On some properties of nearly conservative dynamics of Ikeda map and its relation with the conservative case

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Abstract

The behavior of the well-known Ikeda map with very weak dissipation (so called nearly conservative case) is investigated. The changes in the bifurcation structure of the parameter plane while decreasing the dissipation are revealed. It is shown that when the dissipation is very weak the system demonstrates an "intermediate" type of dynamics combining the peculiarities of conservative and dissipative dynamics. The correspondence between the trajectories in the phase space in conservative case and the transformations of the set of initial conditions in the nearly conservative case is revealed. The dramatic increase of number of coexisting low-period attractors and the extraordinary growth of the transient time while the dissipation decreases have been revealed. The method of plotting a bifurcation trees for the set of initial conditions has been used to classify existing attractors by it's structure. Also it was shown that most of coexisting attractors are destroyed by rather small external noise, and the transient time in noisy driven systems increases still more. The new method of two-parameter analysis of conservative systems was proposed.

1 Introduction

It is well known that the behavior of conservative and dissipative systems differs essentially. E.g., the majority of nonlinear conservative systems can demonstrate the chaotic dynamics practically at all values of parameters, but usually it realizes in very small area of phase space. On the other hand dissipative system demonstrates chaotic behavior only at certain values of parameter, but the basin of that chaotic attractor usually occupies a considerable area in phase space (see e.g. [1, 2]). Furthermore, differences in dynamics lead to significant difference in numerical methods for investigation. So, practically all methods applicable for dissipative systems are based on the analysis of the attractors, while conservative systems have no attractor at all. As a result two practically independent branches studying conservative and dissipative systems correspondingly had been formed in nonlinear dynamics.

But physically dissipative and conservative systems are not isolated and for a big number of systems a transition from dissipative to conservative systems while continuous change of the parameters can occur. In this case dynamics changes smoothly from dissipative to conservative and some "intermediate" behavior should occur "near" the conservative case. Investigation of this process seems to be very interesting because such behavior should demonstrate both conservative and dissipative features. Such investigations were began in [3] for so-called rotor map, or standard map, which conservative modification is the classical model of conservative system (see e.g. [2]). It had been revealed, that the rotor map could demonstrate very peculiar dynamics combining some features of conservative and dissipative dynamics while its Jacobian approaches 1. In particular, a huge number of co-existing low-period periodic attractors can be observed, which leads to significant dependence of the dynamics on the initial conditions. We should note that such dependence is typical for conservative systems.

In this paper we try to investigate the dynamics of another classical model - the Ikeda map - while dissipation decreases and the system evolutes from dissipative to conservative.

2 Ikeda map

The Ikeda map

$$z_{n+1} = A + Bz_n exp(i(|z_n|^2 + \psi)) \tag{1}$$

had been proposed by Ikeda *et al.* [4] to describe the dynamics of light in the ring cavity. Now it is one of the classical models of nonlinear dynamics demonstrating a big number of it's basic phenomena. We would like to emphasize that the Ikeda map is an approximate stroboscopic map for a driven nonlinear oscillator [5] and so can roughly describe a big amount of systems of different nature. Let's consider the connection between map (1) and the pulse driven nonlinear oscillator

$$\ddot{x} + \gamma\dot{x} + {\omega_0}^2 x + \beta x^3 = \sum C\delta(t - nT) \tag{2}$$

more detail. The Ikeda map can be obtained from the Eq. (2) if one solves the autonomic equation between external pulses by method of slow amplitudes. The connection between parameters of Eqs. (1) and (2) is given by following formulae:

$$A = \frac{C}{\omega_0}\sqrt{\frac{3\beta}{8\omega_0}\frac{1 - e^{-\gamma T}}{\gamma}}, B = e^{-\gamma T/2}, \psi = \omega_0 T. \tag{3}$$

The Jacobian of this map is equal to B^2 , hence $B = 1$ corresponds to the conservative system, $B < 1$ - to dissipative and $B \approx 1$ - weakly dissipative (nearly conservative) system. Nowadays the dynamics of the Ikeda map in dissipative case is well studied (see e.g. [5–7]). In particular, it is known that so called "crossroad area" structures [8,9] typical for driven nonlinear oscillator exist in the parameter plane of this map (fig. 1 a).

At the transition to weakly dissipative case the general structure of the parameter plane remains practically the same, but some changes occur (see fig. 1 b). For example, the "crossroad area" structure changes, and the degenerate flip [10] point appears on the period-doubling line, indicating the appearance of the supercritical period-doubling bifurcation. Also it should be mentioned that a transition to chaos occurs now at smaller values of parameter A.

