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Birth of a New Class of Period-Doubling Scaling Behavior as a Result of Bifurcation in the Renormalization Equation

S.P. Kuznetsov · A.A. Mailybaev · I.R. Sataev

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Abstract It is found that a fixed point of the renormalization group equation corresponding to a system of a unimodal map with extremum of power κ and a map summarizing values of a function of the dynamical variable of the first subsystem, undergoes a bifurcation in the course of increase of κ. It occurs at $\kappa_c = 1.984396$ and results in a birth of the period-2 stationary solution of the RG equation. At $\kappa = 2$ this period-2 solution corresponds to the universal period-doubling behavior discovered earlier and denoted as the C-type criticality (Kuznetsov and Sataev in Phys. Lett. A 162:236–242, 1992). By combination of analytical methods and numerical computations we obtain and analyze an asymptotic expansion of the period-2 solution in powers of $\Delta\kappa = \kappa - \kappa_c$. The developed approach resembles the ϵ-expansion in the phase transition theory, in which a "trivial" stationary point of the RG transformation undergoes a bifurcation that gives rise to a new fixed point responsible for the critical behavior with nontrivial critical indices.

1 Introduction

One of challenges in the modern nonlinear dynamics concerns elaboration of the renormalization group (RG) methods for studying transitions to chaos [1, 2]. In many cases, such an approach recovers universal features of the transitions and establishes regularities of self-similarity for the small scale structures in the phase space and in the parameter space. In particular, the RG analysis has been developed for transitions in dissipative and conservative systems via period doubling, quasiperiodic regimes and intermittency [1–8].

A study of transitions to chaos with application of the RG method in the context of multiparameter analysis may be regarded as an approach similar in a spirit to the bifurcation

S.P. Kuznetsov (✉) · I.R. Sataev
Institute of Radio-Engineering and Electronics of RAS, Saratov Branch, Zelenaya str. 38, Saratov 410019, Russia
e-mail: spkuz@rambler.ru

A.A. Mailybaev
Institute of Mechanics, Moscow State Lomonosov University, Michurinsky pr. 1, Moscow 119192, Russia

theory and catastrophe theory. In this respect, the classical Feigenbaum period-doubling critical behavior [1, 2] appears as a phenomenon of codimension one, since it is generic (typical) in one-parameter families of nonlinear dissipative dynamical systems. A number of distinct types of critical behaviors (universality classes) of higher codimensions also has been detected and studied on a basis of the RG analysis [9–12]. Though a search for new types of criticality at the onset of chaos is of principal interest, in a big part it remains a matter of luck. In this situation, developing general principles and methods for search and classification of new universality classes is of great importance.

The RG approach originates from the statistical physics and the phase transition theory, and one of the basic ideas in those fields consists in consideration of the transition phenomena in dependence on the space dimension treated formally as a continuous parameter [13, 14]. In many cases a critical dimension exists, at which the "trivial" fixed point of the RG transformation bifurcates. As a result, a new stationary solution appears, which is responsible for the universal critical behavior with nontrivial (nonclassical) critical indices describing actual phase transitions. The systematic approach based on the perturbation theory, where the deflection from the critical dimension is considered as a small parameter, was developed by Wilson and is called the ϵ-expansion (Nobel Prize in Physics, 1982). It seems that a similar methodology may be useful in nonlinear dynamics for analysis of critical chaotic behaviors at the chaos threshold.[1]

In this paper, we study a bifurcation in the RG equation describing the period-doubling route to chaos in two-dimensional maps. This bifurcation gives rise to a cycle of period 2 developing from a fixed point of the RG equation. As stated, it generates a new universal type of critical behavior at the border of chaos. In Sect. 2, we describe the two-dimensional generalization of the Feigenbaum–Cvitanović equation [3, 4, 12]. In Sect. 3 a critical behavior is considered, which may be regarded as an analog of the "trivial" fixed point in the phase transition theory. The model system consists of two elements represented by one-dimensional maps, with unidirectional coupling. It is assumed that the first map relates to the class manifesting the Feigenbaum period-doubling universality at the chaos threshold. The second map is supposed to be a generator of an integral characteristic: it produces a sum of a function of the dynamical variable of the first subsystem in the course of the dynamics. In Sect. 4, we modify the RG transformation to account degree of the extremum point in the first map as a continuous parameter rather than a constant $\kappa = 2$. It appears that the fixed point of the RG equation undergoes a bifurcation at $\kappa_c = 1.984396$. As κ exceeds this value, the bifurcation gives rise to a new period-2 stationary solution. In Sect. 5, we analyze the critical behavior associated with this solution and find out that at $\kappa = 2$ it corresponds to the universality class detected in two-dimensional irreversible maps and denoted as the C-type critical behavior [16, 17]. Then we present a numerical asymptotic expansion of the period-2 solution in powers of $\sqrt{\kappa - \kappa_c}$. In the Conclusion, we summarize the content of this paper as an example of the bifurcation approach to the analysis of the RG equation solutions, which is analogous to the ϵ-expansion method in the phase transition theory.

2 Two-Dimensional Generalization of the Feigenbaum–Cvitanović Equation

In the context of nonlinear dynamics the content of the RG approach consists in the following.

[1] In the context of nonlinear dynamics, bifurcations in RG equations were discussed earlier in a different aspect: for the problem of transition to chaos in a system described by one-dimensional quadratic map under external excitation with fractal properties [15].

Let us start with an operator describing evolution of a dynamical system in some finite time interval. With application of this operator several times, we construct the evolution operator for a larger time interval and, additionally, perform an appropriate scale change of the dynamical variables. Then, it may occur that the new operator coincides (or is very close) to the original operator at some special values of the parameters of the system. This procedure defines the *renormalization group (RG) transformation* of the evolution operators, and the corresponding parameter values define the *critical point*. Applying the RG transformation repeatedly, one obtains a series of evolution operators for larger and larger time intervals. At the critical point, the behavior of this operator sequence corresponds to a stationary solution of the RG equation, a fixed point or a periodic orbit. The operators representing the stationary solution are determined by structure of the RG transformation rather than by a concrete original system under consideration; it implies *universality* of the dynamics on large-scale time intervals. As the evolution operators obtained by multiple action of the RG transformation are equivalent up to the scale change, the dynamics demonstrates similar behaviors on the associated scales; it is the property of *scaling*.

Transition to chaos through a series of period-doubling bifurcations in a class of unimodal maps $g(x)$, including the quadratic (logistic) map, is described by a fixed point of the RG transformation $g(x) \mapsto \alpha g(g(x/\alpha))$, where α is a scaling constant. This yields the functional equation of Feigenbaum–Cvitanović

$$g(x) = \alpha g(g(x/\alpha)). \tag{1}$$

The well-known solution of this equation with $\alpha = -2.502908$ was obtained numerically by Feigenbaum [1, 2]:

$$\begin{aligned} g(x) = 1 &- 1.5276x^2 + 0.1048x^4 + 0.0267x^6 - 0.0035x^8 \\ &+ 0.0001x^{10} + \cdots . \end{aligned} \tag{2}$$

The universal function (2) represents the asymptotic form of the evolution operators for time intervals 2^m at the Feigenbaum critical point in terms of the rescaled dynamical variable.

