta T_n$ passes the random addition to amplitude of oscillator can be neglected in comparison with the addition to its complete phase. It means, that we come to the following form of Ikeda map with random influence:

$$Z_{n+1} = A + BZ_n \exp\left(i\left(\left|Z_n\right|^2 + \psi + \omega_0 \Delta T_n\right)\right). \tag{14}$$

Maps (13) and (14) are the two-dimensional, because Z is complex variable, containing the real and imaginary parts. It is known, however, that in wide region of parameters values (the more precise the greater A) the Ikeda map admits a description with the help of one-dimensional map [9, 10]. Let us do the similar procedure for map (14). Following [10, 16] let us assign

$$Z = A + \tilde{Z}, \tag{15}$$

where $\tilde{Z}$ is small addition. Let us substitute this expression in the left and right parts of the equation (14) taking into account smallness of $\tilde{Z}$:

$$\tilde{Z}_{n+1} = BA \exp\left(i\left(A^2 + 2A \operatorname{Re} \tilde{Z}_n + \psi + \omega_0 \Delta T_n\right)\right). \tag{16}$$

Having put further $X_n = 2A \operatorname{Re} \tilde{Z}_n + A^2 + \psi$, we come to the following one-dimensional cosine map with a noise influence:

$$X_{n+1} = \lambda \cos\left(X_n + \omega_0 \Delta T_n\right) + \varphi . \tag{17}$$

Here new parameters are used

$$\lambda = 2A^2 B, \quad \varphi = A^2 + \psi. \tag{18}$$

Further we shall suppose, that $\omega_0 \Delta T_n = \varepsilon \xi_n$, where ε is non-dimensional amplitude of modulation of the external influence period, and ξ_n is random variable.

The charts of dynamical regimes and Lyapunov exponent for cosine map (17) at the absence of external noise influence are given in Fig. 2. The range of changing of parameter φ is chosen as ($-\pi/2$, $3\pi/2$) because of 2π-periodicity of cosine function.

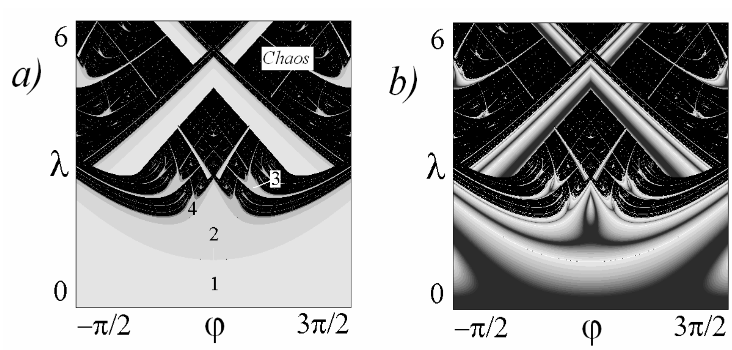

Fig. 2. A chart of dynamical regimes (a) and chart of Lyapunov exponent Λ (b), computed for cosine map (17) in the absence of noise.

3. One-Parametrical Transition to Chaos and Scaling

The cosine map (17) in the absence of fluctuations $X_{n+1} = \lambda \cos X_n + \varphi$ has a set of quadratic extrema. Therefore, such map demonstrates the scenario of transition to chaos via period-doublings. At crossing the border of chaos along a typical route on a parameters plane (λ, φ) the classic cascade of period-doublings that obeying Feigenbaum's laws will be observed. For example, at $\varphi = 0$ the Feigenbaum cascade accumulates to a critical point λ_c = 1.974133… The illustration of scaling for the bifurcation tree in system with noise is shown in Fig. 3. The self-similarity is illustrated by a series of figures; each subsequent figure represents magnified fragment of the previous one. In according with Feigenbaum laws the horizontal scale of enlarged right fragment of Fig. 3 is decreased by Feigenbaum factor α_F relatively to a point $X = 0$ (one of extrema of cosine map), $X \to X/\alpha_F$, and the vertical scale is recalculated by other Feigenbaum factor δ_F relatively to a critical point λ_c = 1.974133.., $\lambda \to \lambda_c + (\lambda - \lambda_c)/\delta$. According to the approach, stated in Introduction, at transition to deeper level of a hierarchical picture the initial amplitude of noise must be decreased by noise factor μ_F = 6.61903.., $\varepsilon \to \varepsilon/\mu_F$, that allows to observe one more level of period-doubling.

