Coexistence of Dissipative and Conservative Regimes in Unidirectionally Coupled Maps

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Abstract. We consider the billiard system consisting of a particle moving between two walls one of which is plane and fixed and the other one is harmonically corrugated (like in Tennyson-Lieberman-Lichtenberg system) and oscillates harmonically. The collisions of the particle and the wall are suggested to be elastic. We assume that the oscillation and the corrugation amplitudes are weak so some significant simplifications of the system are justified which results in the system of two unidirectionally coupled 2D maps. The master system is the original Tennison-Lieberman-Lichtenberg system with fixed walls and the slave system is Ulam map parametrically driven by the master system. The variables of the slave system are the velocity of a particle before the collision and the time between the collisions. We calculate numerically the Jacobian of various trajectories of the system and reveal that the regions of conservative (with the Jacobian very close to zero) and dissipative dynamics coexist in the phase space of the system.

Keywords: Time-dependent billiards · Numerical research · Mixed dynamics.

1 Introduction

Billiard-like dynamical systems are of great interest, both from applied and fundamental points, since a wide variety of nonlinear phenomena is observed in them and they are easy to research [1-4]. Usually billiards are assumed to be conservative (i.e. without loss of energy) but also they can be dissipative [5] (e.g. with energy loss by friction or inelastic collisions). Dissipative dynamics is characterized by the existence of attracting invariant sets [6]. The conservative behavior of billiards is well described by Hamiltonian systems [7]. For example, the phase space of Tennyson-Lieberman-Lichtenberg system [8] is typical for non-integrable two-dimensional Hamiltonian systems. It has regular trajectories that are quasi-periodic (KAM) tori and chaotic ones that are destructed tori as a result of the perturbations of an integrable Hamiltonian system [9].

In this paper we consider a system that consists of a particle moving between two walls one of which is plane and fixed and the other one is harmonically corrugated (like in Tennyson-Lieberman-Lichtenberg system) and oscillates harmonically. The collisions of the particle and the wall are suggested to be elastic. We assume that the oscillation and the corrugation amplitudes are weak so some significant simplifications of the system are justified, which results in the system of two unidirectionally coupled 2D maps. The master system is original Tennison-Lieberman-Lichtenberg system with fixed walls and the slave system is Ulam map [10] parametrically driven by the master system. We calculate numerically the Jacobian of various trajectories of the system and reveal that the regions of conservative (with the Jacobian very close to zero) and dissipative dynamics coexist in the phase space of the system.

2 Model Description

The original system consists of a particle which moves between two boundaries and elastically collides with them. One boundary is fixed and set by the equation:

$$y_1 = 0. \tag{1}$$

The other boundary is corrugated and can oscillate harmonically. Then its equation is:

$$y_2 = F(x, t) = b \cos kx + a \cos wt + h \tag{2}$$

In (2) a – the oscillation amplitude, b – the corrugation amplitude, h – the average distance between the boundaries.

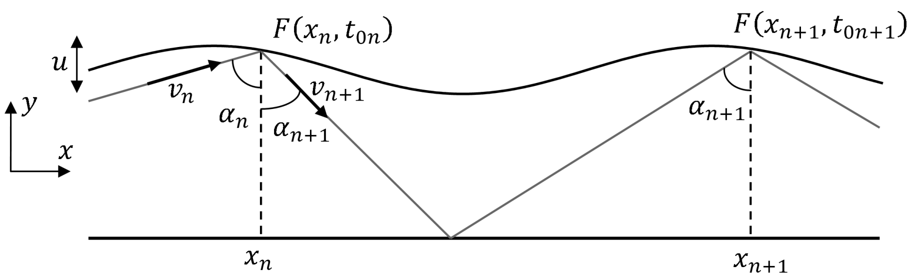

Fig. 1. Illustration of the particle movement between two boundaries. x_n – the coordinate of the n-th collision with the upper boundary; α_n – the angle between the normal to the bottom boundary and the velocity vector at the moment of the n-th collision v_n; v_n – the particle velocity at the moment of the n-th collision with the upper boundary; t_{0_n} – the elapsed time from the moment the particle begins to move until the moment of the n-th collision with the upper boundary.