To generalize the Feigenbaum–Cvitanović RG transformation for a two-dimensional case it is convenient to assume that the coordinate system is chosen in such way that the scale transformation becomes diagonal: $x \to x/\alpha$ and $y \to y/\beta$. (In general, the scaling coordinates x and y may be distinct from the variables in the original system, and one has to perform an appropriate variable change.) Let us assume that

$$\{x, y\} \mapsto \{G_m(x, y), F_m(x, y)\}, \tag{3}$$

is the evolution operator for 2^m steps of discrete time. The functions $G_m(x, y)$ and $F_m(x, y)$ are supposed to be smooth and obey the conditions $G_m(0, 0) = 1$ and $F_m(0, 0) = 1$.

To construct the RG transformation we apply the operator (3) twice and perform the scale change

$$x \to x/\alpha_m, \qquad y \to y/\beta_m.$$

The new evolution operator will be determined by the functions $G_{m+1}(x, y)$ and $F_{m+1}(x, y)$ as follows:

$$\begin{pmatrix} G_{m+1} \\ F_{m+1} \end{pmatrix} = RG \begin{pmatrix} G_m \\ F_m \end{pmatrix}$$
$$= \begin{pmatrix} \alpha_m G_m(G_m(x/\alpha_m, y/\beta_m), F_m(x/\alpha_m, y/\beta_m)) \\ \beta_m F_m(G_m(x/\alpha_m, y/\beta_m), F_m(x/\alpha_m, y/\beta_m)) \end{pmatrix}, \tag{4}$$

where we select the scaling constants α_m and β_m to normalize the new functions to unity at the origin:

$$\alpha_m = 1/G_m(G_m(0,0), F_m(0,0)), \qquad \beta_m = 1/F_m(G_m(0,0), F_m(0,0)).$$

Critical behavior may be associated with a stationary solution of the RG transformation (4) represented by a fixed point or a periodic orbit (cycle). Note that these cases are not essentially different: Indeed, one can interpret a cycle of period p as a fixed point of the modified RG transformation composed of p steps of the original procedure.

In the case of the fixed point $p = 1$ the equations yield

$$\begin{gathered} \begin{pmatrix} G \\ F \end{pmatrix} = \begin{pmatrix} \alpha G(G(x/\alpha, y/\beta), F(x/\alpha, y/\beta)) \\ \beta F(G(x/\alpha, y/\beta), F(x/\alpha, y/\beta)) \end{pmatrix}, \\ \alpha = 1/G(G(0,0), F(0,0)), \qquad \beta = 1/F(G(0,0), F(0,0)). \end{gathered} \tag{5}$$

The system of functional equations (5) or a similar system for a cycle can be solved numerically. For this purpose, one represents all the functions by polynomial approximations. Then, the functional equations are reformulated as a set of algebraic relations for the approximation coefficients, and the set of equations may be solved numerically, for example, by application of the multidimensional Newton method. As a result, one obtains approximate representations for the functions $\{G(x,y), F(x,y)\}$ and numerical estimates for the scaling constants α and β.

The next step in the RG analysis consists in a study of small perturbations (stability analysis) of a stationary solution. The perturbations appear under variation of one or several parameters in the original system in a neighborhood of the critical point. Let $\{G_m(x,y), F_m(x,y)\}$ be a cycle of period p of the RG equation. Substituting the perturbed functions $\{G_m(x,y) + \varepsilon u_m(x,y), F_m(x,y) + \varepsilon v_m(x,y)\}$ into the equations (4) and neglecting higher-order terms in ε, we get a linear system for the perturbations $u_m(x,y), v_m(x,y)$:

$$\begin{pmatrix} u_{m+1} \\ v_{m+1} \end{pmatrix} = L_m \begin{pmatrix} u_m \\ v_m \end{pmatrix}. \tag{6}$$

Here L_m is the linearized operator of the RG transformation at the stationary solution $\{G_m, F_m\}$ (we do not provide here a cumbersome explicit expression for L_m). Since $L_m = L_{m+p}$, we may look for solutions of (6) in the form $\binom{u_{m+pn}}{v_{m+pn}} = \nu^n \binom{u_m}{v_m}$. Then, ν is an eigenvalue of the operator $L = \prod_{i=1}^{p} L_i = L_p L_{p-1} \dots L_1$, and the set of functions $\binom{u_m}{v_m}$, $m = 1, \dots, p$ defines the eigenvector.

Relevant eigenvalues are those with absolute values larger than one; they correspond to a growth of the perturbations, which appear due to variations of parameters in the original system with a shift from the critical point. (It is necessary to exclude those eigenvalues, which are associated with infinitesimal changes of the dynamical variables.) The number of relevant eigenvalues M determines the *codimension* of the critical point. In a generic case, M equals to a minimal number of parameters, which have to be tuned to reach the critical behavior under consideration. Indeed, requiring the coefficients at M relevant eigenvectors to vanish, we impose exactly M conditions for parameters of the dynamical system.

3 Universal Properties of Behavior of Integral Characteristics at the Onset of Chaos

Let us consider a smooth one-dimensional map

$$x_{n+1} = g(x_n), \quad x_n \in \mathbb{R},\ n = 1, 2, \dots \tag{7}$$

and introduce an additional variable governed by the equation

$$y_{n+1} = y_n + f(x_n), \tag{8}$$

where $f(x)$ is a smooth function. Obviously, $y_n = y_0 + \sum_{i=0}^{n-1} f(x_i)$. Hence, the variable y_n accumulates the sum of values of the function of the variable of the first subsystem. In physical applications, the summation can represent, for example, a total work produced by a system or a total emitted radiation.

The maps (7) and (8) together define a two-dimensional dynamical system consisting of two elements with unidirectional coupling.

Let us assume that the functions $g(x)$ and $f(x)$ depend smoothly on real parameters A and B, respectively, and the Feigenbaum period-doubling transition to chaos takes place in the first subsystem with increase of A. At the critical point of the period-doubling accumulation $A = A_c$ the dynamics of the map (7) possesses universal scaling properties [1, 2]. The parameter B in the second equation (8) in the simplest case may be present as an additive term. We can select a value of this parameter $B = B_c$ to have nontrivial scaling regularities in the dynamics of the second variable y_n too.

Let us consider a series of parameter values A_m at which the map (7) has stable cycles of period $k = 2^m$: $x_1 \to \cdots \to x_k \to x_1 = g(x_k)$ and set $y_0 = 0$. So, $y_{Nk} = N \sum_{i=0}^{k-1} f(x_i)$. If $\sum_{i=0}^{k-1} f(x_i) \neq 0$, then $y_{Nk} \to \infty$ for large N. Let us choose $B = B_m$ to ensure the equality $\sum_{i=0}^{k-1} f(x_i) = 0$. As $m \to \infty$, we have $A_m \to A_c$ and expect that the series B_m converges to a limit value B_c.

