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Realization of a Strange Attractor of the Smale–Williams Type in a Radiotechnical Delay-Fedback Oscillator

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Abstract—Based on a nonautonomous delay-feedback van der Pol oscillator, a radiotechnical circuit generating chaotic signals has been proposed and implemented, which alternatively occurs in the states of excitation and decay. This is ensured by the periodic variation of a parameter responsible for the bifurcation creating a limit cycle. The excitation of oscillations at every next activity stage is stimulated by a signal generated at the preceding stage and arriving via a delay line. Owing to a quadratic nonlinear transformation of the delayed feedback signal, the phase variable exhibits doubling at every excitation stage, which implies that the phase is subjected to circle-stretching mapping (Bernoulli mapping) with a chaotic dynamics.

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The mathematical theory of dynamical systems introduces the class of homogeneous hyperbolic strange attractors, examples of which are offered by mathematical models such as Plykin's attractor or the Smale–Williams solenoid [1–3]. These attractors possess strong chaotic properties that admit profound mathematical analysis. From practical standpoint, dynamical; systems with such attractors could be of interest owing to their inherent property of structural stability (robustness). However, it was commonly accepted until recently that attractors of this class are never realized in real systems with chaotic dynamics.

Recently, a physically realizable system was proposed, in which the Poincaré map had an attractor of the Smale–Williams type [4, 5]. Computer verification performed for this system, composed of two van der Pol (vdP) oscillators, confirmed the validity of sufficient conditions for the presence of a hyperbolic attractor as known from the relevant mathematical literature [6, 7]. Other variants of such schemes have been considered in [8–10]. The general principle of functioning of such systems consists in manipulating oscillation phases during the transfer of excitation between coupled vdP oscillators, which are alternatively rendered active, so that the phase transformation would represent map iterations with chaotic dynamics.

In the present Letter, we suggest to implement this principle in a dynamical system based on a single active element, representing a vdP oscillator with delay feedback. From the standpoint of possible realizations—for example, in radio engineering or laser physics—this system admits a simpler technical solution as compared to the scheme with two active elements. On the other hand, it is more complicated for mathematical analysis, since delay-feedback systems correspond to a phase space of infinite dimensionality.

Let us consider a model system (vdP oscillator) described by the following equation:

$$\begin{aligned}\ddot{x} - (A\cos\Omega t - x^2)\dot{x} + \omega_0^2 x \\ = \varepsilon x(t-\tau)\dot{x}(t-\tau)\cos\omega_0 t.\end{aligned} \quad (1)$$

where x is the dynamic variable) and ω_0 is the working frequency of the vdP oscillator (see the left-hand side of this equation). The parameter controlling the bifurcation creating a limit cycle slowly varies with the time with the frequency $\Omega \ll \omega_0$ and the amplitude A, so that the oscillator alternatively occurs in the states of excitation and decay of its natural oscillations. The right-hand side of this equation accounts for the delay feedback, representing a product of the dynamic variable at the moment delayed by τ, the derivative of the variable, and an auxiliary harmonic signal with the frequency ω_0. Factor ε determining the amplitude of the delay feedback will be considered small. For convenience, we assume that the ω_0 is an integer multiple of Ω, so that the external actin is on the whole periodic. 1

The functioning of system (1) as a generator of chaos can be described as follows. Owing to a periodic variation of the parameter responsible for the onset of oscillations (excitation), the oscillator alternatively occurs in the states of excitation and decay of its natural oscillations. By selecting a proper delay time of $\tau = 3T/4$, is possible to realize a situation where a "seed" for the onset of self-sustained oscillations in every excitation stage is provided by a signal generated at the preceding active stage and arriving via a delay line. Let us assume that this signal has a certain phase φ, so that

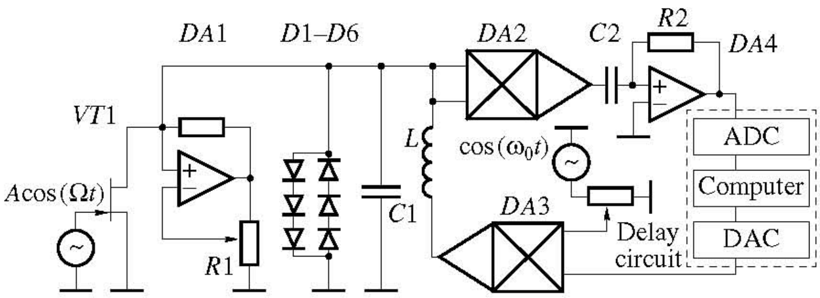

Fig. 1. Block scheme of a hyperbolic chaos generator with delay feedback (see the text for explanations).