The model is mechanical and it is not difficult to obtain expressions for $v_{n+1}, \alpha_{n+1}, x_{n+1}, t_{0_{n+1}}$ (Fig. 1) in the case of the weak amplitudes of the corrugation and the oscillation. So, these expressions form the following 4D map:

$$\begin{cases} v_{n+1} = \sqrt{v^2_{n+1_x} + v^2_{n+1_y}}; \\ \alpha_{n+1} = \arctan \frac{v_{n+1_x}}{v_{n+1_y}}; \\ x_{n+1} = x_n + 2h \frac{v_{n+1_x}}{v_{n+1_y}}; \\ t_{0_{n+1}} = t_{0_n} + \frac{2h}{v_{n+1_y}}. \end{cases} \tag{3}$$

In (3): $v_{n+1_x} = v_n \sin(\alpha_n + 2\gamma) - 2\gamma u$, $v_{n+1_y} = v_n \cos(\alpha_n + 2\gamma) - 2u$, $u = -aw \sin wt_{0_n}$, $\gamma = -kb \sin kx_n$. The number of the parameters can be reduced via the following replacement:

$$\begin{cases} \phi_n = kx_n; \\ \psi = wt_{0_n}; \\ \Omega = \frac{v_{n_x,y}}{2hw}; \\ A = 2hk; \\ B = \frac{a}{h}; \\ C = bk. \end{cases} \tag{4}$$

It results in the 4D map with four parameters:

$$\begin{cases} \Omega_{n+1} = \sqrt{\Omega^2_{n+1_x} + \Omega^2_{n+1_y}}; \\ \alpha_{n+1} = \arctan \left[\frac{\Omega_{n+1_x}}{\Omega_{n+1_y}} \right]; \\ \phi_{n+1} = \phi_n + A \frac{\Omega_{n+1_x}}{\Omega_{n+1_y}}; \\ \psi_{n+1} = \psi_n + \frac{1}{\Omega_{n+1_y}}. \end{cases} \tag{5}$$

In (5): $\Omega_{n+1_x} = \Omega_n \sin(\alpha_n + 2\gamma) - 2\gamma u$, $\Omega_{n+1_y} = \Omega_n \cos(\alpha_n + 2\gamma) - 2u$, $\gamma = -C \sin \phi_n, u = -B \sin \psi_n, \Omega_n$ – the dimensionless velocity, ϕ_n – the dimensionless coordinate, ψ_n – the dimensionless time, A — the dimensionless average distance between the boundaries, B — the dimensionless oscillation amplitude, C — the dimensionless corrugation amplitude.

The system (5) is of interest because the critical velocity appears in the system with weak corrugation and oscillation of the boundary [12] and if the initial velocity is lower than the critical velocity, then the particle has slow Fermi acceleration, otherwise the particle has fast or classical Fermi acceleration [11]. If the particle is moving with slow Fermi acceleration, then $u << \Omega_n$ and (5) can be simplified as follows:

$$\begin{cases} \alpha_{n+1} = \alpha_n - 2C \sin \phi_n \\ \phi_{n+1} = \phi_n + A \tan \alpha_{n+1} \end{cases} \tag{6}$$

$$\begin{cases} \Omega_{n+1} = \Omega_n + 2B \sin \psi_n \cos \alpha_{n+1} \\ \psi_{n+1} = \psi_n + \frac{1}{\Omega_n + 2B \sin \psi_n} \end{cases} \tag{7}$$

The four-dimensional map is split into two two-dimensional ones, where (6) is the Tennison-Lieberman-Lichtenberg map that affects the system (7) which is similar to Ulam map. The first map is the master system and the second one is the slave system.