Let us apply the RG approach of Sect. 2 to the system (7), (8). In this case

$$G(x, y) = g(x), \quad F(x, y) = y + f(x) \tag{9}$$

and the RG transformation is represented as

$$RG\begin{pmatrix} G \\ F \end{pmatrix} = RG\begin{pmatrix} g(x) \\ y + f(x) \end{pmatrix} = \begin{pmatrix} \alpha g\left(g(x/\alpha)\right) \\ y + \beta\Big(f(x/\alpha) + f(g(x/\alpha))\Big) \end{pmatrix}. \tag{10}$$

Note that the RG transformation does not violate the structure of the two-dimensional map as a system of two elements with unidirectional coupling (9). If the new functions $\alpha g(g(x/\alpha))$ and $\beta(f(x/\alpha) + f(g(x/\alpha)))$ coincide with $g(x)$ and $f(x)$, respectively, this will be a fixed point of the RG transformation. As follows from (10), the fixed-point solution may be constructed with the Feigenbaum function $g(x)$ (see (1) and (2)) and the function $f(x)$ satisfying the equation

$$f(x) = \beta(f(x/\alpha) + f(g(x/\alpha))). \tag{11}$$

From numerical solution of the functional equation (11), we have found $\beta = -4.586197$ and

$$f(x) = 1 - 2.4410x^2 + 0.0987x^4 + 0.1445x^6 - 0.0207x^8 + 0.0002x^{10} + \cdots. \quad (12)$$

Perturbation analysis of the fixed point with application of (9), (2), (12) reveals four relevant modes with eigenvalues $|\nu| \geq 1$ and eigenfunctions $\{u, v\}$:[2]

$$\begin{aligned}
&\nu_1 = 2\beta = -9.1724, \quad u_1(x,y) = 0, \quad v_1(x,y) = 1 - f(x);\\
&\nu_2 = \delta = 4.6692, \quad u_2(x,y) = h(x), \quad v_2(x,y) = w(x);\\
&\nu_3 = 2, \quad u_3(x,y) = 0, \quad v_3(x,y) = y + z(x);\\
&\nu_4 = \delta/\beta = -1.0181, \quad u_4(x,y) = yp(x) + q(x), \quad v_4(x,y) = yr(x) + s(x),
\end{aligned} \quad (13)$$

where

$$\begin{aligned}
h(x) &= 1.2642x^2 - 0.1800x^4 - 0.1010x^6 + 0.0174x^8 - 0.0004x^{10}\\
&\quad - 0.0002x^{12} + \cdots,\\
w(x) &= 1.2183x^2 + 0.0346x^4 - 0.4959x^6 + 0.0943x^8 - 0.0007x^{10}\\
&\quad - 0.0020x^{12} + \cdots,\\
z(x) &= -0.8430x^2 - 0.0194x^4 + 0.1394x^6 - 0.0210x^8 + 0.0009x^{10} + \cdots,\\
p(x) &= -2.6480 + 0.8623x^2 + 0.1339x^4 - 0.0386x^6 + 0.0025x^8\\
&\quad + 0.0001x^{10} + \cdots,\\
q(x) &= 0.9500x^2 + 0.2131x^4 - 0.1843x^6 + 0.0212x^8 + 0.0001x^{10}\\
&\quad - 0.0002x^{12} \cdots,\\
r(x) &= -4.1186 + 1.8551x^2 + 0.7732x^4 - 0.2247x^6 + 0.0132x^8\\
&\quad + 0.0031x^{10} - 0.0005x^{12} \cdots,\\
s(x) &= -1.4507x^2 + 5.4327x^4 - 1.7091x^6 + 0.0796x^8 + 0.0216x^{10}\\
&\quad - 0.0033x^{12} + \cdots.
\end{aligned} \quad (14)$$

The mode associated with ν_1 gives rise to appearance of an x-dependent additional term in the second map, presence of which causes an unbounded growth for y_n with n. The eigenvalue $\nu_2 = \delta = 4.6692016$ and the function $h(x)$ are known from the Feigenbaum theory. They correspond to parameter variation that gives rise to the transition to chaos through period-doubling cascade in the first map. The mode with $\nu_3 = 2$ is responsible for variation of the coefficient at the term y in the second map. Finally, the mode with $\nu_4 = \delta/\beta$ breaks the unidirectional nature of coupling by introducing dependence on the variable y in the first map. In contrast to the last mode, the perturbations associated with ν_1, ν_2 and ν_3 preserve the structure of the mapping as a system with unidirectional coupling.

Consider an orbit of the map (7) (with the function (2)) starting at $x_0 = 0$. This orbit belongs to the attractor at the onset of chaos [1, 2]. By using equation (1), we find

$$\begin{aligned}
x_0 = 0, \quad x_1 = g(0) = 1, \quad x_1 = g^2(0) = g(0)/\alpha = 1/\alpha,\\
x_2 = g(x_2) = g(g(0)/\alpha) = g(1/\alpha), \quad \ldots.
\end{aligned} \quad (15)$$

[2]From this list we exclude eigenvalues associated with infinitesimal coordinate changes: $\nu_5 = \alpha = -2.5029$ corresponds to the variable change $x \to x + \varepsilon(1-x)$ and $\nu_6 = \beta/\alpha = 1.8323$ to $y \to y + \varepsilon(x - y)$.

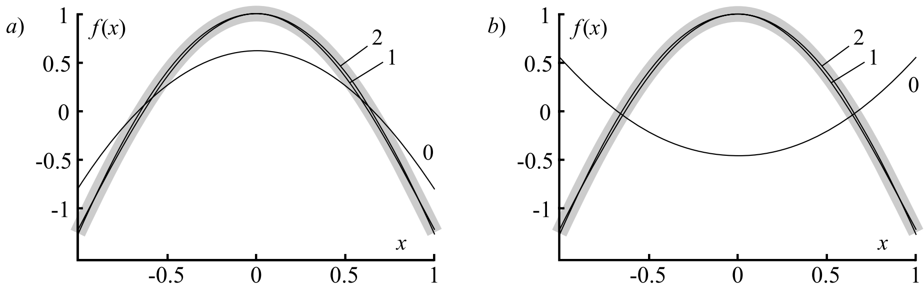

Fig. 1 Convergence of integral characteristics to the universal function for the logistic map

Then, as follows from (11),

$$\begin{aligned} 1 = f(0) &= \beta(f(0) + f(g(0))) = \beta(f(x_0) + f(x_1)) \\ &= \beta^2(f(0) + f(g(0)) + f(g(0)/\alpha) + f(g(g(0)/\alpha))) \\ &= \beta^2(f(x_0) + f(x_1) + f(x_2) + f(x_3)) \\ &\vdots \\ &= \beta^m \sum_{i=0}^{2^m-1} f(x_i) = \beta^m y_{2^m}. \end{aligned} \quad (16)$$

We see that $y_{2^m} = 1/\beta^m$, so the second variable y_n is bounded for large n. (Recall that this property was used in the definition of the critical value $B = B_c$ for the system (7), (8).)

To present a concrete example, let us assume that the first subsystem is governed by the logistic map $x_{n+1} = 1 - Ax_n^2$ at the parameter value corresponding to the period-doubling accumulation point $A = A_c = 1.401155189092$. In the second map we set $f(x) = x - B$. The critical value of parameter B in this case may be estimated as $B_c = \lim_{m\to\infty}(1/2^m)\sum_{i=0}^{2^m-1} x_i$, where $x_0 = 0$. The computations yield $B_c = 0.3760720508$. In order to observe convergence of the two-dimensional map to the universal form (9), we perform a variable change for the second coordinate, $y = \tilde{y} - x$ (this excludes the instability related to the irrelevant eigenvalue ν_5). Then the second map becomes $\tilde{y}_{n+1} - x_{n+1} = \tilde{y}_n - B_c$, or $\tilde{y}_{n+1} = \tilde{y}_n + 1 - A_c x_n^2 - B_c$.

