Invited

EXAMPLES OF MECHANICAL SYSTEMS MANIFESTING CHAOTIC HYPERBOLIC ATTRACTORS OF SMALE-WILLIAMS TYPE

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An interesting kind of dynamical behavior is structurally stable chaotic dynamics associated with uniformly hyperbolic attractors, such as the Smale – Williams solenoid [1]. As we try developing clear and transparent examples [2], the preference surely should be given to mechanical systems. Indeed, it is the mechanical motions that are easily perceived and interpreted due to our everyday experience [3]. I report here on three simple mechanical systems with chaotic attractors of Smale – Williams type.

1. Consider a particle of unit mass on a plane in a stationary potential field possessing rotational symmetry about the origin, with minimum on the unit circle (Fig. 1). An additional potential force field is applied, being switched on and off periodically, with time interval T and producing short-time kicks of magnitude and direction depending on the instantaneous position of the particle. The model equations are

$$\begin{aligned} \ddot{x} &= \mu x(1-x^2-y^2) + (-x+x^2-y^2)\sum\nolimits_{n=-\infty}^{\infty}\delta(t-nT) - \alpha\dot{x} \\ \ddot{y} &= \mu y(1-x^2-y^2) + (-y-2xy)\sum\nolimits_{n=-\infty}^{\infty}\delta(t-nT) - \alpha\dot{y}. \end{aligned} \tag{1}$$

In practice, a pulsed potential field can be produced using six electromagnets (the hexapole) and an additional magnet G that creates a centrally symmetric part of the potential. All these magnets are switched on periodically by current pulses (Fig. 1a). The particle should be a permanent magnet, one pole of which is involved in the interaction, and the other one is removed outside the interaction region. Panel (b) depicts the trajectory of the particle, and panel (c) shows the attractor in the stroboscopic section in a two-dimensional projection. The attractor is a kind of Smale – Williams solenoid, with distinguishable Cantor-like transversal structure. As checked, the angular coordinate behaves in accordance with the expanding circle map (one bypass for the pre-image implies two detours for the image in the opposite direction).

2. Consider two disks arranged horizontally, one above the other, which rotate about the common axis alternately (while one is rotating, the other is at rest and vice versa) with angular velocity ω (Fig. 2a). On each disk a particle is placed, capable to slide with the friction force proportional to its relative velocity. Suppose that there is a potential field symmetric about the axis, with potential minimum at the center and strongly growing to the edges of the disks. In addition, assume that the two particles interact via the appropriately chosen potential $V(\mathbf{r}_1-\mathbf{r}_2)$, where $\mathbf{r}_1$ and $\mathbf{r}_2$ are the position vectors. Model equations are

$$\begin{aligned} \ddot{x}_1 &= x_1(k+x_1^2+y_1^2) + \alpha(-\omega_1 y_1 - \dot{x}_1) - \varepsilon[(x_1-x_2)^2-(y_1-y_2)^2], \\ \ddot{y}_1 &= y_1(k+x_1^2+y_1^2) + \alpha(\omega_1 x_1 - \dot{y}_1) + 2\varepsilon(x_1-x_2)(y_1-y_2), \\ \ddot{x}_2 &= x_2(k+x_2^2+y_2^2) + \alpha(-\omega_2 y_2 - \dot{x}_2) + \varepsilon[(x_1-x_2)^2-(y_1-y_2)^2], \\ \ddot{y}_2 &= y_2(k+x_2^2+y_2^2) + \alpha(\omega_2 x_2 - \dot{y}_2) - 2\varepsilon(x_1-x_2)(y_1-y_2), \end{aligned} \tag{2.5}$$

where

$$\omega_1(t) = \begin{cases} \omega_0, & nT \le t < nT + T/2 \\ 0, & nT + T/2 \le t < nT, \end{cases} \qquad \omega_2(t) = \begin{cases} 0, & nT \le t < nT + T/2 \\ \omega_0, & nT + T/2 \le t < nT. \end{cases} \tag{2.6}$$

With appropriate parameters, the chaotic dynamics in the map for the state transformation in successive periods of switching corresponds to the attractor of the Smale – Williams type.

3. The third example is related to a simplified model for parametric oscillations of a string. In the classic wave equation $\rho y_{tt} = G y_{xx}$ with periodic boundary conditions we assume the tension force to depend on time as $G(t) = 1 + a_2^0 \sin^2 \pi(t/T - 1/4)\sin 2\omega_0 t + a_6^0 \cos^2 \pi(t/T - 1/4)\sin 6\omega_0 t$, wheread the density depends on the spatial coordinate as $\rho(x) = 1 + \varepsilon \sin 4k_0 x$ [4]. Next, we introduce dissipation by the terms $-(\alpha + \beta u^2) y_t$ and $-\gamma y$ added to the right-hand side of the equation. Nonlinear dissipation is needed to stabilize the parametric instability; moreover, it is also important that the cubic

nonlinearity provides generation of the third harmonic in the wave-oscillatory motion. Taking into consideration that the parametric excitation alternates at k_0 and $3k_0$, we use the ansatz $y = u_1 \cos k_0 x + v_1 \sin k_0 x + u_3 \cos 3k_0 x + v_3 \sin 3k_0 x$ to compose the low-dimensional model as a set of second-order ordinary differential equations for the variables $u_{1,3}$ and $v_{1,3}$ [3]. Figure 3 illustrates results of the numerical simulation for the finite-dimensional model which agree well with those relating to the partial-differential equations [4].