3 Jacobian of System

The Jacobian of the system (6) is identically equal to one, which means that the system is conservative. However the Jacobian of the system (7) depends on the variables:

$$J = 1 - 2B\left(\frac{\sin(\alpha_n - 2C\sin\phi_n)}{\Omega_n\cos(\alpha_n - 2C\sin\phi_n) + 2B\sin\psi_n}\right)^2\cos\psi_n. \quad (8)$$

The Jacobian of the full system consisting of these two maps is the same. Since the Jacobian depends on the state of the system, we should calculate the iteration-averaged (average over the number of iterations) Jacobian along the trajectory to find out if the regime is conservative or dissipative.

Let us fix the parameters $A = 2, B = 0.03, C = 0.05$ and plot the map of the Jacobian values, on which: vertical α_0 – the initial angle, horizontal ϕ_0 – the initial coordinate. The color indicates the absolute value of the Jacobian averaged along the trajectory: orange - impossible to determine (we will discuss why below), red - less than one, blue - equal to one, green - larger than one. The initial velocity Ω_0 is different for each figure. Initial time $\psi_0 = 0$ is selected for all figures.

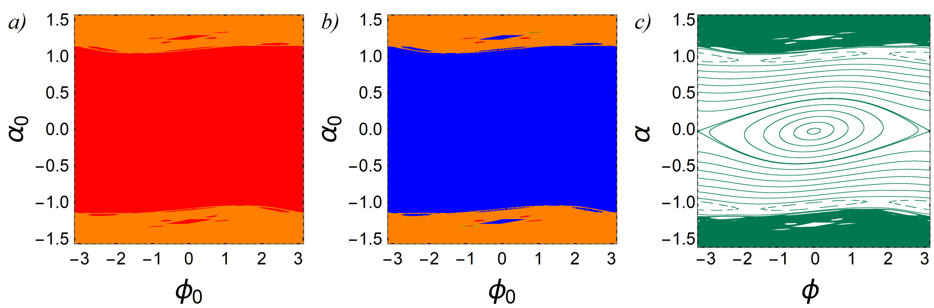

Fig. 2. Map of the Jacobian of the system (7) with the parameters: $A = 2, B = 0.03, C = 0.05$ and $\psi_0 = 0$, (*a*) $\Omega_0 = 0.1$; (*b*) $\Omega_0 = 1.1$; α_0 and ϕ_0 are coordinates of the map and colors marked the value of Jacobian. (*c*) Phase portrait of the Tennyson-Lieberman-Lichtenberg system with the same parameters.

The map of the Jacobian at the initial velocity $\Omega_0 = 0.1$ is shown in Fig. 2a. The red parts are the areas of dissipative dynamics and the orange ones are the areas of the initial conditions in which the numerical calculation of the Jacobian gives significant errors, since the Jacobian changes significantly when the number of iterations for its averaging is changed. If we compare the map of the Jacobian with the phase portrait of the system (6) we can see that the orange regions are situated in the chaotic layer of the phase space. The chaotic trajectories are located close to the angle $\frac{\pi}{2}$. It means that the collisions of a particle with the wall occur almost tangentially to it. Due to this reason the time between hits becomes extremely large. This situation seems to be similar to Levy's flight [13] i.e. the extremely long flight without collisions. It causes that the values of the time derivatives become large and they significantly affect the value of the local Jacobian, thereby leading to a significant change in averaged Jacobian.

If we increase the initial velocity to $\Omega_0 = 1.1$ then most of the dissipative regions become the conservative ones, the numerical value of the Jacobian is equal to one with adequate accuracy (Fig. 2b). We assumed that the region is conservative when $|\overline{J}| < 1 - |\epsilon|$. If ϵ is gradually reduced, then at one moment all values of the average Jacobians that fell into this region stop doing this. Also, this region is resistant to weak variation of the parameters or the initial conditions. We calculated the largest Lyapunov exponent, which is zero with adequate accuracy for this region, to validate that these areas are not the regions of unstable initial conditions. All this makes it possible to confirm that the blue parts are the areas of conservative dynamics. Note that there are small green areas where attractors exist, but their Jacobian is also calculated incorrectly due to Levy flights.