$x(t) \propto \sin(\omega_0 t + \varphi)$ and $\dot{x}(t) \propto \cos(\omega_0 t + \varphi)$. Then, the term in the right-hand side of Eq. (1) will contain a component with the main frequency ω_0 and the double phase. Indeed, since $x(t)\dot{x}(t) \sim \sin 2(\omega_0 t + \varphi)$, we have

$$x(t-\tau)\dot{x}(t-\tau)\cos\omega_0 t \sim \sin 2(\omega_0(t-\tau)+\varphi)\cos\omega_0 t$$
$$= \frac{1}{2}\sin(\omega_0 t - 2\omega_0\tau + 2\varphi) + (\ldots),$$

where the dots in parentheses denote a term with the triple frequency.

As a result, the phase of oscillations in every new stage of activity will be determined by the phase of the resonance component of the seeding signal. Thus, the phases of the sequential stages of activity obey the following relation:

$$\varphi_{n+1} = 2\varphi_n + \mathrm{const}(\mathrm{mod}2\pi), \tag{2}$$

where symbol mod2π indicates that the phase is determined to within an additive that is multiple of 2π. Relation (2) is known as the stretching mapping of the circle or the Bernoulli map [11].[1] As is known, this relation generates chaotic dynamics with a Lyapunov exponent of $\Lambda = \ln 2$.

Assuming that the other directions in the phase space are featuring contraction, this dynamics must correspond to the presence of the Smale–Williams attractor in the self-map of the phase space for the period of external action.

In accordance with the concept presented above, we have designed and constructed a radiotechnical circuit schematically depicted in Fig. 1. The active element was implemented as an *LC* circuit, a negative-resistance element on amplifier DA1, and a nonlinear dissipative element consisting of D1–D6 diodes. The dynamics of this circuit generally corresponds to that of the vdP oscillator with a natural autooscillation frequency of $f_0 = \omega_0/2\pi = 3$ kHz. The modulation of a parameter responsible for the excitation of autooscillations is achieved by introducing additional dissipation via field-effect transistor VT1, whose resistance slowly varies with the time under the action of an external periodic signal as $\Delta R \sim A\cos\Omega t = A\cos(2\pi t/T)$. During one half-period, the oscillator occurs in the regime of generation, while in the other half-period, it is below the generation threshold. The generation of oscillations at a certain phase during every next stage of activity is excited by a signal arriving via a delay feedback line from the output of multiplier DA3.

In the feedback line, the signal is subject to a quadratic transformation (by a multiplier implemented on amplifier DA2) and differentiation (by a standard scheme implemented on R2, C2, and DA4). Then, the signal is passed via a delay line comprising an analog-to-digital converter (ADC), a computer, and a digital-to-analog converter (DAC), where the delay is realized by a computer program algorithm. The delayed signal is multiplied by an auxiliary signal of frequency f_0 supplied from an external generator. Owing to the selection of a delay time at $\tau = 3T/4$, the signal generated at the moment of maximum excitation arrives at the oscillator via the delay line exactly at the beginning of the next stage of activity, thus providing a seed for the onset of generation. Owing to the presence of the quadratic transformation, the system produces phase doubling on the passage from one to another train of oscillations. The ratio of external signals in our experiments, was $N = \omega_0/\Omega = f_0 T = 6$.