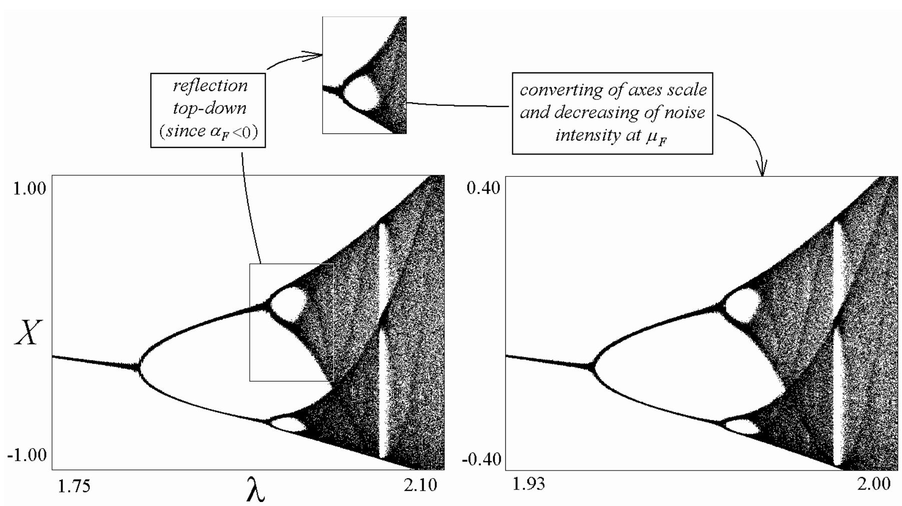

Fig. 3. An illustration of Feigenbaum scaling property for bifurcation tree of cosine map. The initial noise intensity is equal to 0.003.

However, the map (17) is characterized by two parameters (λ, φ), therefore it is possible to observe tricritical points on its plane. As we have already noted in Introduction, for Feigenbaum critical lines tricritical points are "attributes" of maps having two and more quadratic extrema. At these points it is possible to observe period-doubling cascades along curves, on which the maximum is mapped exactly onto a minimum [17]. The reason of non-Feigenbaum character of convergence in this case is that twice iterated map under such condition has no quadratic extremum, but has extremum of the fourth order. So for cosine map (17) under condition $\lambda = \pi - \varphi$ the quadratic maximum $X = 0$ is mapped exactly onto a quadratic minimum $X = \pi$. Thus along the line $\lambda = \pi - \varphi$ on a parameters plane (λ, φ) a non-Feigenbaum cascade of period-doubling bifurcations, accumulating to tricritical point λ_T = 2.18603861533.., φ_T = 0.9555540392…[9], should be observed. Fig.4 demo nstrates an illustration of scaling for the bifurcation tree of map (17) with typical for tricriticality universal constants δ_T = 7.284686217… along ë-axis and α_T = −1.6903029714… along *X*-axis. At transition from one level of period-doublings to another the noise magnitude ε must be decreased by ì$_T$ = 8.2439… [18] in order to observe scaling. So scaling in system with noise is well executed. It is the evidence of universality of scaling property concerning a concrete form of map with renormalization of noise intensity both for Feigenbaum, and for tricritical dynamics.

To study sensitivity of the obtained results to noise we used several types of random number operations. The cases of binary noise, a noise with uniform distribution and a noise with Gauss distribution are executed. It turned out that in all this cases the property of self-similarity is well realized. This fact indicates universality of scaling property also in relation to a type of noise. In the present computations for illustration of all results we choose a noise with uniform distribution. We define ξ_n as random variable uniformly distributed over an interval [–0.5; 0.5]. (Hence, the mean is zero and the standard deviation is equal to $1/\sqrt{12}$.) Generally speaking, if the amplitude of noise is small, and the dynamics of noisy system is considered on a large time scale, the concrete form of the probability distribution for random sequence ξ_n appears not to be essential, and the behavior of the noisy system will be of a universal nature. We have already drawn similar conclusion in our previous papers dedicated to problems of determination of noise factor μ (see [6], [18]). Earlier analogous conclusion was made by D. Fiel [19].