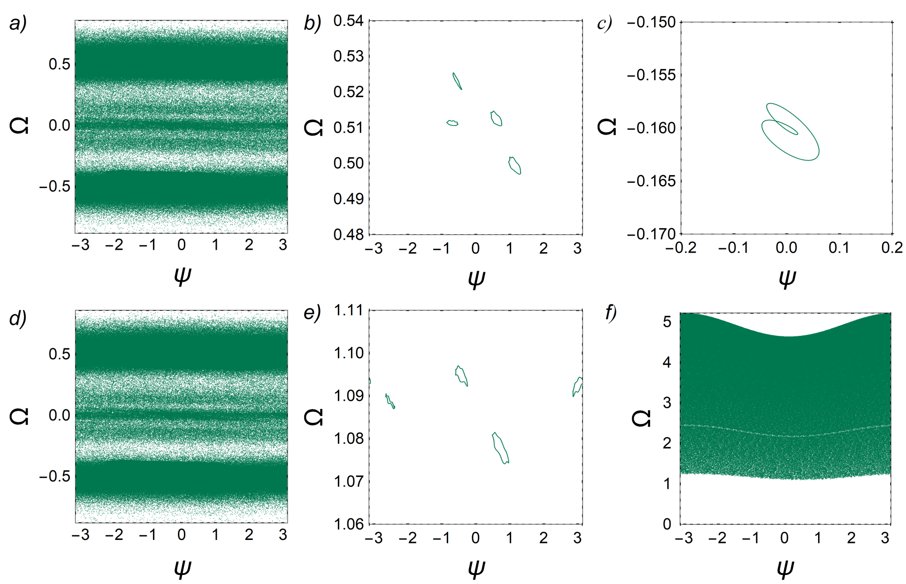

Fig. 3. Some trajectories of the system (7) with the parameters: $A = 2, B = 0.03, C = 0.05$ and $\psi_0 = 0$; (*a, b, c*) $\Omega_0 = 0.1$; (*d, e, f*) $\Omega_0 = 1.1$; (*a, d*) $\alpha_0 = 0.41\pi$, $\phi_0 = -0.57\pi$; (*b, e*) $\alpha_0 = 0.382\pi$, $\phi_0 = -0.19\pi$; (*c, f*) $\alpha_0 = 0.05\pi$, $\phi_0 = 0$.

It should be noted that we also calculated values of the largest Lyapunov exponent for such initial conditions and we found that it is positive in the orange areas and is equal to zero with adequate accuracy in the other areas. Examples of numerical values of the largest Lyapunov exponent are provided below.

Let us consider some of the phase trajectories of the system (7). For all trajectories in Fig. 3 the initial time $\psi_0 = 0$, in Fig. 3a, b, c $\Omega_0 = 0.1$, and in Fig. 3d, e, f $\Omega_0 = 1.1$. The trajectory in Fig. 3a with $\phi_0 = -0.57\pi$ and $\alpha_0 = 0.41\pi$ (chaotic region) has the largest Lyapunov exponent $\Lambda = 1.49$ (the Jacobian is incorrect). This is a chaotic trajectory and it has not undergone significant changes while the initial velocity increases. (Fig. 3e). The trajectory in Fig. 3b with $\phi_0 = -0.19\pi$ and $\alpha_0 = 0.382\pi$ (the stability island) has the dissipative Jacobian $J = -0.61$ and the largest Lyapunov exponent $\Lambda = 6.76 \times 10^{-6}$ which can be assumed equal to zero. Since there are only two Lyapunov exponents of the system (7), the second one has to be negative because for dissipative Jacobians the sum of the Lyapunov exponents has to be negative. In 2D maps one zero and one negative Lyapunov exponents indicate that there is a two-frequency torus in the phase space. The attractor changes with the increase of the initial velocity (Fig. 3e) and this trajectory has $J = 0.95$ and $\Lambda = 6.85 \times 10^{-6}$, which indicates the existence of the multistability in the system. The trajectory in Fig. 3c with $\phi_0 = 0$ and $\alpha_0 = 0.05\pi$ (close to the elliptic point) has the $J = 0.97$ and $\Lambda = 5.29 \times 10^{-6}$, which indicates the existence of the attractor. The attractor disappears as the initial velocity increases (Fig. 3f), $J = 1 - 0.2 \times 10^{-6}$ and $\Lambda = 1.10 \times 10^{-6}$. We believe that the Jacobian is equal to one with adequate accuracy, which indicates either conservative dynamics or instability. Since the largest Lyapunov exponent is equal to zero with adequate accuracy, it means that blue parts are regions of the conservative dynamics. Note that a part of the trajectory is shown in Fig. 3f. In fact, it continues to move upwards in the same way and its velocity increases without limit.

