ABOUT LANDAU-HOPF SCENARIO IN ENSEMBLES OF REGULAR AND CHAOTIC OSCILLATORS

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Introduction

As is known, the Landau-Hopf scenario implies that the oscillatory modes with incommensurable frequencies emerge subsequently, which leads to increasingly more complicated oscillatory regimes [1, 2]. In this case, we can speak about a cascade of quasi-periodic Hopf bifurcations [3] responsible for the soft birth of the tori of increasingly higher dimension. The possibility of such a scenario remains largely debated for several decades. In this paper we discuss the conditions and situations, in which such a scenario can actually occur.

When choosing a model to analyze, we have to take into account several important physical aspects. First, the oscillatory modes responsible for the dynamics must be characterized by different degrees of activation. For example, if we consider an ensemble of coupled van der Pol oscillators, we must introduce a set of parameters λ_i controlling the negative friction and select them appropriately to ensure conditions for gradual involving of the modes in the motion in the course of decreasing dissipation level. (In the context of the hydrodynamic problems it just corresponds to increasing Reynolds number.) Second, all the oscillatory modes have to be separated in frequency in sufficient degree. Otherwise, essential interaction of the modes will occur, which can destroy the Landau-Hopf picture. Third and finally, it is desirable to have a situation, where the coupled oscillators are arranged in such way that they all are involved essentially in the interaction; it means that no preferable interaction of each concrete partial oscillator should occur, say, with spatially close neighbors, or with a different number of neighbors. If these three conditions are met, the control parameter of each single oscillator λ_i will regulate a certain quasi-periodic bifurcation. Otherwise, it might happen that due to the different number of interacting neighbors the elements with a close level of activation would have essentially different influence on the picture of the multi-frequency dynamics.

Network of five coupled van der Pol oscillators

Let us consider a model satisfying the above requirements. It will be a kind of network of five oscillators with global coupling and equidistant spectrum [4]:

$$\begin{aligned}
&\ddot{x}-(\lambda_1-x^2)\dot{x}+x+\frac{\mu}{4}(4\dot{x}-\dot{y}-\dot{z}-\dot{w}-\dot{v})=0,\\
&\ddot{y}-(\lambda_2-y^2)\dot{y}+(1+\frac{\Delta}{4})y+\frac{\mu}{4}(4\dot{y}-\dot{x}-\dot{z}-\dot{w}-\dot{v})=0,\\
&\ddot{z}-(\lambda_3-z^2)\dot{z}+(1+\frac{\Delta}{2})z+\frac{\mu}{4}(4\dot{z}-\dot{y}-\dot{x}-\dot{w}-\dot{v})=0,\\
&\ddot{w}-(\lambda_4-w^2)\dot{w}+(1+\frac{3\Delta}{4})w+\frac{\mu}{4}(4\dot{w}-\dot{y}-\dot{x}-\dot{z}-\dot{v})=0,\\
&\ddot{v}-(\lambda_5-v^2)\dot{v}+(1+\Delta)v+\frac{\mu}{4}(4\dot{v}-\dot{y}-\dot{x}-\dot{z}-\dot{w})=0
\end{aligned} \tag{1}$$

Here λ_i are control parameters responsible for excitation of the partial oscillators, Δ determines the frequency detuning of the oscillators, and the frequency of the first oscillator is unity. We set hereafter $\lambda_1=0.1$, $\lambda_2=0.2$, $\lambda_3=0.3$, $\lambda_4=0.4$, $\lambda_5=0.5$.

Figure 1 shows the Lyapunov chart obtained for the model (1) on the parameter plane (Δ,μ). Because of the structure of the system, the main resonance effects are actually excluded. In particular, no tongues of resonant tori of different dimensions similar to those mentioned, e.g., in Ref. [5] are observed. The only pronounced tongue corresponding to a resonant two-frequency torus may be seen in the region $\Delta\le 0.5$ between the domains of the five-frequency tori and of the complete synchronization. Interestingly, it is immersed in the region of chaos that occurs at small coupling. Here the three-frequency, four-frequency, and partially the five-frequency tori are destroyed, although the chaos is weak. On the other hand, in the case of large detuning of the oscillators, $\Delta\ge 1$, with decreasing dissipation parameter μ one can observe the appearance of all the tori of higher dimensions. Boundaries of the relevant areas in the asymptotic $\Delta \to \infty$ correspond to the values of the control parameters $\mu = \lambda_i$. Thus, a decrease in the parameter of dissipative coupling qualitatively corresponds to the pattern expected for the Landau-Hopf scenario.