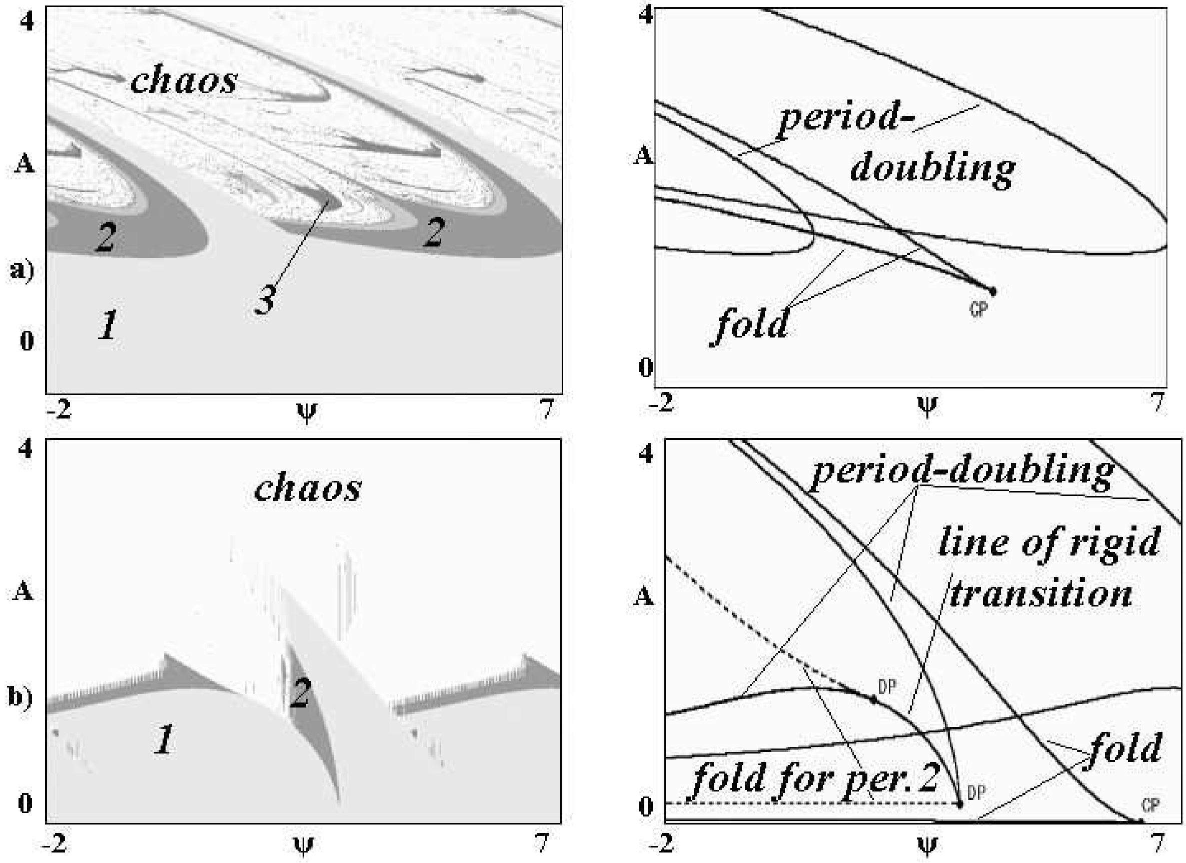

Figure 1: The structure of the parameter plane of the Ikeda map (1) with essential (a, $B = 0.3$) and weak (b, $B = 0.99$) dissipation. On the left side there are so called charts of dynamical regimes where the stability regions of the cycles of different periods are shaded with different gray shades. On the right side there are bifurcation lines of Ikeda map (1) plotted by the *Content* program. CP means a cusp point, DP — the degenerated flip point.

3 Evolution of attractors with the decrease of dissipation

Now let's turn to the analysis of the phase space structure in the nearly conservative case. For this we have taken "cloud" of points in the phase space and have observed the consecutive stages of its condensing on the attractor (see fig. 2). We can see that at the first stages of cloud evolution (fig. 2 a) structures similar to the phase portrait in conservative case (fig. 2 d) arise. At further stages of the dynamics several focuses which attract other points can be seen (see fig. 2 b, where these focuses are marked by stars). The location of those focuses corresponds to the location of the elliptic fixed points in conservative case. It shows us that on small times of evolution a weakly dissipative system demonstrates nearly conservative dynamics with further transition to dissipative dynamics.

When all transients have died away, several attracting points can be seen in the phase space (see fig. 2 c) although this point on the parameter plane is inside the region of the period 1 at the fig. 1 b. This allows us to suppose that several periodic attractors coexist at this point of the parameter plane, i.e. multistability exists.

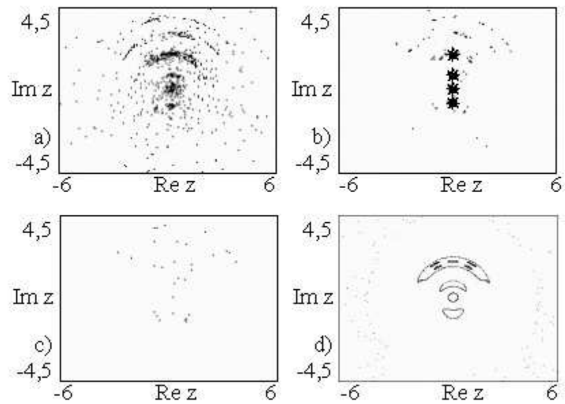

Figure 2: The different stages of the evolution of cloud of initial conditions for the Ikeda map (1) in weakly dissipative case (a–c) and phase portrait in the conservative case (d). The figures a–c differs by the number of missed iterations: a) 200; b) 660; c) 6000. The locations of focuses are marked by stars at fig. b. Values of parameters: $A = 0.5; \psi = 3\pi/4; B = 0.99$ (a-c), $B = 1$ (d).

For investigation of this phenomenon in weakly dissipative case a method of drawing bifurcation trees for a set of initial conditions on one diagram had been proposed in [3]. At each value of the control parameter one should take a set of the initial conditions, make a sufficient number of iterations to cut off all transients and then plot several consequent iterations at the "parameter - variable" plane. Such diagrams allow us to obtain the number of coexisting attractors and to trace their transformations while changing the parameter. It seems natural to choose a set of initial conditions in the domain where an attractor exists to decrease the amount of calculations. From (1) we can obtain $|z_{k+1}| \leq A + B|z_k|$. It is obvious that an attractor can't exist in the domain, where $|z_{k+1}| \leq |z_k|$ because there $|z_k|$ decreases. The boundary of this domain we can determine from the condition $|z_{k+1}| \leq A + B|z_k| \leq |z_k|$. Hence, $|z_k| \geq A/(1-B)$ is the domain where an attractor can't exist. Therefore, the domain of attractor existence is bounded by the condition $|z| \leq A/(1-B)$, i.e. $|z|_{max} = A/(1-B)$. We'll take initial conditions on the mesh in the rectangle $[-x_{max}, x_{max}] \times [-y_{max}, y_{max}]$, where $x_{max} = y_{max} = A/(1-B)$.