4 Conclusion

The research shows numerically that in the Ulam-like map parametrically driven by Tennison-Lieberman-Lichtenberg map the dissipative and the conservative regimes coexist in the phase space. For both maps the Jacobian was found. The Tennison-Lieberman-Lichtenberg map is conservative while the other one has the Jacobian which depends on the system's state and has to be either larger than one or less than one. When we calculated the Jacobian averaged along the trajectory for different initial conditions, it turned out that if the initial velocity is small then the average Jacobian is less than one and if the initial velocity is large enough then the system has to behave conservatively. This consideration is true for trajectories that are not located in the area of chaos of the Tennison-Lieberman-Lichtenberg system. Another problem is that for chaotic trajectories it is impossible to calculate the Jacobian numerically.

The similar phenomenon of conservative regimes and attractors coexisting in the phase space was studied in [14] and was called mixed dynamics there. It was shown that such dynamics occur if the system is time-reversible with involution. That means that some homeomorphism in the phase space exists when there is some transformation of the orbits into themself with the reversion of time. However there is no such homeomorphism (at least, the evident one) in our system, so we think that the phenomenon in our system is not exactly the same.

References

1. Bunimovich, L. A.: Mushrooms and other billiards with divided phase space. Chaos **11**, 802 (2001)
2. Chernov, N., Markarian, R.: Chaotic Billiards. Mathematical Surveys and Monographs, USA (2006)
3. Loskutov, A.: Dynamical chaos: systems of classical mechanics. Phys.-Usp **50**, 939–964 (2007)
4. Sinai, Ya. G.: Dynamical systems with elastic reflections. Russ. Math. Surv. **25**, 137.
5. Oliveira, D. F. M., Leonel, E. D.: Evolution to the equilibrium in a dissipative and time dependent billiard. Phys. A **465**, 043122 (2011)
6. Kuznetsov, Yu. A.: Elements of Applied Bifurcation Theory. 2st Ed. Applied Mathematical Sciences, New York (1998)
7. Zaslasvsky, G. M.: Hamiltonian Chaos and Fractional Dynamics. Hamiltonian Chaos and Fractional Dynamics. Oxford University Press, Oxford (2005)
8. Tennyson, J. L., Lieberman, M. A., Lichtenberg, A. J.: Diffusion in near-integrable hamiltonian systems with three degrees of freedom. AIP Conf. Proc. **57**, 272 (1980)
9. Lichtenberg, A. J., Lieberman, M. A.: Regular and Chaotic Dynamics. 1st Ed. Applied Mathematical Sciences, New York (1983)
10. Ulam, S. M.: On Some Statistical Properties of Dynamical Systems. Berkeley Sympos. Math. Stat. Prob. **4.3**, 315–320 (1961).
11. Fermi, E.: On the origin of the cosmic radiation. Phys. Rev. **75**, 1169 (1949)
12. Lubchenko, D. O., Savin A. V.: Critical Velocity for the Onset of Fast Fermi Acceleration. IJBC **32**, No. 12, 2250177 (2022)
13. Lévy, P.: Téorie de l'addition des variables aléatoires. Gauthier-Villars, Paris (1937)
14. Gonchenko, S. V., Turaev, D. V.: On three types of dynamics and the notion of attractor. Trudy Mat. Inst. Steklova **297**, 133–157 (2017)