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Regional Structure of Two- and Three-Frequency Regimes in a Model of Four Phase Oscillators

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We consider the hierarchical organization of the parameter space of four coupled phase oscillators. The structure of the complete synchronization and quasi-periodicity regions of different dimensions are studied. The bifurcation mechanisms of destruction of full and partial synchronizations are discussed.

Keywords: Synchronization; coupled phase oscillators; quasi-periodic oscillation; bifurcation.

1. Introduction

In recent years, the problems of multifrequency quasi-periodicity have been objects of interest in several studies in physics and mathematics [Anishchenko *et al.*, 2009, 2006, 2007, 2008; Emelianova *et al.*, 2013; Emelianova *et al.*, 2014; Kuznetsov *et al.*, 2013; Broer *et al.*, 2008a, 2008b; Vitolo *et al.*, 2010; Broer *et al.*, 2011; Banerjee *et al.*, 2012; Sekikawa *et al.*, 2014; Kamiyama *et al.*, 2012]. Such problems are essential for different areas of natural science. For example radio-physics, electronics, biophysics, optics, etc. In a certain sense, this class of problems can be referred to as the problem of synchronization of quasi-periodic oscillations. One of the simplest systems used to study this problem is an ensemble of coupled van der Pol oscillators. It is a universal model whose dynamics is widely investigated [Pikovsky *et al.*, 2001; Landa, 1996; Balanov *et al.*, 2009; Guckenheimer & Holmes, 1983]. In the case of small coupling parameter multifrequency quasi-periodic dynamics are typical for such ensemble. The study of systems with three-frequency and higher-dimensional quasi-periodicity is a complex and multifaceted problem. The cases of three oscillators with dissipative coupling and two nonautonomous oscillators with dissipative coupling are considered in the following papers [Anishchenko *et al.*, 2009, 2006, 2007, 2008; Emelianova *et al.*, 2013; Emelianova *et al.*, 2014; Kuznetsov *et al.*, 2013; Broer *et al.*, 2008a, 2008b; Vitolo *et al.*, 2010; Broer *et al.*, 2011; Banerjee *et al.*, 2012; Sekikawa *et al.*, 2014; Kamiyama *et al.*, 2012; Landa, 1996]. In one of the most fundamental works [Landa, 1996] a system of two coupled rotation maps is discussed as a model of three coupled oscillators. The frequency detuning parameters plane is analyzed and a variety of bifurcations observed in coupled maps are discussed. The problem of the dynamics of ensembles of three phase oscillators is discussed, for example, in [Emelianova *et al.*, 2013; Emelianova *et al.*, 2014; Ashwin *et al.*, 1990; Ashwin *et al.*, 2008], and four or more oscillators, for example, in [Emelianova *et al.*, 2013; Emelianova *et al.*, 2014; Maistrenko *et al.*, 2004; Kuznetsov *et al.*, 2011].

Revealing the bifurcation mechanisms of the destruction of complete synchronization is one of the important tasks [Anishchenko *et al.*, 2009; Emelianova *et al.*, 2013]. One of the features, in the case of three dissipatively coupled oscillators, is a degenerate saddle-node bifurcation. In this case, two pairs of singular points — a stable equilibrium point and a saddle, and an unstable equilibrium point and a second saddle — are merging simultaneously. The reason for this behavior is the presence in the phase space of a common heteroclinic contour including all of the four equilibrium points. In the case of a system of three coupled oscillators, the bifurcation curves of this type border the region of complete synchronization in the parameter plane of coupling strength and frequency detuning. It takes a form of a characteristic tongue [Emelianova *et al.*, 2013]. In this case, the synchronization region has a threshold for the coupling parameter value. The point defining the threshold value lies at the intersection of two lines of degenerate saddle-node bifurcations. This point corresponds to a codimension two bifurcation. All four possible equilibrium states merge simultaneously at this point [Emelianova *et al.*, 2013; Landa, 1996]. In turn, it was shown in [Ashwin *et al.*, 1990; Ashwin *et al.*, 1993] that the emergence of three-frequency quasi-periodicity is due to the saddle-node bifurcation of a stable and unstable two-frequency tori.