Bifurcation diagrams for different values of dissipation parameter B plotted by this method are shown in fig. 3. On all of them we can see the "basic" attractor, which arises at $A = 0$ and demonstrates the first period-doubling bifurcation at values A near 1. This attractor has the largest basin so usually it is represented on the charts (fig. 1). Besides it there are some "secondary" attractors. They arise at non-zero values of parameter A and demonstrate classical transition to chaos by period-doubling cascade. Their bifurcation trees have rather simple "classical" structure. Let's refer such attractors as the attractors of first type.

At relatively strong dissipation ($B = 0.5$) a number of such "secondary" attractors is not very big but it increases while decreasing of dissipation and they arise at smaller values of A. Also the distance between them along the A axe decreases. Furthermore, fragments with essentially more complex dynamics arise on bifurcation diagrams.

Attractors corresponding to these trees arise at rather large values of A and are characterized with a smaller interval of their existence on A axe than attractors of the first type.We shall refer them as the attractors of the second type. The number of such at-

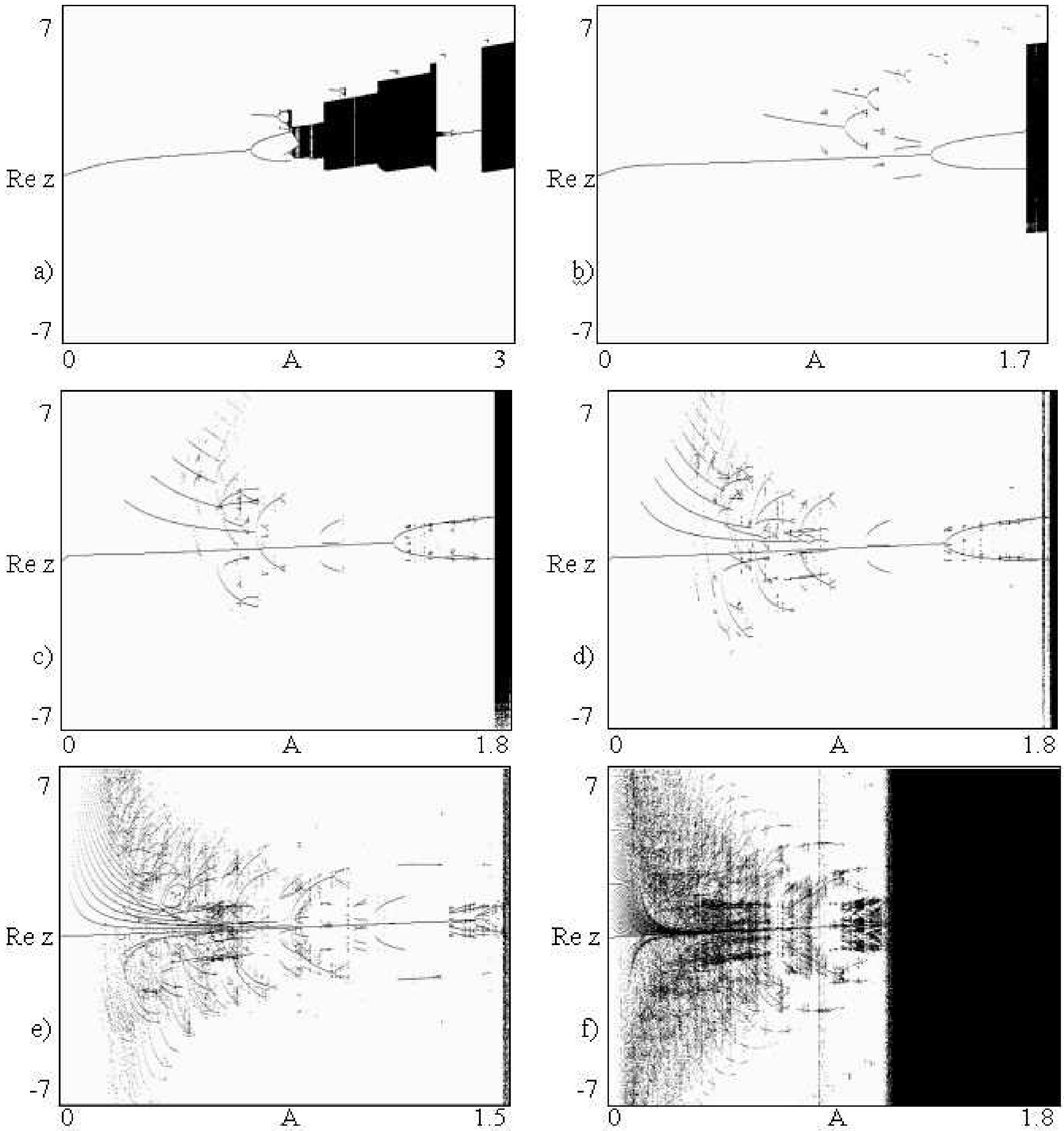

Figure 3: Bifurcation diagrams for the map (1) for different values of dissipation parameter: a) $B = 0.5$; b) $B = 0.75$; c) $B = 0.9$; d) $B = 0.95$; e) $B = 0.99$; f) $B = 0.999$. $\psi = 0$.

tractor also increases while decreasing the dissipation.

Now let's investigate a weakly dissipative case (fig. 3 f,e) in more detail. First we consider the case $B = 0.99$ (fig. 3 f). It should be marked that the transient time becomes extremely long (up to 500000 iterations) at this case and more than ten times exceeds the transient time for essentially dissipative system. In the fig. 4 the bifurcation diagrams for different transient time are shown. It can be seen that besides it's extreme length, the transient time essentially depends on the parameter A. This dependence is extremely irregular so the areas where transient time is less then 20000 iterations interchange with areas where it is more than 200000 iterations.