Figure 1(a) illustrates convergence of the function $f(x)$ to the universal form (12) under the multiple action of the RG transformation. The plot of the universal function is shown by the wide grey line. One can see that already under one or two iterations the deflections from the universal function become hardly visible.

The variable y_n may be interpreted as a sum of deviations from the average value B_c for all preceding states of the first subsystem x_n. The universality implies that the sum of the deviations at the 2^m-th step coincides asymptotically with the analogous quantity at the 2^{m+1}-th step under the renormalization with the factors $1/\alpha$ for x and $1/\beta$ for y. For large m, these quantities are expressed via the universal function (12). Figure 1(b) demonstrates analogous convergence to the universal function in the case of $f(x) = x^2 - B$, at the critical value of the parameter $B_c = 0.4452953917$.

4 Bifurcation of the Fixed Point of the RG Transformation

Now let us consider the above studied fixed point of the RG transformation (see (9), (2), (12)) in the space of general two-dimensional maps $x \mapsto G(x, y)$, $y \mapsto F(x, y)$. Obviously, it defines a universality class of codimension 4, since there are four relevant eigenvalues (13). Let us denote this universality class as FS (F stands for the Feigenbaum universality of the first map, and S for the summation nature of the second map).

Notice that one of the relevant eigenvalues is very close to -1: $\nu_4 = \delta/\beta = -1.018$. This indicates that the fixed point of the RG transformation is about to bifurcate (the eigenvalue -1 just corresponds to a period-doubling bifurcation). A natural idea is to modify the RG transformation by introducing some special parameter to have the bifurcational situation at a definite value of this parameter. In this bifurcation, a period-2 solution will appear near the fixed point of type FS.

In order to find the desired modification of the problem, let us turn to one of the generalizations of the Feigenbaum theory discussed in the literature. It concerns one-dimensional maps with a single extremum of degree κ: $x_{n+1} = 1 - \lambda|x_n|^\kappa$, where $\kappa > 1$ is a real parameter, and $\kappa = 2$ corresponds to the original Feigenbaum case. It turns out that the transition to chaos in such maps is qualitatively of the same nature as in the quadratic map (a period-doubling cascade), but the constants α and δ are distinct and depend on κ [18–20]. We conclude that κ is an appropriate parameter for the required modification of the problem.

The FS fixed-point solution of the two-dimensional RG equation can be rewritten as $G(x, y) = g(x) = \tilde{g}(x^2)$, $F(x, y) = y + f(x) = y + \tilde{f}(x^2)$, where $\tilde{g}$ and $\tilde{f}$ are smooth functions. Consider a more general form of the solution, namely,

$$G_\kappa(x, y) = \tilde{g}(|x|^\kappa), \qquad F_\kappa(x, y) = y + \tilde{f}(|x|^\kappa). \tag{17}$$

The particular $\kappa = 2$ corresponds to the previously studied case. The functions $\tilde{g}(|x|^\kappa)$ and $\tilde{f}(|x|^\kappa)$ are assumed to have an extremum at $x = 0$; they are once-differentiable functions of x for $1 \le \kappa < 2$ and twice-differentiable for $\kappa \ge 2$. With variation of the parameter κ, the functions $\tilde{g}$ and $\tilde{f}$, determining the fixed point of the RG equation evolve, and the constants δ and $-\beta$ as well as the eigenvalues of the linearized RG transformation depend on κ.

We computed the constants δ and β numerically at different κ. One approach is to find δ by means of the Feigenbaum method (limit of ratios for subsequent bifurcational values of the control parameter), and to estimate β by using the algorithm described in Sect. 2. The second approach is based on the numerical solution of the two-dimensional RG equation. As checked, the both methods lead to the same results.

In Fig. 2, the plots of δ and $-\beta$ versus κ are shown. At $\kappa_c = 1.984396$ we have $\delta = -\beta = 4.644444$ and, hence, $\nu_4 = \delta/\beta = -1$. Therefore, κ_c corresponds to a period-doubling bifurcation in the RG equation. At lower values, $\kappa < \kappa_c$, the eigenvalue becomes irrelevant: $|\nu_4| < 1$. In that region, codimension of the universality class associated with the fixed point (17) equals 3.

5 The Birth of a New Universality Class

It is not so convenient to deal with the RG transformation we have now: it does not depend on κ, but the space, in which it acts, does (the parameter κ determines the type of singularity of solution (17) at $x = 0$). It is possible to reformulate the problem in such way that κ will appear explicitly in the RG transformation, which acts in the space of smooth two-dimensional

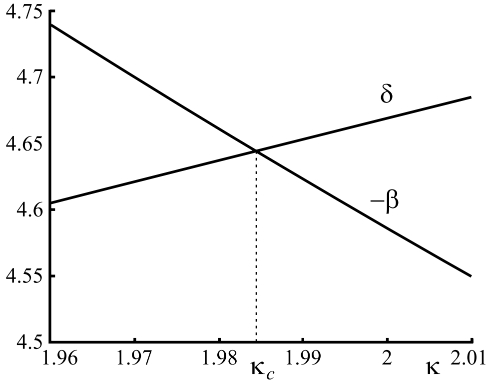

Fig. 2 Eigenvalue δ and renormalization constant β versus the degree of extremum κ

maps. For this purpose, we introduce the functions $\tilde{G}_\kappa$ and $\tilde{F}_\kappa$ by the expressions

$$G_\kappa(x, y) = \tilde{G}_\kappa(|x|^\kappa, y), \qquad F_\kappa(x, y) = \tilde{F}_\kappa(|x|^\kappa, y). \tag{18}$$

In terms of these new functions, the fixed point solution (17) will be represented as

$$\tilde{G}_\kappa(x, y) = \tilde{g}(x), \qquad \tilde{F}_\kappa(x, y) = y + \tilde{f}(x), \tag{19}$$

where $\tilde{g}$ and $\tilde{f}$ are smooth functions. By substituting (18) into (4), one obtains an equivalent form for the RG transformation:

$$\begin{gathered} RG_\kappa \begin{pmatrix} \tilde{G}_\kappa \\ \tilde{F}_\kappa \end{pmatrix} = \begin{pmatrix} \alpha \tilde{G}_\kappa(|\tilde{G}_\kappa(x/|\alpha|^\kappa, y/\beta)|^\kappa, \tilde{F}_\kappa(x/|\alpha|^\kappa, y/\beta)) \\ \beta \tilde{F}_\kappa(|\tilde{G}_\kappa(x/|\alpha|^\kappa, y/\beta)|^\kappa, \tilde{F}_\kappa(x/|\alpha|^\kappa, y/\beta)) \end{pmatrix}, \\ \alpha = 1/\tilde{G}_\kappa(|\tilde{G}_\kappa(0,0)|^\kappa, \tilde{F}_\kappa(0,0)), \qquad \beta = 1/\tilde{F}_\kappa(|\tilde{G}_\kappa(0,0)|^\kappa, \tilde{F}_\kappa(0,0)). \end{gathered} \tag{20}$$

It is easy to check that the new operator RG_κ transforms smooth functions $\tilde{G}_\kappa$, $\tilde{F}_\kappa$ in some neighborhood of the fixed point (19) into smooth functions. (Singularities associated with the non-integer power κ appear outside the domain of interest $|x| < 1$, $|y| < 1$.) Now the operator RG_κ depends smoothly on κ, and $\tilde{G}_\kappa$, $\tilde{F}_\kappa$ are smooth functions for all κ.