Network of five coupled chaotic Rössler oscillators

Now let us consider a network of five coupled Rössler oscillators:

$$\dot{x}_n = -(1+\frac{n-1}{4}\Delta)y_n - z_n,$$
$$\dot{y}_n = (1+\frac{n-1}{4}\Delta)x_n + py_n + \frac{\mu}{4}\sum_{i=1}^{5}(y_i - y_n), \qquad (2)$$
$$\dot{z}_n = q + (x_n - r_n)z_n.$$

Here the parameter Δ is responsible for the frequency detuning of the oscillators. Parameter *n* varies from 1 to 5.Values of the parameters $p = 0.15$, $q = 0.4$, $r = 8.5$ correspond to the chaotic regime in individual subsystems [6]. Lyapunov exponents chart for the system (2) is shown in Figure 2. At low coupling the hyperchaotic mode HC5 dominates with five positive Lyapunov exponents. With the increasing of coupling the number of positive exponents gradually decreases, and after the transition via HC4, HC3 and HC2 regimes usual chaos C arises. For even stronger coupling a complicated picture of alternating modes of different types may be observed, and tori of different dimensions are quite typical. At high frequency detuning, a cascade of quasiperiodic Hopf bifurcation is observed. It corresponds to the Landau-Hopf scenario [1, 2, 4]. But now the bifurcations follow quickly one after another, so that the scenario develops in a narrow range of coupling parameter. Nevertheless, we note the fact of possible Landau-Hopf scenario and a cascade of quasiperiodic bifurcations of invariant tori in coupled chaotic oscillators.

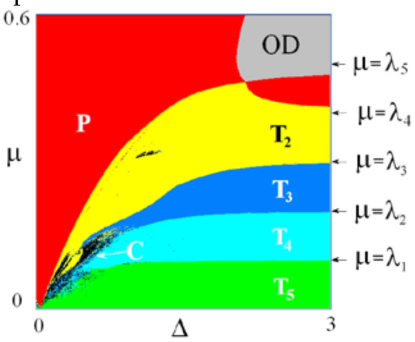

Fig. 1. Chart of Lyapunov exponents for the network of five globally coupled non-identical van der Pol oscillators (1), where $\lambda_1 = 0.1$, $\lambda_2 = 0.2$, $\lambda_3 = 0.3$, $\lambda_4 = 0.4$, $\lambda_5 = 0.5$

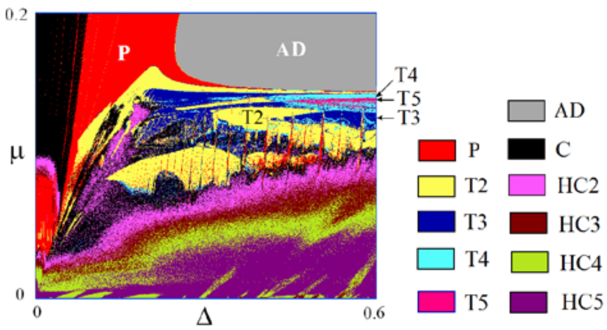

Fig. 2. Chart of Lyapunov exponents for the network of five chaotic oscillators (2). $p = 0.15$, $q = 0.4$, $r_1 = 7.3$, $r_2 = 7.6$, $r_3 = 7.9$, $r_4 = 8.2$, $r_5 = 8.5$

Conclusion

Thus, under certain conditions, e.g. in the case of non-identical parameters of the active modes and of detuning for the modes in an ensemble of self-oscillating elements, one can observe a sequential cascade of soft quasi-periodic bifurcations involving tori of increasing dimension that can be regarded as occurrence of the Landau-Hopf scenario. At the same time, high-dimensional quasiperiodicity is typical not only for ensembles of oscillators with regular behavior, but also for dissipatively coupled chaotic oscillators.

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