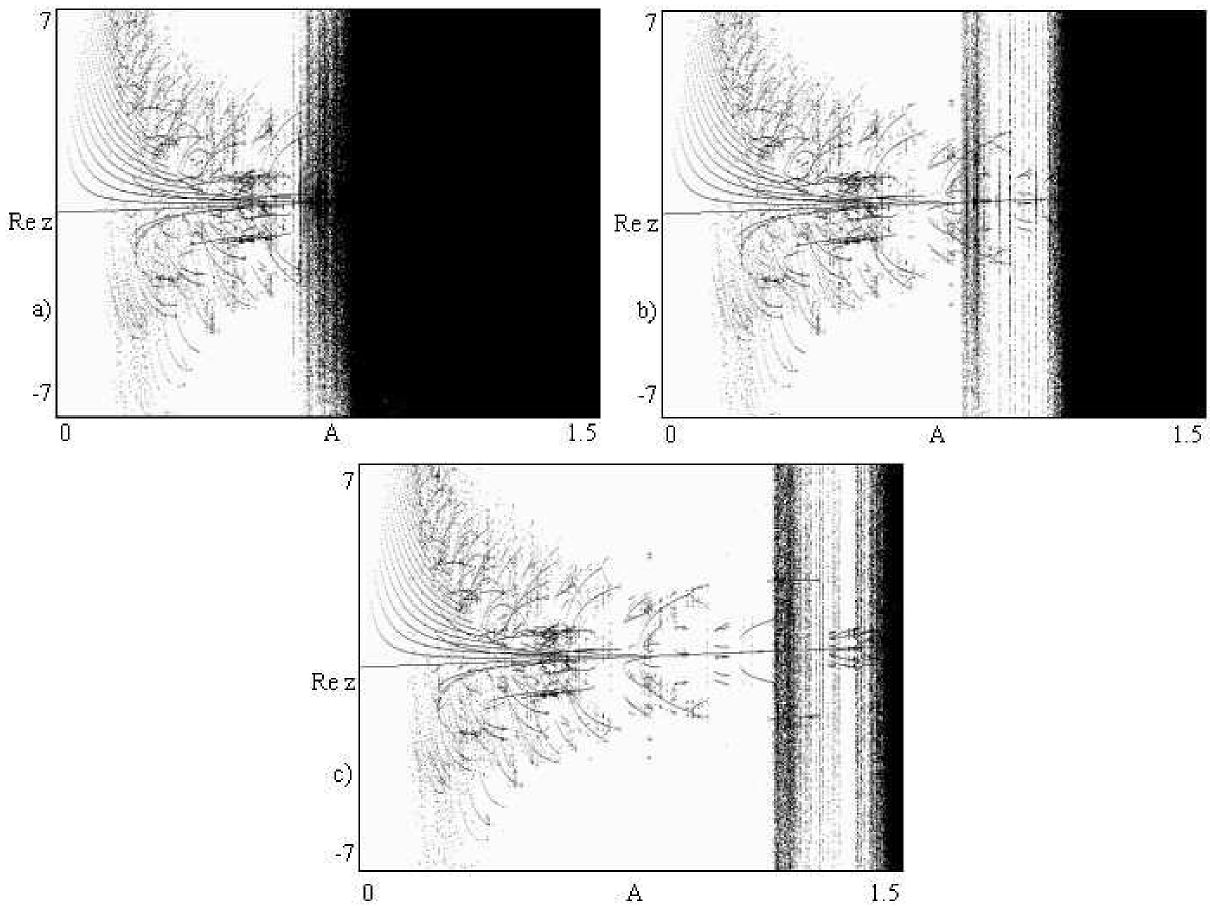

Figure 4: Bifurcation diagram for the map (1) with a different number of missed iterations (transient process): a) 5000; b) 20000; c) 200000. Parameters $B = 0.99; \psi = 0$.

Now let us discuss the diagram structure when all transients have died away (fig. 3 f). It demonstrates a big number of attractors both of the first and second types. It should be noted that in fact there exist a considerably greater number of attractors than it can be seen on the bifurcation diagram because many attractors have the basin smaller than a period of the initial conditions mesh. It is confirmed by the fact that the structure of the bifurcation diagram complicates essentially when a number of points of the mesh increases from 400 to 10000 points (in particular, a number of attractors of the second type increases). At the same time there are no changes constrained with the attractors of the first type, which shows that they have larger basins and, consequently, their observation in realistic system is more probable.

There are no chaotic attractors on the bifurcation diagram for the mesh with 400 initial conditions. We think that it is also because their basins are too small. The period-doubling cascade for the majority of attractors is observed only up to period 2 which can be caused by two reasons. First is that in conservative systems the distance between two consecutive period-doubling points decreases much faster than in dissipative (corresponding constant δ=8,7210972... is essentially greater than well-known Feigenbaum constant 4,6692016...), so the regions of high periods can't be represented on the bifurcation diagram. The second is that the attractors can undergo a crisis reaching the boundary of it's basin. This assumption seems to be more realistic; some arguments in its favor will be discussed in section 4.

Besides of the bifurcation diagrams for variable $x = Re\ z$ diagrams for other variables such as $y = Im\ z$ and $|z|^2$ has been built (fig. 5). On $|z|^2$ diagram (fig. 5 b,c) it is clearly seen that the attractors of the second type are attractors of period 2 and higher because corresponding bifurcation trees always consists of two and more branches. Also it should be noted that while the x value for the attractors of the first type increases with the decrease of the parameter A, the y value decreases, and $|z|^2$ value remains practically constant. It means the point in the phase plane moves on a circle, approaching the real axe while A tends to zero.