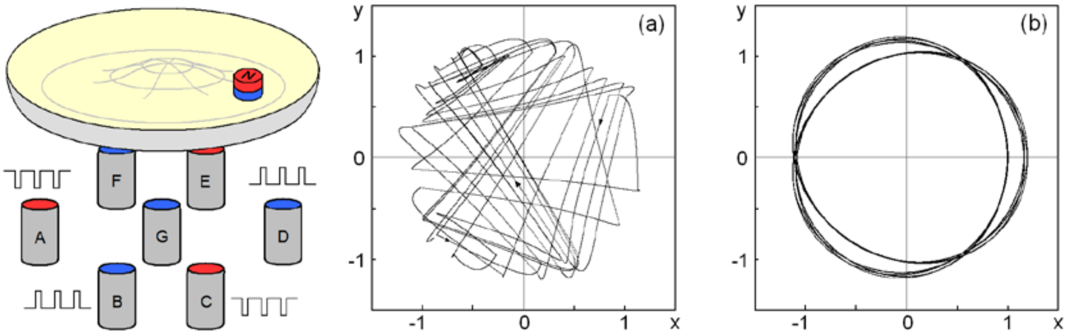

Fig. 1. Schematic setup (a), trajectory for the particle motion under periodic kicks (b) and stroboscopic portrait of the attractor (c). Parameters in equations (1): $\mu = 0.3$, $T = 4$

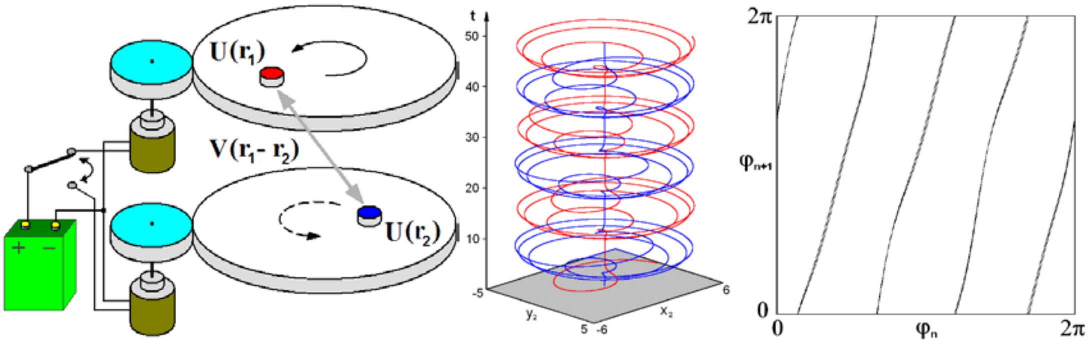

Fig. 2. Schematic setup (a), trajectories for the particles moving on the discs (b) and diagram for the angular variable of one particle (c). Parameters in equations (2): $k = 3,\ \alpha = 3,\ \varepsilon = 0.03,\ T = 16,\ \omega_0 = 2\pi$

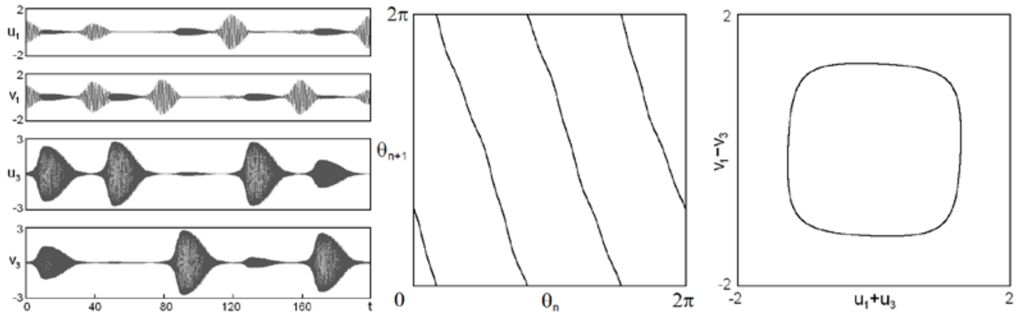

Fig. 3. Plots of the amplitude coefficients versus time (a), diagram for the angular variable for the amplitude ratio for cosine and sine components (b) and stroboscopic portrait of attractor in projection on the plane (c) at $\omega_0 = k_0 = 2\pi$, $T = 40,\ L = 1,\ a_2^0 = 0.4,\ a_6^0 = 0.2,\ \varepsilon = 0.2, \alpha = 0.4, \gamma = 0.03$

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