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Stabilizing the Rössler System by External Pulses on a Runaway Trajectory

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Abstract—The action of external pulses on the Rössler system prior to the onset of a saddle–node type bifurcation on a runaway trajectory is considered. It is shown that a pulsed drive can lead to the appearance of stable periodic and quasi-periodic regimes in the nonautonomous system.

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In recent years, the problem of managing chaos has attracted much attention due to both its basic significance and a number of its important applications [1–5]. This research is traditionally concentrated in the two main directions pertaining to the stabilization of (i) periodic orbits of a dynamical system embedded into a chaotic attractor and (ii) unstable equilibrium states (immobile points). Within the framework of the first direction, the problem has been rather extensively studied [5, 6]. One of the most popular ways of solving the task is offered by the Pyragas method based on the autosynchronization with a delay-time feedback [6]. This approach can be generalized so as to provide the stabilization of unstable equilibrium states [7]. Recently, we have demonstrated that stable periodic and quasi-periodic regimes can be realized in a dynamical system with unstable limit cycle under the pulsed drive action [8].

A nontrivial task is to provide stabilization in the case of a system having no equilibrium states, whereby the imaging point in the phase space runs to infinity. This runaway situation can arise, for example, immediately behind the saddle–node type bifurcation threshold, where the stable and unstable equilibrium states merge together and disappear. A special feature of three-dimensional (3D) dynamical systems is that the saddle–node bifurcation point may correspond to the merge of equilibria of the stable focus and saddle–focus types, which are characterized by the presence of one real and two complex-conjugate roots of the characteristic equation. In this case, the bifurcation results in the appearance of a set of phase trajectories with rotation. Figure 1 gives an example of the phase trajectory of such an autonomous system in projection onto the (x, z) plane of variables. Note that this situation also provides certain prerequisites for the stabilization of a system on a runaway trajectory by means of an external action, despite the fact that the autonomous system features neither stable nor unstable regimes. Such situations cannot be realized in 2D systems. It turns out that external drive of a certain type, namely, a periodic sequence of δ functions, can actually induce stable periodic and quasi-periodic regimes in the system under consideration.

Below we consider this problem in application to the Rössler system—a standard model in nonlinear dynamics—described by the following equations:

$$\dot{x} = -y - z, \quad \dot{y} = x + py, \quad \dot{z} = q + z(x - r), \quad (1)$$

where x, y, and z are the dynamic variables and p, q, and r are the system parameters. Equations (1) describe an autooscillatory system with a 3D phase space.

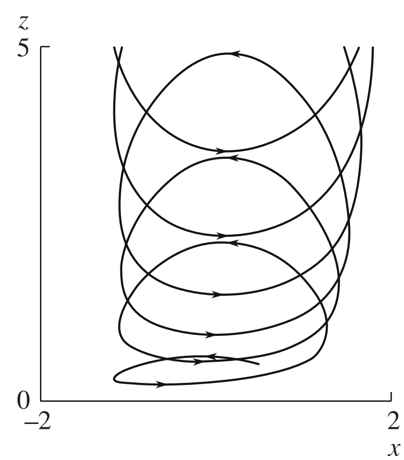

Fig. 1. The (x, z) projection of the typical phase trajectory of an autonomous Rössler system (1) with the parameters $p = 0.2$, $q = 0.2$, and $r = 0.2$ before the onset of a saddle–node bifurcation.