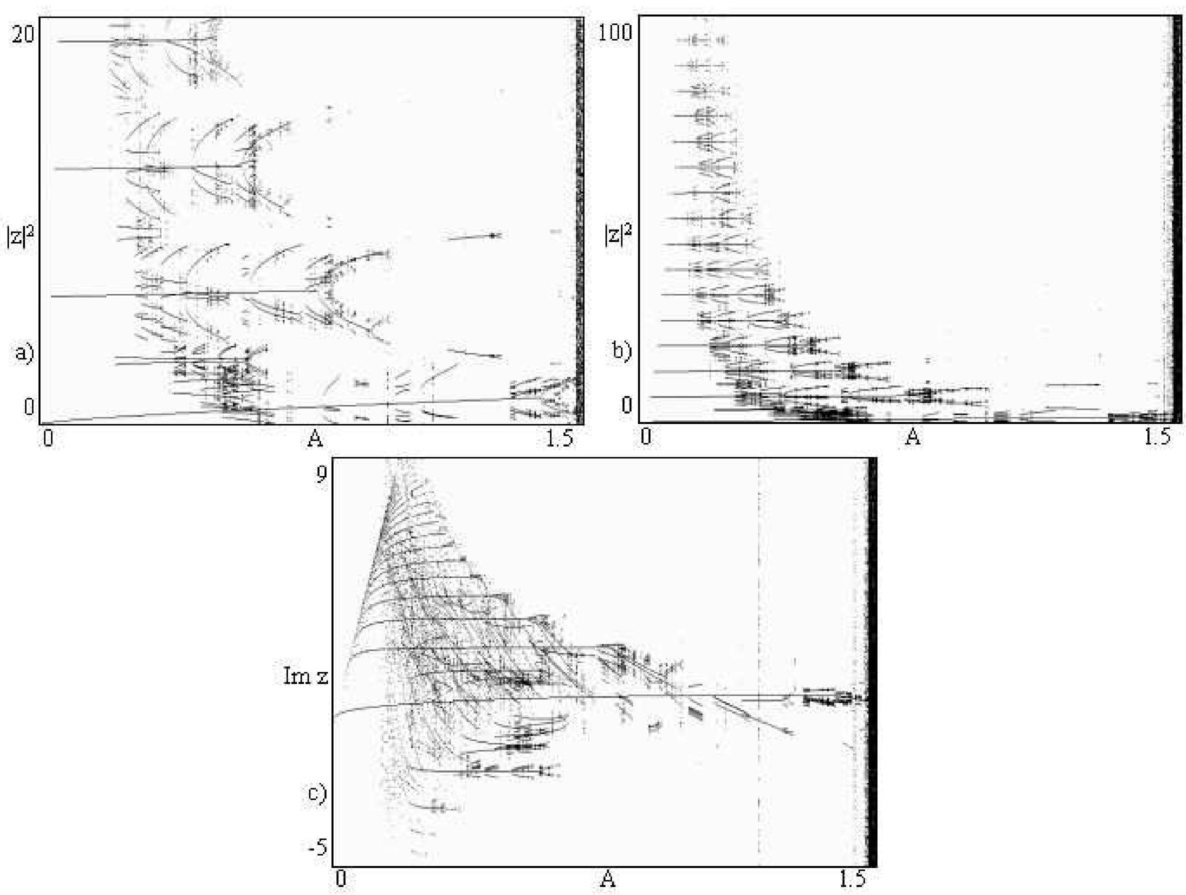

Figure 5: Bifurcation diagrams for variables $y = Im\ z$ and $|z|^2$. The parameters $B = 0.99; \psi = 0$.

For weaker dissipation ($B = 0.999$, fig. 3 f) transient time reaches 5000000 iterations and a number of coexisting attractors extends extremely, but evidently there are no qualitative changes in the structure of the diagram, e.g. all attractors can be divided into the same two types.

In this case the attractors of the first type form several "families" which tends to the horizontal lines with the increase of the parameter A. We can say that in previous case there are only one "family" which includes all attractors of the first type and in this case there are several "families". The "center" of each "family" is the attractor (stable fixed point) with very weak dependence on the parameter represented by horizontal line on the diagram. Empty lines between the "families" may be seen on the diagram so it is naturally to suppose that also some unstable fixed points with very weak dependence on the parameter exists being the boundaries between the "families" of the attractors.

4 Noisy driven weakly dissipative Ikeda map

In the previous sections we have shown that the Ikeda map demonstrates an exceptional variety of coexisting low-period periodic attractors in the case of weak dissipation. But it seems significant to explore how the dynamics of the system will change with adding an external noise, because it always exists in real systems. Let's consider noisy driven system as follows:

$$z_{n+1} = A + Bz_n exp(i(|z_n|^2 + \psi)) + \varepsilon\xi_n \tag{4}$$

where ξ_n is a random real value (uniformly distributed on the segment [-1;1] in our numerical experiments) and ε can be interpreted as an amplitude of noise. It should be noted that in this form the system can describe the nonlinear oscillator driven by external pulses with fixed intervals but random amplitude.

In noisy driven system the transient time becomes even more longer and approaches 700000 for $B = 0.99$. In the fig. 6 a,b bifurcation diagrams for different amplitudes of noise (a, b) are shown. In the fig. 6 c they are laid one on another to compare it's structure. It is well seen that a large amount of attractors (and the larger the amplitude of noise is, the larger is this amount), and between them the majority of the attractors of the second type, is destroyed by noise influence. The destruction of the attractors can be explained by the fact that their basins are too small and a noise influence simply "throw" the point out of the basin.