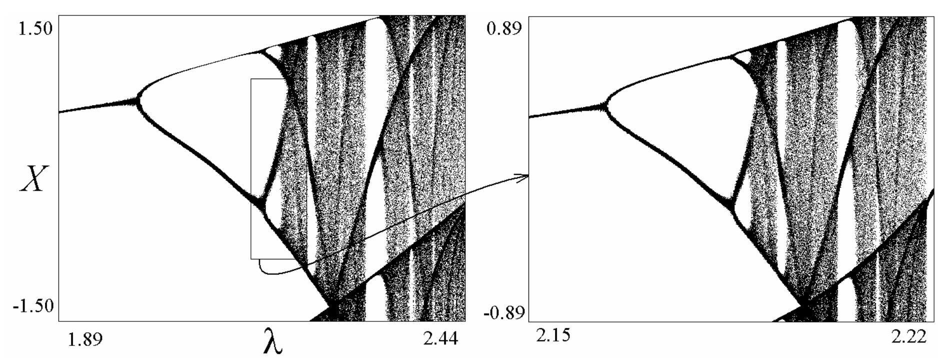

Fig. 4. "Tricritical" scaling on bifurcation tree of cosine map (17) at the presence of noise with initial amplitude 0.01.

4. Two-Parametrical Transition to Chaos and Scaling

Let us turn now to the two-parametrical analysis. For systems without noise it assumes bifurcation analysis (drawing of bifurcation curves on parameters plane) or analysis of charts of dynamical regimes. For systems with noise both these approaches are inapplicable. Therefore, we use drawing of above-mentioned Lyapunov exponent charts. The family of Lyapunov exponent charts for map (17) at various values of noise intensity ε is shown in Fig. 5.

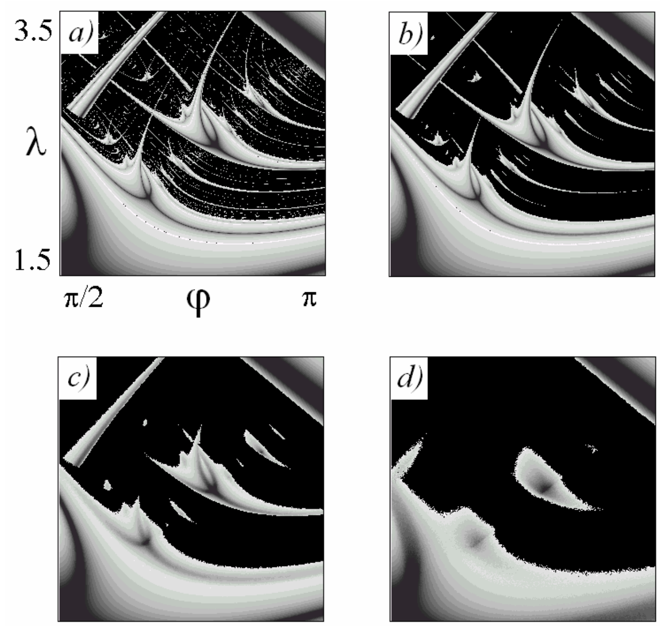

Fig. 5. Family of Lyapunov charts for map (17) for various values of noise amplitude ε = 0.005 (a), 0.01 (b), 0.05 (c) and 0.2 (d).

In the case of small noise ε we see rather clear structure of Lyapunov space (Fig. 5(a)): the borders between certain values of Lyapunov exponent are well defined; the areas with zero Λ are sharply allocated; in chaos region the areas of periodic regimes are well visible. The more intensive noise leads to disappearance of some regular regimes in the chaos area, though keeps general structure of a chart (Figs. 5(b), (c)). At the further increasing of noise amplitude an almost complete disappearance of regular regimes is observed in the chaos region, the structure of a chart becomes more smeared, values of Lyapunov exponent have increased on all parameters plane (Fig. 5(d)).