In the present paper, we consider a similar problem for the case of a chain of four coupled oscillators. The aim of the work is to reveal the hierarchical structure of synchronization regions of multifrequency modes of different dimensions, as well as to analyze the bifurcation mechanisms of the destruction of two- and higher frequency quasi-periodicity. The study will be carried out in terms of the phase model obtained analytically.

The work is structured as follows. In Sec. 2, we derive a system of phase equations for a chain of coupled van der Pol equations using the method of complex amplitudes. In Sec. 3, we present a detailed description of the structure of the parameter plane for this model and discuss the typical dynamical regimes. In Sec. 4, we discuss a sequence of bifurcations which lead to a system of hierarchically organized regions of multifrequency regimes in the parameter plane.

2. The System Equations

We consider a chain of four van der Pol oscillators with dissipative coupling:

$$
\begin{aligned}
&\ddot{x} - (\lambda - x^2)\dot{x} + x + \mu(\dot{x} - \dot{y}) = 0,\\
&\ddot{y} - (\lambda - y^2)\dot{y} + (1 + \Delta_1)y\\
&\qquad + \mu(\dot{y} - \dot{x}) + \mu(\dot{y} - \dot{z}) = 0,\\
&\ddot{z} - (\lambda - z^2)\dot{z} + (1 + \Delta_2)z\\
&\qquad + \mu(\dot{z} - \dot{y}) + \mu(\dot{z} - \dot{w}) = 0,\\
&\ddot{w} - (\lambda - w^2)\dot{w} + (1 + \Delta_3)w\\
&\qquad + \mu(\dot{w} - \dot{z}) = 0.
\end{aligned}
\tag{1}
$$

Here λ is the control parameter responsible for the excitation of the partial oscillators; Δ_1, Δ_2 and Δ_3 are frequency detunings between the second and the first oscillators, the third and the first oscillators, the fourth and the first oscillators respectively; μ is the parameter of the dissipative coupling. The frequency of the first oscillator is assumed to be normalized by one.

Following the method presented in the works [Emelianova *et al.*, 2013; Emelianova *et al.*, 2014], we obtain the corresponding Landau–Stewart equations. For this purpose, we will use the method of complex amplitudes[1] [Pikovsky *et al.*, 2001; Landa, 1996]. We present dynamic variables in the form:

$$
\begin{aligned}
x &= a\exp(it) + a^*\exp(-it),\\
y &= b\exp(it) + b^*\exp(-it),\\
z &= c\exp(it) + c^*\exp(-it),\\
w &= d\exp(it) + d^*\exp(-it)
\end{aligned}
\tag{2}
$$

and use additional conditions traditional for this method:

$$
\begin{aligned}
\dot{a}\exp(it) + \dot{a}^*\exp(-it) &= 0,\\
\dot{b}\exp(it) + \dot{b}^*\exp(-it) &= 0,\\
\dot{c}\exp(it) + \dot{c}^*\exp(-it) &= 0,\\
\dot{d}\exp(it) + \dot{d}^*\exp(-it) &= 0.
\end{aligned}
\tag{3}
$$

Here $a(t)$, $b(t)$, $c(t)$, $d(t)$ are the slow complex amplitudes of four oscillators, which are varying

[1] It is a standard method. Therefore, we do not reproduce complete calculations here.

slowly in comparison with the basic oscillations with the unit frequency. Then we substitute expressions (2) and (3) into Eqs. (1), multiply the equations by factor $\exp(-it)$, and perform averaging over a period of the basic oscillations. Then, we get the set of truncated equations:

$$\begin{aligned} 2\dot{a} &= a - |a|^2 a - \mu(a-b), \\ 2\dot{b} &= b - |b|^2 b + i\Delta_1 b - \mu(2b-a-c), \\ 2\dot{c} &= c - |c|^2 c + i\Delta_2 c - \mu(2c-b-d), \\ 2\dot{d} &= d - |d|^2 d - \mu(d-c). \end{aligned} \tag{4}$$

A control parameter λ is removed by renormalization of variables and parameters.