Also it should be noted that some attractors undergo sharp expansion before the disappearance so we can suppose that the destruction of such attractors is a result of a collision of the attractor with the boundary of its basin. Just before the collision the attractor is very close to the basin boundary and it seems likely that the trajectory can be thrown out of the basin by noise which can lead to significant growth of the variable. The realization of this dynamics confirms our suggestions that some attractors undergo crisis.

5 Conservative case of the Ikeda map

Now let's present a brief analysis of the conservative case of the Ikeda map. The Ikeda map (1) becomes the conservative system at $B = 1$. Plotting of the phase portraits is one of the basic methods for its investigation. Phase portraits of the map (1) are presented in the fig. 7. Their form is typical for driven conservative nonlinear oscillator — the families of invariant tori corresponding to the existence of elliptic fixed points, some "hollows" on

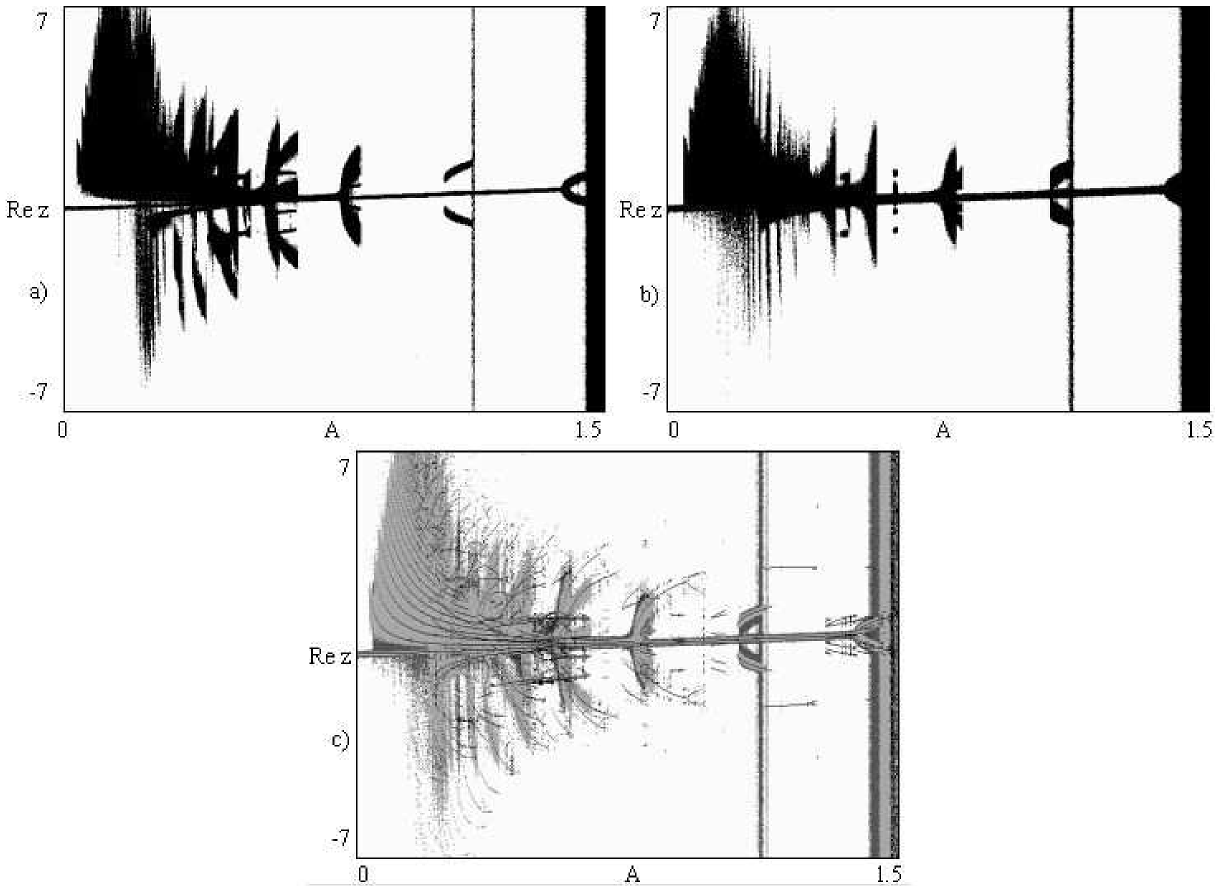

Figure 6: Bifurcation diagrams for the map (4) for various values of noise amplitude ε: a) $\varepsilon = 0.005$; b) $\varepsilon = 0.01$; c) three diagrams are plotted over each other: dark gray - $\varepsilon = 0.005$; light gray - $\varepsilon = 0.01$; black - without noise. $B = 0.99; \psi = 0$.

them corresponding to the hyperbolic (saddle) fixed points on the outside [1] and periodic islands surrounded by the domains of irregular dynamics, or the "chaotic sea", exist. It can be clearly seen that at some parameter values the "chaotic sea" exists not only outside, but also inside the periodic islands. Furthermore, structures that are typical for phase oscillations at nonlinear resonance [1] can be seen on the portraits.

For the investigation of the conservative system we propose a method of plotting of so-called "divergence chart" that in some sense is an analog to the chart of dynamical regimes for the conservative systems. The procedure of its plotting is as follows. For each point of (A, ψ) plane we choose a set of points in the phase space and fix the number of points that have stayed in the finite region of phase space after a big number of iterations (we use 15000 in numeric simulations). The different numbers of non-diverged points correspond to the different shadows of gray color. The comparison of "divergence chart" (fig. 8 b) and the chart of dynamical regimes for nearly conservative case (fig. 8 a) shows some correspondence between structures at the parameter plane. For example, the border of the domain where practically all points have gone to infinity in the conservative case corresponds to the chaos border in dissipative system.