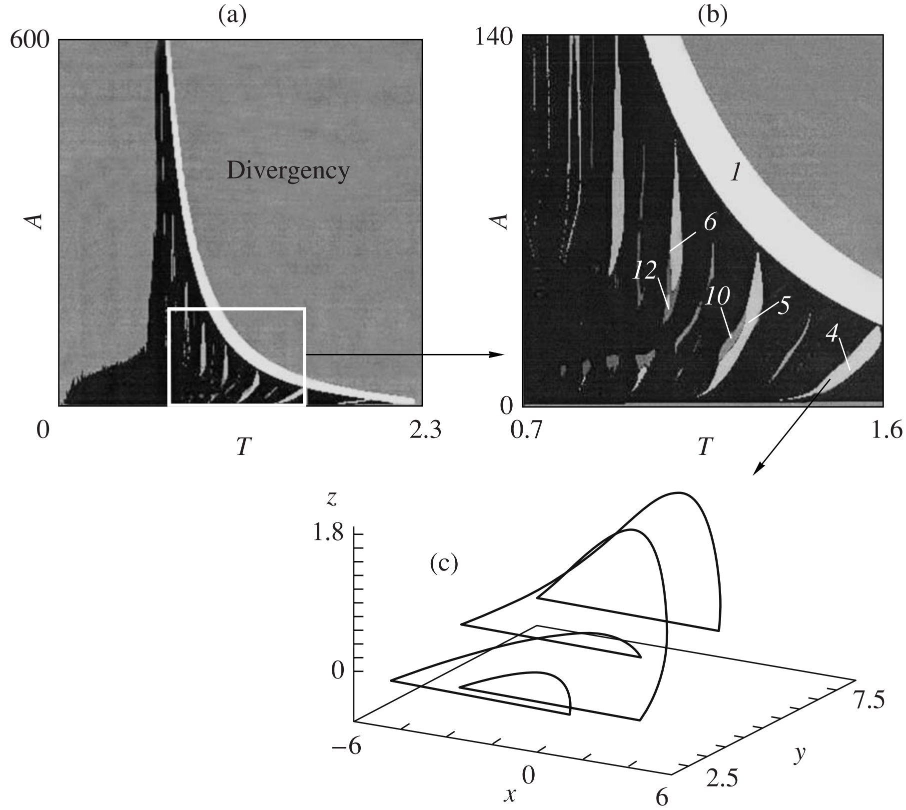

Fig. 2. A nonautonomous Rössler system (3) with the parameters $p = 0.2$, $q = 0.2$, and $r = 0.2$: (a) the map of dynamical regimes on the A–T plane of the drive parameters; (b) a fragment of the map presented on a greater scale (figures give the values of periods in the regions indicated by the arrows); (c) attractor of the period-4 system at the indicated point.

Equating to zero the Jacobian determined for the immobile points of system (1), we obtain the following condition of the saddle–node bifurcation:

$$r^2 = 4pq. \tag{2}$$

Accordingly, we have two immobile points (stable and unstable) for $r^2 - 4pq > 0$ and a single immobile point for $r^2 - 4pq = 0$. In the case of $r^2 - 4pq < 0$, there are no immobile points and the system runs away to infinity in the phase space.

In order to analyze the case of $r^2 - 4pq < 0$, let us select a point ($p = 0.2$, $q = 0.2$, $r = 0.2$) on the plane of parameters of system (1), which corresponds to this situation. The task is to find an appropriate external action. The problems of synchronization are most frequently solved using harmonic systems. An alternative variant is offered by a pulsed action, whereby the external signal has the form of a periodic sequence of pulses with a large amplitude and short duration. Recently, we have demonstrated that such a drive is preferred in solving the tasks of stabilization and initiation of stable synchronous and quasi-periodic regimes. A convenient model is provided by a periodic sequence of δfunctions such as that used in [8, 10–14].

Let us consider the case of external force acting along the x axis, that is, an additional term responsible for the drive is introduced into the first equation of system (1). In this case, the corresponding nonautonomous system of equations can bed written as follows:

$$\dot{x} = -y - z + A\sum_{n=-\infty}^{+\infty}\delta(t - nT), \quad \dot{y} = x + py, \quad \dot{z} = q + z(x - r), \tag{3}$$

where A is the amplitude and T is the period of the external signal, respectively.