At the bifurcation, for $\kappa = \kappa_c$, the fixed point of the RG transformation (19) is represented by the functions

$$\begin{aligned} \tilde{g}(x) &= 1 - 1.5233x + 0.1038x^2 + 0.0257x^3 - 0.0033x^4 + 0.0001x^5 + \cdots, \\ \tilde{f}(x) &= 1 - 2.4366x + 0.1003x^2 + 0.1403x^3 - 0.0198x^4 + 0.0004x^5 + \cdots, \end{aligned} \tag{21}$$

which were estimated numerically.

The eigenfunctions $\tilde{u}(x,y) = y\tilde{p}(x) + \tilde{q}(x)$ and $\tilde{v}(x,y) = y\tilde{r}(x) + \tilde{s}(x)$ corresponding to the critical eigenvalue $\nu_4 = -1$ of the linearized operator RG_κ, are expressed via

$$\begin{aligned} \tilde{p}(x) &= -2.6331 + 0.8555x + 0.1285x^2 - 0.0367x^3 + 0.0025x^4 \\ &\quad + 0.0001x^5 + \cdots, \\ \tilde{q}(x) &= 0.9407x + 0.2196x^2 - 0.1802x^3 + 0.0199x^4 - 0.0002x^7 + \cdots, \\ \tilde{r}(x) &= -4.0983 + 1.8633x + 0.7482x^2 - 0.2168x^3 + 0.0145x^4 \\ &\quad + 0.0016x^5 + \cdots, \\ \tilde{s}(x) &= -1.7429x + 5.8275x^2 - 1.7503x^3 + 0.0692x^4 + 0.0209x^5 \\ &\quad - 0.0024x^7 + \cdots. \end{aligned} \tag{22}$$

The period-doubling bifurcation at κ_c gives rise to a periodic solution (cycle) of period 2:

$$(\tilde{G}^+_\kappa, \tilde{F}^+_\kappa) \xrightarrow{RG_\kappa} (\tilde{G}^-_\kappa, \tilde{F}^-_\kappa) \xrightarrow{RG_\kappa} (\tilde{G}^+_\kappa, \tilde{F}^+_\kappa). \tag{23}$$

For the functions $\tilde{G}^\pm_\kappa(x,y)$ and $\tilde{F}^\pm_\kappa(x,y)$ one can write down the asymptotic expansions as follows [21]:

$$\begin{aligned} \tilde{G}^\pm_\kappa(x,y) &= \tilde{G}_\kappa(x,y) \pm \chi \tilde{G}^{(1)}(x,y) + \chi^2 \tilde{G}^{(2)}(x,y) \pm \chi^3 \tilde{G}^{(3)}(x,y) \\ &\quad + \chi^4 \tilde{G}^{(4)}(x,y) \pm \cdots, \\ \tilde{F}^\pm_\kappa(x,y) &= \tilde{F}_\kappa(x,y) \pm \chi \tilde{F}^{(1)}(x,y) + \chi^2 \tilde{F}^{(2)}(x,y) \pm \chi^3 \tilde{F}^{(3)}(x,y) \\ &\quad + \chi^4 \tilde{F}^{(4)}(x,y) \pm \cdots, \end{aligned} \tag{24}$$

where $\chi = \sqrt{c(\kappa - \kappa_c)}$, and c is a real constant. In the formula (24), the functions $\tilde{G}_\kappa$, $\tilde{F}_\kappa$ are evaluated at the bifurcation point, and $\tilde{G}^{(1)}(x,y) = \tilde{u}(x,y)$ and $\tilde{F}^{(1)}(x,y) = \tilde{v}(x,y)$ are the eigenfunctions corresponding to the critical eigenvalue $\nu_4 = -1$. We choose the normalization condition for the eigenfunctions as $\tilde{u}(1,0) = \tilde{h}(1) = 1$. With account of (21) and (22) the constant may be evaluated from the perturbation theory (Appendix 1):

$$c = 1.607. \tag{25}$$

Since $c > 0$, the bifurcation is supercritical, i.e., the period-2 cycle appears for $\kappa > \kappa_c$.

By using the first-order approximation (24), (25), we obtained the functions $\tilde{G}^\pm_\kappa$, $\tilde{F}^\pm_\kappa$ for small positive $\kappa - \kappa_c$. For larger values of $\kappa - \kappa_c$, we can find out them numerically by continuation. Results of these computations are presented in Fig. 3, where the bold line shows the first-order approximation (24), and the thin line corresponds to the data of numerical computations. The numerical analysis was carried out up to the value of $\kappa = 2$, where the period-2 solution of the RG equation has been found. By using (18), this periodic solution can be rewritten in the original form: $G^\pm(x,y) = \tilde{G}^\pm_\kappa(x^2,y)$, $F^\pm(x,y) = \tilde{F}^\pm_\kappa(x^2,y)$; see Fig. 4.

The approximations of the universal functions and constants with the lowest-order term proportional to $\sqrt{\kappa - \kappa_n}$ are not very accurate at $\kappa = 2$. On the other hand, analytical expressions for higher-order terms are very cumbersome. For that reason, we have developed a numerical method for computing polynomial approximations for the functions constituting the expansions (24) (see Appendix 2).

Figure 5 shows the scaling factors α_* and β_* versus κ for the period-2 cycle; dotted and dashed lines represent the first and the fourth order approximations, respectively. Note

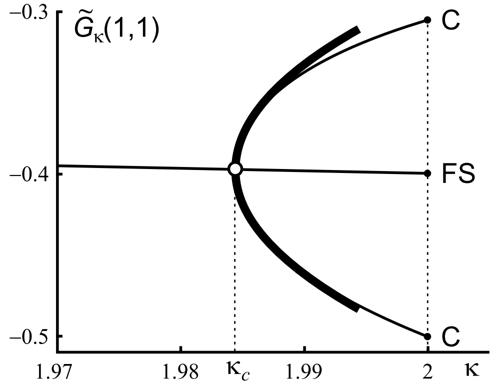

Fig. 3 Bifurcation diagram for the modified RG_κ transformation

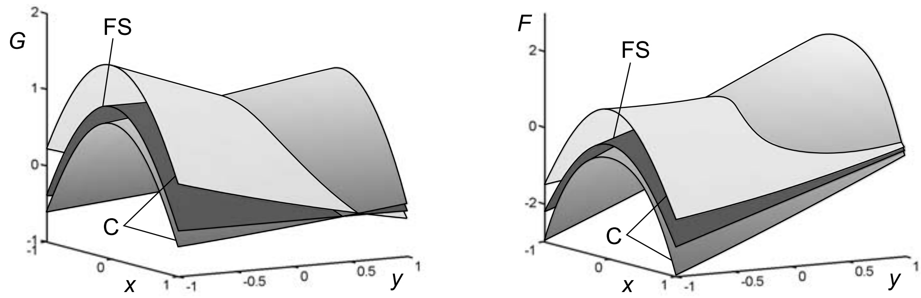

Fig. 4 Fixed point solution of type FS and period-2 solution corresponding to type C criticality