Let us set $a = R\exp(i\psi_1)$, $b = r\exp(i\psi_2)$, $c = v\exp(i\psi_3)$, $d = w\exp(i\psi_4)$. Here R, r, v and w are real oscillator amplitudes, and ψ_i are oscillator phases. If we now assume that the parameter of the coupling is small and all oscillators move in the vicinity of their unperturbed orbits of unit radius then similar to the [Emelianova *et al.*, 2013; Emelianova *et al.*, 2014] we obtain the phase equations for the relative phases of the oscillators:

$$\begin{aligned} \dot{\theta} &= -\frac{\Delta_1}{2} - \mu\sin\theta + \frac{\mu}{2}\sin\varphi, \\ \dot{\varphi} &= \frac{\Delta_1 - \Delta_2}{2} + \frac{\mu}{2}\sin\theta - \mu\sin\varphi + \frac{\mu}{2}\sin\phi, \\ \dot{\phi} &= \frac{\Delta_2 - \Delta_3}{2} + \frac{\mu}{2}\sin\varphi - \mu\sin\phi. \end{aligned} \tag{5}$$

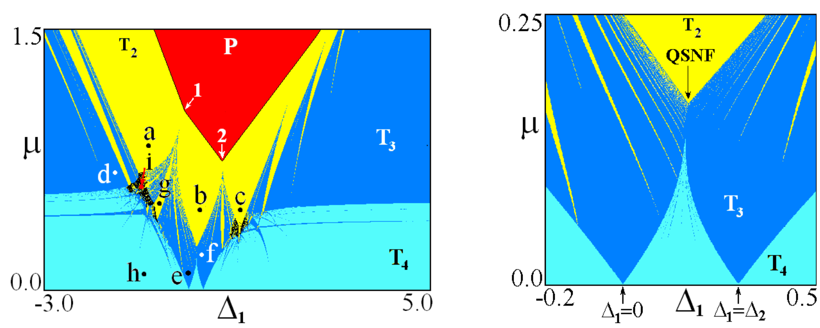
Here

$$\theta = \psi_1 - \psi_2, \quad \varphi = \psi_2 - \psi_3, \quad \phi = \psi_3 - \psi_4.$$

3. The Typical Dynamic Regimes

Figure 1 shows the chart of Lyapunov exponents[2] of the system (5) on the parameter plane of frequency detuning and parameter of coupling (Δ_1, μ). To draw the chart, we use the following method. We calculate the Lyapunov exponents at each green point of the parameter plane (Δ_1, μ). Then we distinguish the following dynamic regimes depending on the signature of the spectrum of Lyapunov exponents:

(1) the complete synchronization of four oscillators P (red color), in this case, all Lyapunov exponents are negative;
(2) the two-frequency quasi-periodicity T_2 (yellow color), in this case, the first Lyapunov exponent equals zero and the other l Lyapunov exponents are negative;
(3) the three-frequency quasi-periodicity T_3 (blue color), in this case, the first and the second Lyapunov exponents equal zero and the third Lyapunov exponent is negative;
(4) the four-frequency quasi-periodicity T_4 (light blue color), in this case, the first, the second and the third Lyapunov exponents equal zero;
(5) the chaotic attractor C (the black color), in this case, the first Lyapunov exponent is positive.



(a) (b)

Fig. 1. Chart of Lyapunov exponents for the system of four coupled van der Pol oscillators (5) with its magnified fragment. $\Delta_2 = 0.3$, $\Delta_3 = 1$. P is the region of complete synchronization of four oscillators; T_2 is the region of two-frequency quasi-periodicity; T_3 is the region of three-frequency quasi-periodicity; T_4 is the region of four-frequency quasi-periodicity; the region of chaotic attractor is indicated by black color; *QSNF* is the quasi-periodic point saddle node fan; points 1 and 2 are intersection points of lines (7) which are shown in black color.

[2]When calculating the exponents, we chose the initial phase values in the vicinity of the origin and the time interval was equal to 10000 units of normalized time. This type of computation leads to what should be called "local finite-time Lyapunov exponents", but for brevity we will use the term "Lyapunov exponents" below.

Here the types of regimes are indicated as for the original system (1), while the signature of the spectrum is indicated for the system (5). In this case (as we will show below) there exists a certain correspondence between the regimes of the system (5) and of the unabridged system (1) which is due to the averaging procedure.