Figure 2 presents the results of numerical solution of Eq. (1) in comparison to the experimental data. Figure 2a shows the time series of the dynamic variable representing sequential trains of oscillations. The chaos is manifested by irregular variation of the position of filling within the envelope in different trains. Figure 2b shows diagrams of the phase of oscillations for sequential trains calculated as $\Phi_n = \arg[x(t_n) + i\omega_0^{-1}\dot{x}(t_n)]$, where the time t_n corresponds to the middle of the active stage and is fixed relative to the slow process of variation of the excitation parameter. In our experiments, the map $\varphi_{n+1} = f(\varphi_n)$ was constructed using time series characterizing the sampling at a discrete time step of the generated alternating signal, which were recorded in computer via the ADC. As can be seen the phase transformation over one period T approximately corresponds to the Bernoulli map, representing circle-stretching map of the same topological type.

[1] Note that the constant factor can be eliminated by shifting the origin of phase count.

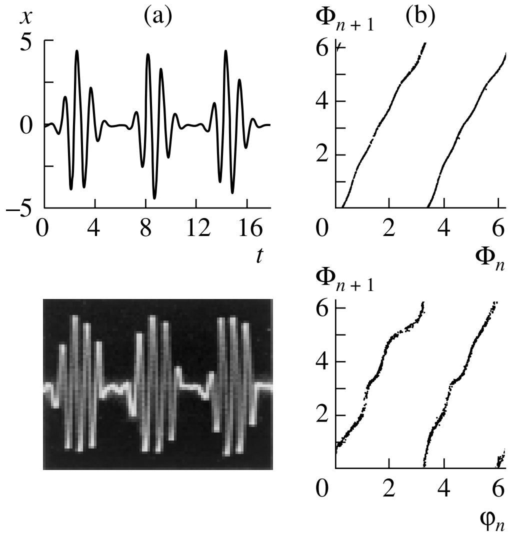

Fig. 2. (a) Time series of dynamic variable x and (b) diagrams of phase transformation in sequential stage of oscillator activity according to (top) results of numerical calculations using Eq. (1) with $\omega_0 = 2\pi$, $T = 2\pi/\Omega = 6$, $\tau = 3/4T$, $A = 5.5$, $\varepsilon = 90.2$ and (bottom) results obtained in a physical experiments with a circuit depicted in Fig. 1.

Figure 3 shows the oscillographic patterns illustrating the regime of chaotic dynamics in the system under consideration, representing a two-dimensional projection of the phase portrait and a stroboscopic pattern. The spectrum of observed noise oscillations had a full width at half maximum (FWHM) of 20% at an average frequency of 3 kHz. The stroboscopic image resembles the Smale–Williams solenoid with a well-resolved transverse fractal structure characteristic of the attractors of this type.

We believe that the realization and experimental investigation of hyperbolic attractors is an interesting and important direction of research in nonlinear dynamics. From the practical standpoint, the main advantage of these attractors is the inherent property of structural stability. For the delay-feedback system under consideration, the statement concerning the hyperbolic nature of the attractor embedded into the phase space of infinite dimensionality presently has a hypothetical character. However, features of the system dynamics observed in calculations and experiments agree with this hypothesis. From the standpoint of practical realizations, the scheme with one active element is obviously simpler that the schemes proposed previously, which include two or even greater number of active elements. The advantages of this scheme will be more significant for devices functioning at higher frequencies, where the proposed delay line and the quadratic phase nonlinearity can be quite readily realized.

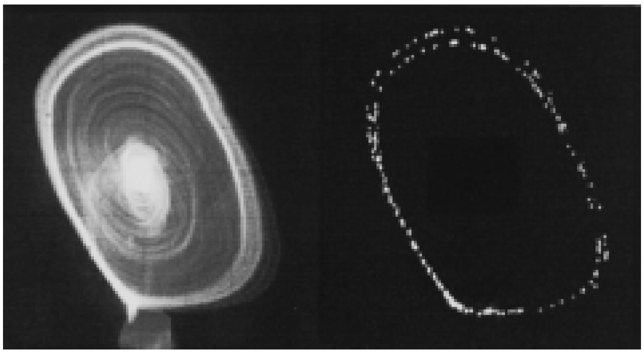

Fig. 3. Oscillographic patterns illustrating the regime of chaotic dynamics: (a) a two-dimensional projection of the phase portrait; (b) a stroboscopic pattern on the (V_{C1}, dV_{C1}/dt) plane.

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