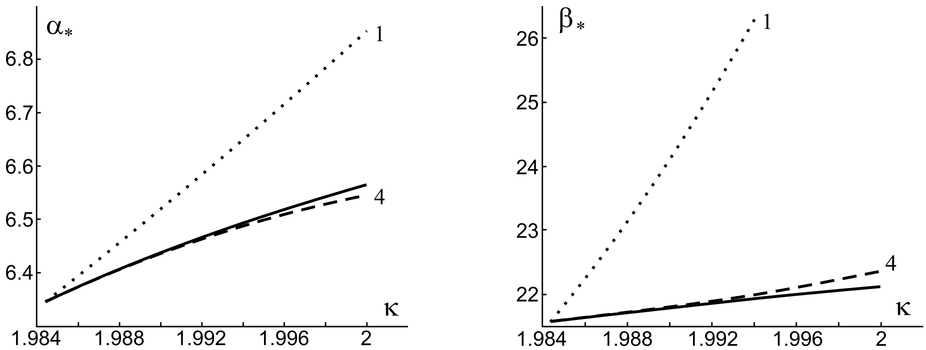

Fig. 5 Scaling factors of the period-2 stationary solution

that the first-order corrections are useless for evaluating corrections to the scaling factors: they cancel each other due to the opposite signs in (24). In Fig. 6, analogous results are presented for the relevant eigenvalues δ_1 and δ_2. Note that the fourth-order approximations demonstrate good agreement with the accurate numerical calculations (solid lines).

Analysis of the data at $\kappa = 2$ shows that the obtained period-2 solution corresponds to the universal critical behavior detected earlier in Refs. [12, 16, 22] for two-dimensional irreversible maps and denoted by C (abbreviation of "Cycle"). Table 1 contains the numerical data for scaling factors, eigenvalues, and multipliers of periodic solutions evaluated at

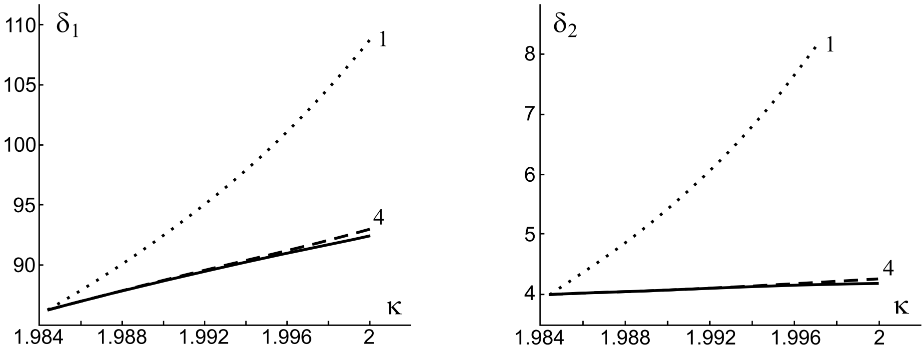

Fig. 6 Relevant eigenvalues of the period-2 stationary solution

Table 1 Universal constants of the period-2 stationary solution of the RG equation at $\kappa = 2$

Universal constant	Fourth-order approximation	Exact value
α_*	6.5452	6.56534993
β_*	22.3701	22.12022737
δ_1	93.0037	92.43126367
δ_2	4.2639	4.19244406
$\mu_1^{(1)}$	0.8501	0.84745051
$\mu_2^{(1)}$	−0.8150	−0.72525604
$\mu_1^{(2)}$	1.2019	1.17445893
$\mu_2^{(2)}$	−0.8667	−0.84886546

$\kappa = 2$. The results, obtained by using the fourth-order approximation, are compared to the quantities corresponding to the type C criticality [12, 16, 22].

The constants δ_1 and δ_2 are relevant eigenvalues of the linearized RG transformation defined over the complete period of the stationary solution (period 2) and correspond to a quadrupling of the time scale. These constants determine regularities of departure from the stationary solution under multiple action of the RG transformation for the evolution operators with the variation of parameters in the original map near the critical point. Additionally, there are three irrelevant eigenvalues with absolute values greater than one:

$$\delta_3 = \beta_* = 22.1202, \qquad \delta_4 = \alpha_* = 6.5653, \qquad \delta_5 = \beta_*/\alpha_* = 3.3692,$$

which correspond to infinitesimal coordinate changes, respectively,

$$y \mapsto y + \varepsilon, \qquad x \mapsto x + \varepsilon, \qquad x \mapsto y + \varepsilon x.$$

The universality class of C-type is of codimension 2 [11, 16, 17]. On the other hand, in accordance with our analysis, the universality class FS has codimension 4. Thus, the codimension decreases by 2 as a result of the doubling bifurcation in the RG equation. This surprising fact is linked with irrelevance of the eigenvalue δ_3 for the universality class C. At the bifurcation point κ_c, this eigenvalue coincides with the squared relevant eigenvalue ν_2 of the class FS. Thus, the transformation of the relevant eigenvalue ν_2 into the irrelevant eigenvalue δ_3 takes place.

This phenomenon can be described as follows. The eigenfunction, corresponding to the irrelevant eigenvalue δ_3, is associated with an infinitesimal shift along the second coordinate: $y \mapsto y + \varepsilon$. This shift does not influence the first terms $\tilde{G}_\kappa$, $\tilde{F}_\kappa$ in expansions (24), which have the form (19). Thus, the perturbation of the map under the coordinate shift is of order $O(\varepsilon\chi)$. Hence, at the bifurcation point κ_c, at which $\chi = 0$, the relation of the eigenvalue δ_3 to the coordinate shift is lost.

6 Conclusion

In this paper, we applied the bifurcation analysis to the RG equation describing critical behavior at the onset of chaos (two-dimensional generalization of the Feigenbaum–Cvitanović equation). It is shown that the fixed-point solution of this RG equation relating to two subsystems with unidirectional coupling (a unimodal map with an extremum of degree κ and a map accumulating the sum of function values at the states of the first subsystem), with variation of κ undergoes a bifurcation. This bifurcation gives rise to a period-2 stationary solution of the RG equation. At $\kappa = 2$ this solution corresponds to the universal critical behavior of type C at the onset of chaos, which was detected and studied in Refs. [12, 16]. Therefore, we establish the origin of the type C criticality through the bifurcation of the RG equation.

By using analytical and numerical methods, we obtained the asymptotic expansion of the period-2 cycle solution as a function of κ near the critical value $\kappa_c = 1.984396$. The fact that the stationary solution is close to the bifurcation point, explains a number of characteristic features of the type C criticality. First, it reveals the nature of the solution as a period-2 cycle. Then, it explains an approximate equality of the universal constants for the type C criticality to certain combinations of Feigenbaum's constants. Finally, the reason for a slow convergence typically intrinsic to the regularities of universality and scaling associated with the type C criticality becomes clear, as a consequence of the closeness to the bifurcation situation.

The study we undertake in the present article is analogous to an approach in the theory of phase transitions known as ϵ-expansion. The idea of this approach consists in the analysis of critical phenomena in dependence on space dimension treated as a continuous parameter. In this way, a "trivial" fixed point of the RG transformation undergoes a bifurcation, and a new stationary solution appears responsible for the universal critical behavior describing a phase transition. In our study, instead of the space dimension, we consider a degree of extremum of the map, and the solution of the RG equation relating to the system of the unidirectionally coupled two maps appears as an analog of the trivial fixed point. Then, the period-2 cycle arising in the bifurcation corresponds to the non-trivial stationary solution of the RG equation.