The presented charts of Lyapunov exponent are characterized by thin and complicated organization containing a set of small (in absence of noise indefinitely small) details. The most representative in this respect are tricritical points, in which vicinity the parameters plane is characterized by self-similar hierarchical organization. Illustration of scaling on a parameters plane requires a determination of special scaling coordinate system. To base on the earlier obtained results, we shall slightly simplify map (17). For this purpose let us assume, that

$$X_n = \pi/2 + y_n - \varepsilon\xi_n \,. \tag{19}$$

As a result, we shall get the next expression

$$y_{n+1} = -\lambda \sin y_n + \varphi - \pi/2 + \varepsilon\xi_n \,. \tag{20}$$

Let us expand sine function up to the cubic term inclusively. Then we shall introduce new variable and parameters according to relations

$$X = y\sqrt{\lambda/6}, \quad a = (\varphi - \pi/2)\sqrt{\lambda/6}, \quad b = \lambda.$$

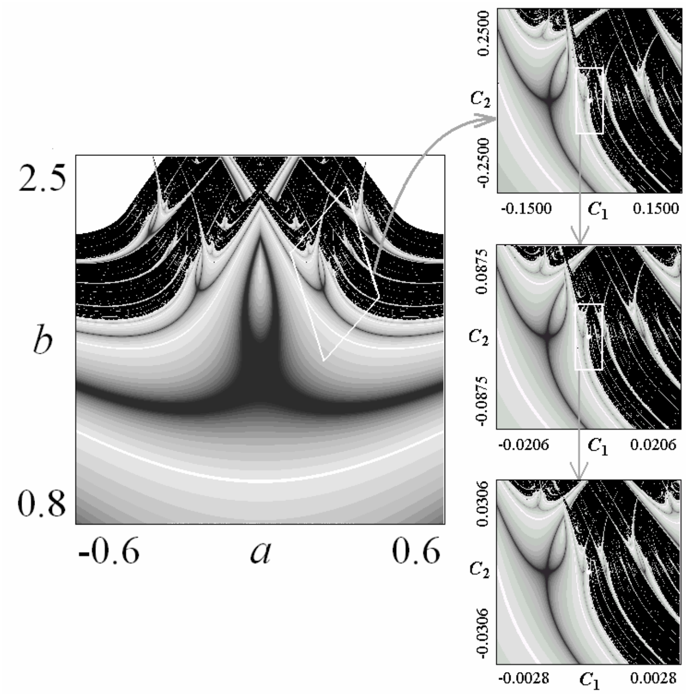

Fig. 6. Scaling on Lyapunov exponent chart for map (21) in absence of noise.

Amplitude of noise is normalized on the factor $\sqrt{\lambda/6}$. Then we come to cubic map

$$X_{n+1} = a - bX_n + X_n^3 + \varepsilon\xi_n . \tag{21}$$

The Lyapunov exponent chart of the cubic map in case of noise amplitude $\varepsilon = 0$ is shown at the left part of Fig. 6. It may be shown, that above-mentioned approximations are not essential from the point of view of chart's structure. (This statement concerns also characteristics of tricritical dynamics, because such dynamics is a "rough" phenomenon in system with two quadratic extrema. The transition from cosine map (17) to cubic map (21) only slightly moves coordinates of two such essential extrema.)

The illustration of scaling property for map (21) on a parameters plane for a Lyapunov exponent chart is given in Fig.6. On large left fragment the coordinates axes are parameters of initial map (a, b). Depicted parallelogram (at which center the tricritical point is placed) is formed by coordinate lines $C_1 = \pm 0.15$ and $C_2 = \pm 0.25$. Here (C_1, C_2) are special coordinates, which are necessary to use for observation of self-similarity. Scaling coordinates (C_1, C_2) in the parameter plane of initial map (a, b) are determined by expressions [17, 20]:

$$a - a_T = 0.5998610C_1 - 0.2192807C_2, \quad b - b_T = C_1 + C_2. \tag{22}$$

Tricritical point a_T, b_T has coordinates (0.2426987573.., 1.9513857778…).