Figure 2a shows the map of dynamical regimes in system (3) constructed on the A–T plane of parameters. In these maps, the regions painted white correspond to the regime of period 1, the bright gray color corresponds to the regimes of period 2, and so on with darker gray tints; black color refers to chaos, and one of the gray tints indicates a region where the trajectory goes to infinity. The Poincaré section was selected with respect to the external drive period, which is a traditional approach to the nonautonomous flow systems. The map presented in Fig. 2 was constructed so that the initial point for every value of the control parameter was "inherited" from an established regime for the preced-

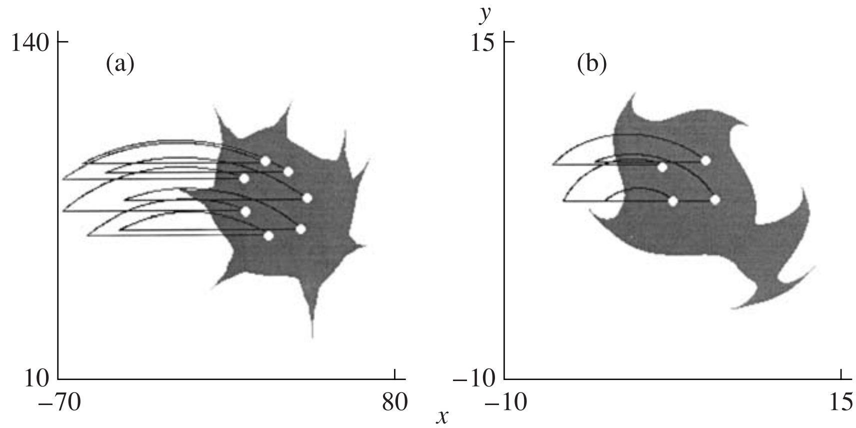

Fig. 3. Phase portraits of period-7 and -4 attractors of the nonautonomous Rössler system (3) on the (x, y) plane, showing the corresponding points in the Poincaré section and the basins of attraction: (a) $A = 77.5$, $T = 0.9$; (b) $A = 8.0$, $T = 1.41$.

ing value of this parameter. Figure 2b presents a fragment of the map on a greater scale.

An analysis of the map presented in Fig. 2 indicates that the system under consideration features stable regimes, although the autonomous system exhibited only phase trajectories going to the infinity. Figure 2c shows a possible phase portrait of the attractor of the nonautonomous system, which corresponds to a regime with period 4.

Let us consider the map of stable regimes in more detail. Figures 2a and 2b reveal a pattern characteristic of stable states, which contains a set of regions representing periodic and quasi-periodic regimes. This structure occupies a rather small interval with respect to the external drive period, but extends over a very broad range of amplitudes. The "tongs" of periodic regimes are acute at both top and bottom ends. An analogous structure was previously observed for the tongs of synchronization at subharmonics on the external force amplitude–period maps of autooscillatory systems under a pulsed drive action [10–14]. In the case under consideration, almost all tongs contain period-doubling regions.

As the drive period increases, the tongs acquire more complicated internal structure. In the region of quasi-periodic behavior, the tongs become undistinguishable because the autonomous system (1) has no periodic attractors and the imaging point moves along a helical trajectory (Fig. 1), where the number of turns increases with the time. Upon the introduction of an external drive, the pulses return the imaging point to the origin of the trajectory. This nature of stabilization is, to a certain extent, similar to the nature of synchronization with delayed feedback, which returns the system to a previous state. In the case under consideration, the feedback is replaced by external pulses, which also return the system trajectory to the previous state. Depending on how far will the imaging point go away during the absence of an external action (i.e., on the period of the external force) the arriving pulse either stabilizes the system or not. For example, at $T > 2.3$ (Fig. 2), the trajectory runs far away so that the external pulse cannot stabilize it. Note also a rather wide band of period 1 touching the top ends of the synchronization tongs. This band separates the region of runaway to infinity from the region of quasi-periodic regimes.

Figure 3 shows the projections of periodic attractors of the nonautonomous system (3) onto the plane (x, y), the corresponding points in the Poincaré section (colored white), and the basins of attraction for a fixed initial value of the third variable ($z_0 = 0.5$). As can be seen, the dimensions of the basins of attraction on the plane of dynamic variables are rather large, which is evidence for the robustness of the stabilized regimes with respect to a change in the initial conditions.

Thus, it was demonstrated that a sequence of external δ pulses, which arrive before the threshold of bifurcation leading to the appearance of stable and unstable immobile points, induces stable periodic and quasi-periodic regimes in a nonautonomous Rössler system. The map of dynamic regimes on the drive amplitude–period plane in this case exhibits the characteristic pattern of synchronization tongs.

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