As pointed out in Refs. [22–24] the type ? criticality may be expected in a wide class of systems in a situation, when variation of one parameter gives rise to period doublings, and variation of another one to a saddle-node bifurcation.

A particular example is a Rössler system under external periodic driving [22]. In the space of three parameters (the control parameter of the period doubling, and the amplitude and frequency of the external force), the synchronization domain (referred to as the Arnold tongue) is bounded by two surfaces of the saddle-node bifurcations. Each period-doubling bifurcation surface has an edge curve at the intersection with the border of the Arnold tongue. These curves associated with the successive period doublings converge to a limit, the curve of the C-type criticality.

As known, the Rössler oscillator manifests dynamical behavior typical for a wide class of low-dimensional dissipative chaotic systems. It means that the dynamical properties analogous to those found in the forced Rössler oscillator will occur in other systems of this class under external periodic driving. It may be expected, hence, that the critical behavior of C-type could be observed in carefully organized experiments on synchronization of period doubling dissipative systems (e.g. convective systems, electronic oscillators, etc.). As may be conjectured, this is a universal attribute of the synchronization breakup corresponding to the limit of period-doubling at the edge of Arnold tongue. Of course, in the experiments only a finite number of levels of the intrinsic hierarchical structures in the phase space and in the parameter space will be observable because of a finite resolution of the measurements and of presence of an inevitable noise.

We hope that the bifurcation analysis of the RG equations may be productive as a general constructive method for search and study of different critical behaviors in multiparameter chaotic systems. Beside the period-doubling route to chaos, it is reasonable to expect similar phenomena in other situations of transitions to chaos, e.g. through intermittency and quasiperiodicity.

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Appendix 1

We can represent the functions $\tilde{G}_\kappa, \tilde{F}_\kappa$ by a vector of their values $\mathbf{x} = (x_1, \ldots, x_n)^T \in \mathbb{R}^n$ on a grid of interpolation nodes, and by a definite interpolation procedure with respective interpolation polynomials between the nodes. Then, we come to a discrete approximation of the RG transformation acting in the space of finite-dimensional vectors $\mathbf{x} \mapsto RG_\kappa(\mathbf{x})$.

Let $\mathbf{x}_c = RG_\kappa(\mathbf{x}_c)$ be a stationary solution of the RG equation at the period-doubling bifurcation point $\kappa = \kappa_c$. The Jacobian matrix $\mathbf{L} = \partial RG_\kappa/\partial \mathbf{x}$ evaluated at $\mathbf{x} = \mathbf{x}_c$, $\kappa = \kappa_c$ has the eigenvalue $\nu = -1$ (and this is the only eigenvalue with the unit absolute value). The right and the left eigenvectors $\mathbf{r}$ and $\mathbf{l}$ corresponding to $\nu = -1$ are determined by the equations $\mathbf{Lr} = -\mathbf{r}$ and $\mathbf{l}^T\mathbf{L} = -\mathbf{l}^T$. As a result of the bifurcation, a period-2 cycle $\mathbf{x}^+ \to \mathbf{x}^- = RG_\kappa(\mathbf{x}^+) \to \mathbf{x}^+ = RG_\kappa(\mathbf{x}^-)$ appears for parameter values in the neighborhood of κ_c. The vectors $\mathbf{x}^+$ and $\mathbf{x}^-$ can be represented as expansions [21]:

$$\mathbf{x}^\pm = \mathbf{x}_c \pm \chi\mathbf{x}^{(1)} + \chi^2\mathbf{x}^{(2)} \pm \chi^3\mathbf{x}^{(3)} + \cdots, \qquad \chi = \sqrt{c(\kappa - \kappa_c)}. \tag{26}$$

Sign of the real constant c determines a type of the bifurcation: supercritical for $c > 0$ (the period-2 cycle exists for $\kappa > \kappa_c$), and subcritical for $c < 0$ (the period-2 cycle exists for $\kappa < \kappa_c$).

The constant c may be estimated from a standard bifurcation analysis technique. We substitute (26) into the equation $RG_\kappa^2(\mathbf{x}^+) = \mathbf{x}^+$ for the period-2 cycle, where the functions are represented by Taylor expansions, and compare the coefficients of equal powers of $\sqrt{\kappa - \kappa_c}$. The equation for the first-order terms yields $\mathbf{x}^{(1)} = \mathbf{r}$. Comparing the second-order terms, we obtain the vector $\mathbf{x}^{(2)}$ in the form

$$\begin{aligned} &\mathbf{x}^{(2)} = (\mathbf{I} - \mathbf{L})^{-1}(\mathbf{y}_2 + \mathbf{y}_\kappa/c) + \alpha\mathbf{r}, \\ &\mathbf{y}_2 = \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 RG_\kappa}{\partial x_i \partial x_j} r_i r_j, \qquad \mathbf{y}_\kappa = \frac{\partial RG_\kappa}{\partial \kappa}, \end{aligned} \tag{27}$$

where $\mathbf{I}$ is the identity matrix, α is an unknown factor, r_i is the i-th component of the vector $\mathbf{r}$, and the derivatives are taken at $\mathbf{x} = \mathbf{x}_c$, $\kappa = \kappa_c$. Finally, the equation for the third order terms, multiplied by the vector $\mathbf{l}^T$, yields the unknown constant c as

$$c = -\frac{\mathbf{l}^T \left(D_2(\mathbf{r}, (\mathbf{I}-\mathbf{L})^{-1}\mathbf{y}_\kappa) + \mathbf{y}_{1\kappa}\right)}{\mathbf{l}^T \left(D_2(\mathbf{r}, (\mathbf{I}-\mathbf{L})^{-1}\mathbf{y}_2) + \mathbf{y}_3\right)}, \tag{28}$$

where

$$D_2(\mathbf{r}, \mathbf{w}) = \sum_{i,j=1}^{n} \frac{\partial^2 RG_\kappa}{\partial x_i \partial x_j} r_i w_j,$$
$$\mathbf{y}_{1\kappa} = \sum_{i=1}^{n} \frac{\partial^2 RG_\kappa}{\partial x_i \partial \kappa} r_i, \qquad \mathbf{y}_3 = \frac{1}{6} \sum_{i,j,k=1}^{n} \frac{\partial^3 RG_\kappa}{\partial x_i \partial x_j \partial x_k} r_i r_j r_k. \tag{29}$$

In numerical computations, analytic formulae for the linearization of the RG_κ operator (the matrix $\mathbf{L}$) were used (see also (6)). Derivatives of the second and the third order were calculated numerically via finite differences. As a result, we obtain $c = 1.607$.