The part of a parameters plane that got inside of mentioned parallelogram is shown separately on the top right fragment in "scaling coordinates". Then it is twice reproduced with recalculation of scale along axis C_1 by factor δ_T and along axis C_2 by factor α_T^2. For observation of scaling at transition from one level to another the value of Lyapunov exponent needs to be rescaled by factor 2 in comparison with the previous fragment. According to this rule the color palette is changed.

To observe the scaling properties for cubic map in the presence of noise (21) on a chart of Lyapunov exponent it is also necessary to rescale the noise amplitude by the factor $\mu_T = 8.2439\ldots$ The appropriate illustration of scaling for Lyapunov exponent chart in the presence of noise for initial noise intensity $\varepsilon = 0.02$ are given in Fig.7. It is easy to see, that each fragment with high accuracy repeats structure of previous fragment. That is an illustration of two-parametrical scaling in system with noise.

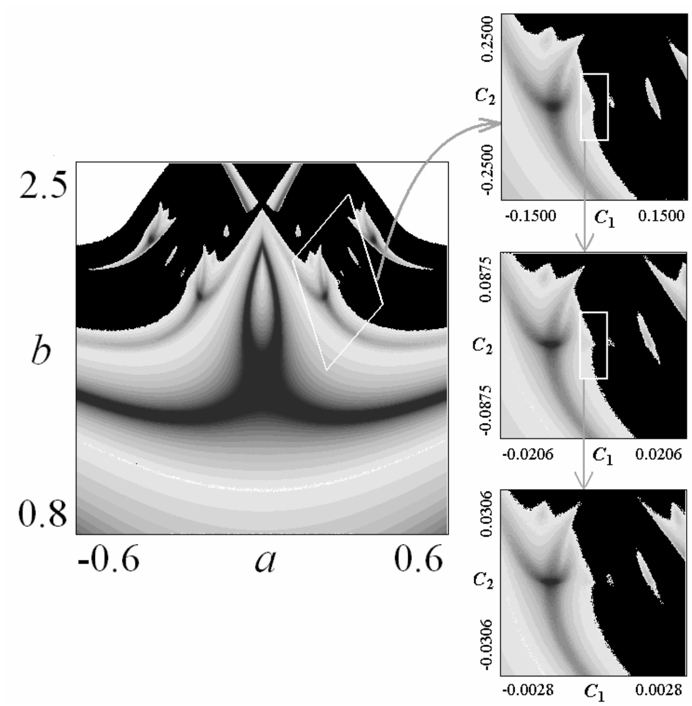

Fig. 7. Scaling on Lyapunov exponent chart for cubic map (21) in the presence of noise with initial intensity ε = 0.02.

5. Conclusion

We have discussed scaling regularities associated with the effect of additive noise on the maps obtained as a result of application of slowly varied amplitude method to the Duffing oscillator forced by the periodic sequence of kicks. It seems, that periodically kicked Duffing oscillator with random modulated amplitude or period of kick sequence is convenient model for study of the critical phenomena at transition "order – chaos" in systems with noise. The maps obtained as approximation of this system demonstrate both one-parametrical Feigenbaum scaling (but with an additional noise constant $\mu_F = 6.61903\ldots$), and scaling on a plane of parameters with a noise constant $\mu_T = 8.2439...$

It is necessary to note that similar investigations of scaling characteristics in the presence of noise can be realized in case of other types of external effect, for example, sinusoidal driving. However, in given considered case of influence in form of periodic kicks it was materially that such type of influence permits discretization and reduction of differential equation to maps. We believe that described procedure of the noise supplement and demonstration of scaling properties with help of noise factor μ can be generalized to other models and other types of noise addition owing to simplicity of procedure and universal nature of factor μ which arising from conclusions of renormalization group approach.

Acknowledgements

The work has been performed under support of Russian Science Support Foundation, Russian Fund of Basic Researches (grant 04-02-04011), Ministry of Education and Science of Russian Federation and CRDF (grant CRDF BRHE REC-006 SR-006-X1/BF5M06 Y3-P-06-07) and grant of President of Russian Federation (MK-4162.2006.2).

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