Now let's turn to the noisy driven conservative system (see (4) with $B = 1$). On the "divergence charts" (fig. 9) noise destroys some small-scale structures, and the more noise amplitude is, the more large-scale structures are destroyed.

On the phase portraits in noisy driven systems (fig. 10), as we can predict, large-scale structures destroys a little but in general stay unchanged, and more small structures

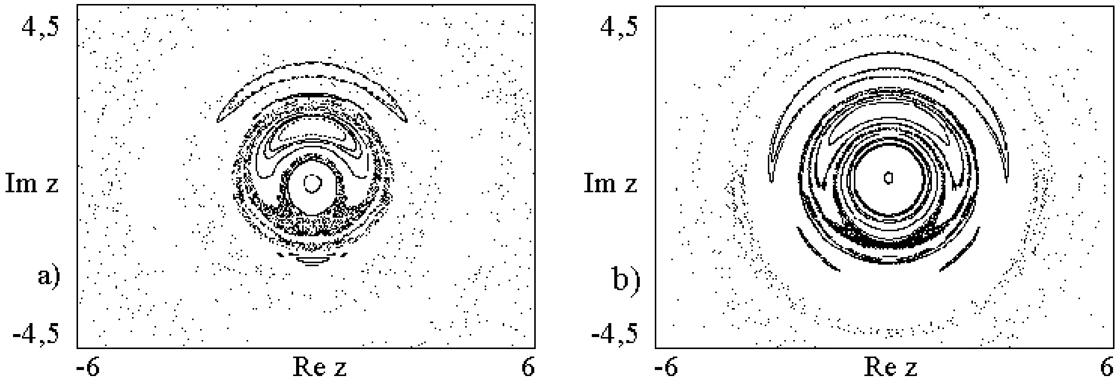

Figure 7: Phase portraits of the map (1) in the conservative case ($B = 1$). Parameters: a) $A = 0.3, \psi = 3\pi/2$; b) $A = 0.2, \psi = \pi$.

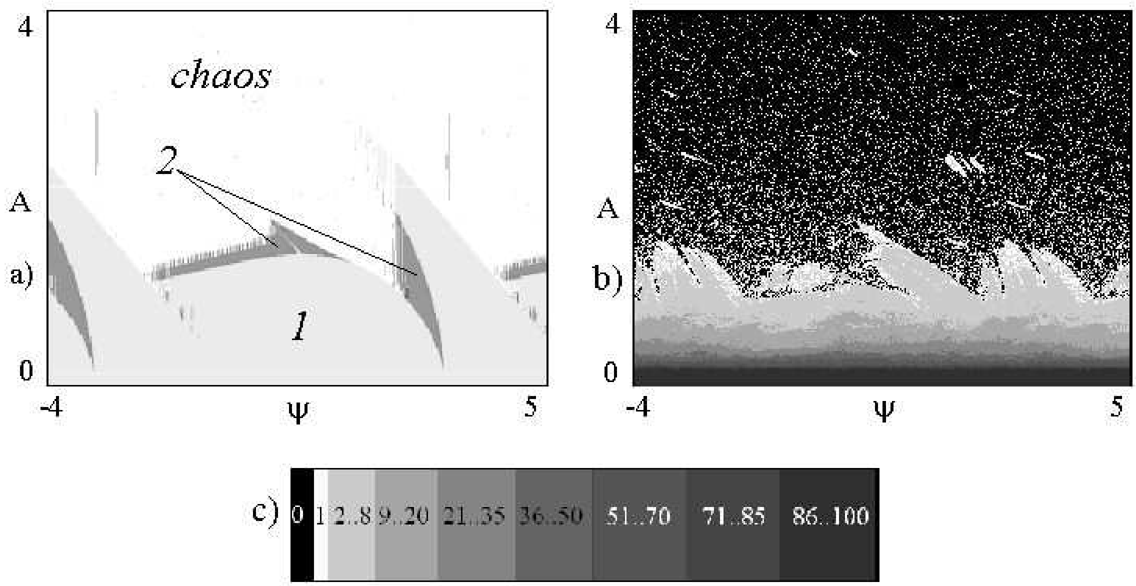

Figure 8: Chart of dynamical regimes for dissipative map (1) (a, $B = 0.99$) and "divergence chart" for conservative map (1) (b, $B = 1$). Table of correspondence of colors to the numbers of points that haven't gone to infinity is presented in fig. c. In fig. b transient time is equal to 15000 iterations.

destroy. It should be noted as a remarkable fact that phase portraits of noisy driven system at fixed parameter values can be significantly different (fig. 10 b, c). It can be explained as follows. If a point in phase space lies near the separatrix bounding two domains with different dynamics, it can be "thrown" by noise influence from one domain to another so it will demonstrate different dynamics on further stages of evolution. So the more noise amplitude is the more wide is the band in which a dynamics of the point can be changed.