Table 2 Coefficients of polynomial expansions (30) for the maps $\tilde{G}_\kappa^{\pm}$

$\tilde{G}_\kappa$	1	x	x^2	x^3	x^4	x^5	x^6	x^7
1	1.0000	−1.5233	0.1038	0.0257	−0.0033	0.0001	0.0000	0.0000
$\tilde{G}^{(1)}$	1	x	x^2	x^3	x^4	x^5	x^6	x^7
1	0	1.1923	0.2784	−0.2283	0.0252	0.0002	−0.0003	0.0000
y	−3.3373	1.0844	0.1627	−0.0462	0.0028	0.0003	−0.0001	0.0000
$\tilde{G}^{(2)}$	1	x	x^2	x^3	x^4	x^5	x^6	x^7
1	0	10.3376	−4.6218	0.1091	0.0794	−0.0144	0.0007	0.0001
y	−11.3919	3.9357	−1.1647	0.2021	−0.0014	−0.0041	0.0005	0.0000
y^2	2.1128	0.2376	−0.2247	0.0271	0.0013	−0.0006	0.0000	0.0000
$\tilde{G}^{(3)}$	1	x	x^2	x^3	x^4	x^5	x^6	x^7
1	0	−16.0014	−0.0440	7.1634	−1.3962	0.0647	0.0152	−0.0024
y	35.2183	−27.2956	−4.4609	2.1916	−0.2756	0.0011	0.0040	0.0000
y^2	14.4261	−2.5007	0.2422	0.0341	−0.0224	0.0033	0.0000	0.0000
y^3	−0.0629	−0.4572	0.1126	−0.0002	−0.0033	0.0000	0.0000	0.0000
$\tilde{G}^{(4)}$	1	x	x^2	x^3	x^4	x^5	x^6	x^7
1	0	−250.886	135.838	−14.306	−0.833	0.601	−0.093	0.0000
y	265.233	−139.381	54.163	−13.140	0.675	0.268	−0.051	0.0000
y^2	−23.228	−19.108	14.140	−2.327	−0.014	0.055	0.000	0.0000
y^3	−3.843	−0.985	0.453	−0.087	0.009	0.000	0.000	0.0000
y^4	−0.325	0.216	−0.016	−0.009	0.000	0.000	0.000	0.0000

Appendix 2

The period-2 stationary solution in the neighborhood of the bifurcation point $\kappa_c = 1.9843964$ can be represented as an expansion

$$\{\tilde{G}_\kappa^\pm, \tilde{F}_\kappa^\pm\} = \{\tilde{G}_\kappa, \tilde{F}_\kappa\} \pm \chi\{\tilde{G}^{(1)}, \tilde{F}^{(1)}\} + \chi^2\{\tilde{G}^{(2)}, \tilde{F}^{(2)}\} \pm \chi^3\{\tilde{G}^{(3)}, \tilde{F}^{(3)}\} + \cdots \quad (30)$$

with $\chi = \sqrt{\kappa - \kappa_c}$. Here $\{\tilde{G}_\kappa, \tilde{F}_\kappa\}$ corresponds to the fixed point of the operator RG_κ at the bifurcation point, and $\tilde{G}^{(1)}, \tilde{F}^{(1)}, \tilde{G}^{(2)}, \tilde{F}^{(2)}, \ldots$ are unknown functions. Here we prefer to use a form of expansions, which differs from that in (26) by absence of the factor c.

The formulae of Appendix 1 relate to the first-order correction terms in the expansions (30). Unfortunately, application of the analytical bifurcation theory for the second and higher order terms becomes complicated and cumbersome. Hence, for finding these higher-order expansions we prefer to develop a numerical approach.

For approximation of the unknown functions we use Chebyshev interpolation polynomials. Values of the functions were evaluated at the points corresponding to zeros of Chebyshev polynomials. In calculations, we used the grid 9×11 in the (x, y)-plane.

The approximate RG equation was solved by means of the multidimensional Newton method. Then, by a standard continuation technique, we computed the solutions of the RG

Table 3 Coefficients of polynomial expansions (30) for the maps $\tilde{F}_\kappa^\pm$

$\tilde{F}_\kappa$	1	x	x^2	x^3	x^4	x^5	x^6	x^7
1	1	−2.4366	0.1003	0.1402	−0.0197	0.0003	0.0002	0.0000
y	1	0	0	0	0	0	0	0
$\tilde{F}^{(1)}$	1	x	x^2	x^3	x^4	x^5	x^6	x^7
1	0	−2.2089	7.3860	−2.2184	0.0870	0.0288	−0.0041	−0.0003
y	−5.1943	2.3618	0.9470	−0.2726	0.0162	0.0035	−0.0005	0.0000
$\tilde{F}^{(2)}$	1	x	x^2	x^3	x^4	x^5	x^6	x^7
1	0	35.7683	−52.4970	7.6173	1.6843	−0.4273	0.0087	0.0079
y	−18.3448	33.6336	−16.3566	1.1736	0.3819	−0.0774	−0.0001	0.0019
y^2	5.0091	1.7337	−1.3104	0.1700	0.0182	−0.0060	−0.0001	0.0001
$\tilde{F}^{(3)}$	1	x	x^2	x^3	x^4	x^5	x^6	x^7
1	0	−34.2483	−2.2272	55.4843	−15.4733	0.7247	0.3685	−0.0673
y	60.9298	−119.6105	12.3885	19.6307	−5.6060	0.1734	0.1410	−0.0241
y^2	57.8457	−42.1787	4.6398	2.2900	−0.6054	0.0148	0.0184	−0.0031
y^3	0.4640	−2.6362	0.7223	0.0277	−0.0328	0.0014	0.0000	0.0000
$\tilde{F}^{(4)}$	1	x	x^2	x^3	x^4	x^5	x^6	x^7
1	0	−686.919	1226.006	−388.275	−23.576	21.866	−2.842	−0.156
y	455.974	−1033.898	675.544	−128.217	−5.386	6.657	−1.090	−0.025
y^2	−10.469	−114.529	102.617	−30.080	0.996	1.161	−0.223	0.000
y^3	−42.472	6.997	7.832	−2.673	0.128	0.100	−0.023	0.000
y^4	−1.861	1.381	−0.067	−0.101	0.013	0.000	0.000	0.000

equation depending on κ, at about 150 values of the parameter $\chi = \sqrt{\kappa - \kappa_c}$. The positive and negative values of χ were associated with two different solution branches. By taking a finite number of terms (≈20) in the approximation (30), we used the numerical data for evaluating the unknown functions (by means of the method of the least squares). The coefficients corresponding to lower order terms, which show good convergence, yield the functions $\tilde{G}_\kappa$, $\tilde{F}_\kappa$, $\tilde{G}^{(1)}$, $\tilde{F}^{(1)}$, $\tilde{G}^{(2)}$, $\tilde{F}^{(2)}, \ldots$ at the interpolation nodes.

The obtained numerical data for the approximate functions are summarized in Tables 2 and 3 presenting coefficients for the expansions in powers of x and y. Note that the polynomial representations for $\tilde{G}^{(m)}$ and $\tilde{F}^{(m)}$ contain powers of y up to m only. This fact, discovered numerically, can also be proved analytically. The constant c, obtained by means of the semi-analytical approach from the bifurcation theory (25), agrees well with the numerical results that give $c = [h_1(1,0)]^2 = 1.6064$.

The scaling factors in phase space are evaluated by the formulae

$$\alpha_* = 1\big/\big(\tilde{G}_\kappa^+(1,1)\tilde{G}_\kappa^-(1,1)\big), \qquad \beta_* = 1\big/\big(\tilde{F}_\kappa^+(1,1)\tilde{F}_\kappa^-(1,1)\big).$$

The eigenvalues δ_1 and δ_2 (scaling factors in parameter space) are evaluated for the approximate linearized operator $D[RG_k[\tilde{G}_\kappa^+, \tilde{F}_\kappa^+]]D[RG_k[\tilde{G}_\kappa^-, \tilde{F}_\kappa^-]]$ with the use of 7×9 interpolation grid. The results of these calculations were discussed in Sect. 5.

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