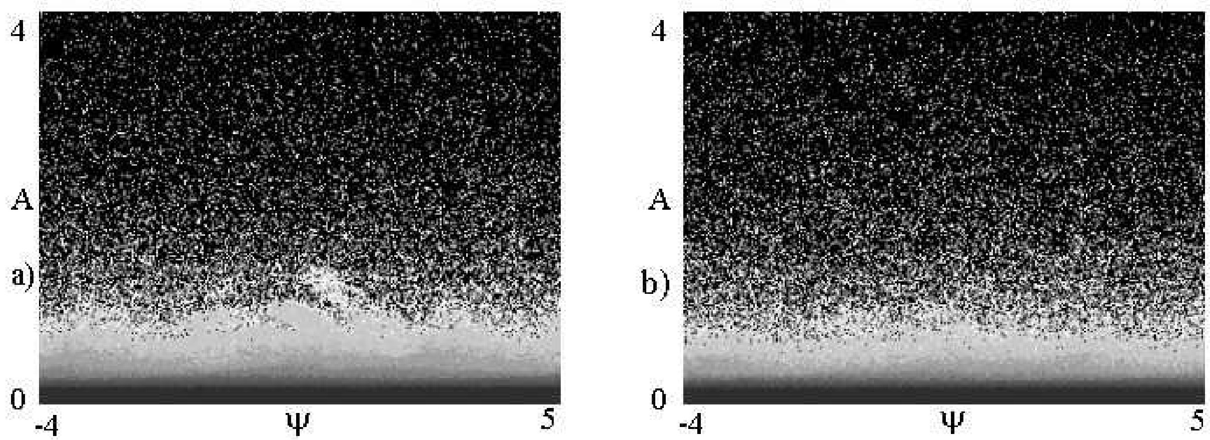

Figure 9: "Divergence charts" for map (4) at different values of noise amplitude: a) $\varepsilon = 0.005$; b) $\varepsilon = 0.01$. Transient time is equal to 15000 iterations.

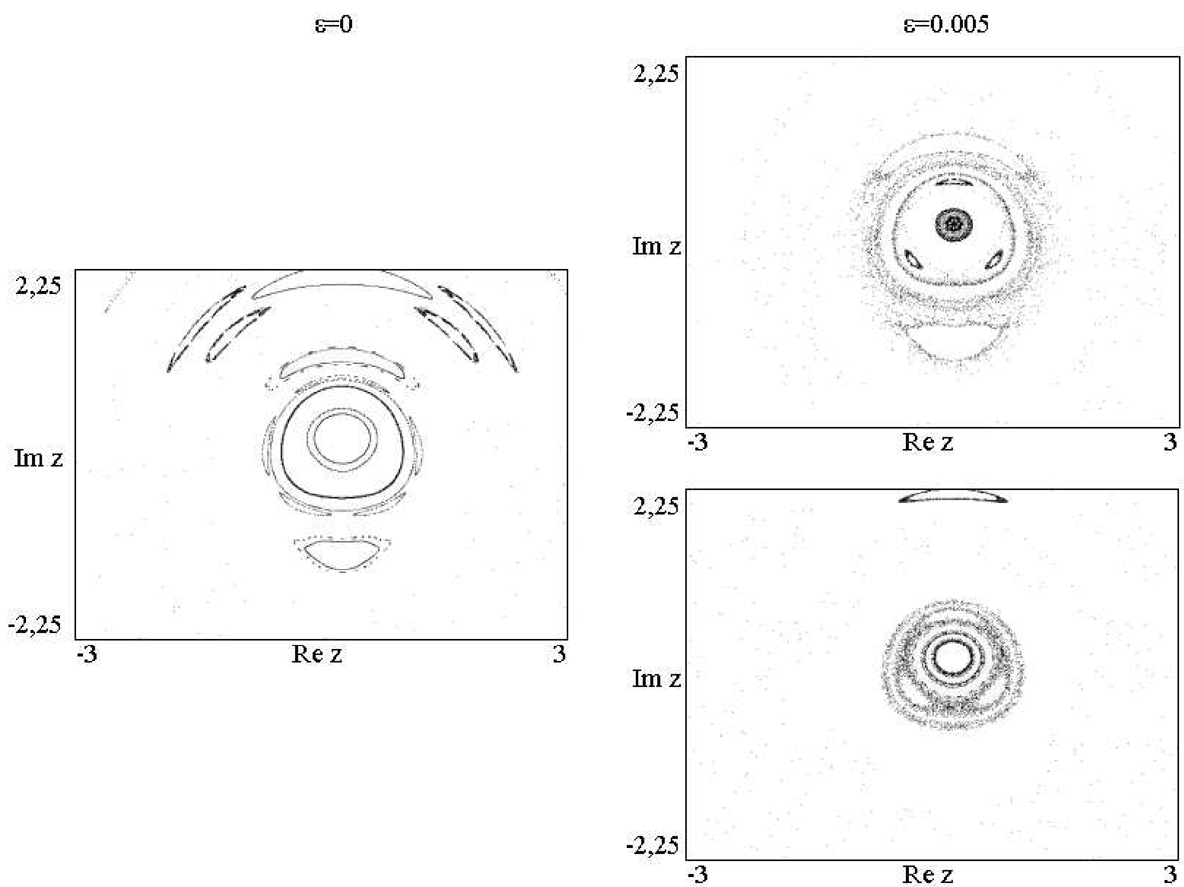

Figure 10: Phase portrait for map (1) (a) and its different realization for map (4) (b, c) with noise amplitude $\varepsilon = 0.005$. Parameters: $A = 0.5; \psi = \pi/2; B = 1$.

6 Conclusions

Thus we have shown that the Ikeda map demonstrates a big number of coexisting periodic attractors in the case of weak dissipation and their number increase with the decreasing of dissipation. These attractors can be divided into two types with different structure and different length of the interval of the parameter A where they exist.

The sharp increasing of transient time has been revealed with the approaching of conservative case. At the beginning of the transient process the system behavior is similar to conservative and in the end to dissipative one. Moreover, it should be noted that transient time depends essentially on the value of the parameter A.

Also the sensitivity of the weakly dissipative system to the external noise has been revealed: many of attractors are destroyed by the noise of rather small amplitude. It can be explained as follows. It is known that the noise effects the first stage of the evolution much more then the stable regime, and the more dissipative the system is, the faster it "forgets" initial conditions. The system with very weak dissipation "remembers" initial conditions for a very big time, hence, an external noise influences on such systems more strongly.

At last, the new method for investigation the conservative case was proposed. It was shown that structures similar to typical for dissipative system arise at the parameter plane of conservative system. Also it was shown that the conservative Ikeda system demonstrates strong sensitivity to the noise influence.

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