In Fig. 1(a), one can see hierarchical organization of regions corresponding to different types of synchronization regimes. Its enlarged fragment in Fig. 1(b) presents the case of weak coupling. In this figure one can see two tongues of three-frequency quasi-periodic modes immersed in the domain of four-frequency quasi-periodicity. The tops of these tongues lie on the axis Δ_1 at points determined by the following resonance conditions. The frequency of the first oscillator coincides with the frequency of the second oscillator ($\Delta_1 = 0$). And the frequency of the second oscillator coincides with the frequency of the third oscillator ($\Delta_1 = \Delta_2$).[3]

To illustrate some typical regimes of the system (5), examples of attractors in the space of three relative phases are shown in Fig. 2. The letters correspond to the points on the chart of Lyapunov exponents [Fig. 1(a)]. Figures 2(b) and 2(c) demonstrate the simplest invariant curves embedded in the three-dimensional phase space. They correspond to two-frequency quasi-periodic modes.[4]

In Fig. 2(a) the phases θ and ϕ fluctuate around a certain equilibrium value, while the phase φ varies across the whole range of values $(0, 2\pi)$. It means that oscillators are mode-locked pairwise, namely the first is synchronized with the second oscillator and the third with the fourth, as schematically shown in Fig. 3(a). We can classify the two-frequency quasi-periodic partial synchronization modes using three-component rotation numbers $w = p : q : r$, where p, q and r correspond to the numbers of essential intersections of the invariant curve with the cube sides in the space of relative phases of the oscillators (θ, φ, ϕ). The above mode has a rotation number $w = 0 : 1 : 0$ [Fig. 2(a)].

In Fig. 2(b) one can see a similar regime. Now the phases θ and φ fluctuate around a certain equilibrium value, while the phase ϕ varies across the whole range of values. This situation corresponds to the situation when first, second and third oscillators are partially mode-locked simultaneously, as shown in Fig. 3(b). Rotation number of the observed regime is $w = 0 : 0 : 1$.

Figure 2(c) shows the case when second, third and fourth oscillators are mode-locked. In this case, the rotation number is $w = 1 : 0 : 0$. And the resulting clustering is shown schematically in Fig. 3(c).

Note that in each of the three considered cases, each locked pair of phase variables fluctuates around a zero (or 2π) values. This corresponds to "in-phase" mode-locking of each pair of oscillators.

The schematic representation of emerging simplest two-frequency modes T_2 (Fig. 3) is presented as the first step in the construction of a "clustering tree", which is a useful guide in studying the regimes of four coupled oscillators.

Let us now pass to the region of three-frequency quasi-periodicity[5] and consider the points (d)–(f) in Fig. 1(a). Now attractors are invariant surfaces which are everywhere densely covered with phase trajectories. In case Fig. 2(d) the phase ϕ fluctuates weakly. This means that the third and fourth oscillators are partially mode-locked. In Figs. 2(e) and 2(f) one can see two other possible types of partial mode-locking of pairs of oscillators. These figures show phase portraits differently oriented in phase space invariant surfaces. In the "clustering tree", the qualitative configuration of emerging clusters is shown as in Figs. 3(d)–3(f).[6]

A more complex case is illustrated in Fig. 2(g). In this case, an invariant curve with the rotation number $w = 0 : 1 : 1$ is observed. It may be considered as arising on the surface shown in Fig. 2(e) as a result of "condensation of trajectories". This is another example of a resonant two-frequency regime that occurs on the surface of a three-frequency regime shown in Fig. 2(e). The system under consideration exhibits many other tongues of two-frequency modes with different rotation numbers.

As the coupling decreases, clusters corresponding to three-frequency modes will be destroyed with the appearance of a four-frequency quasi-periodicity.[7] In this case, the phase trajectories fill

[3]Since we vary the frequency of one oscillator (the second), the number of such resonances is equal to the number of its nearest neighbors, i.e. in this case two. For a network, it will depend on the number of partial systems and the topology of coupling.

[4]In the system (1), this regime corresponds to a two-frequency torus.

[5]In the system (1), this regime corresponds to a three-frequency torus.

[6]The fourth figure in this series is qualitatively similar to the case Fig. 2(d).

[7]In the system (1), this regime corresponds to a four-frequency torus.

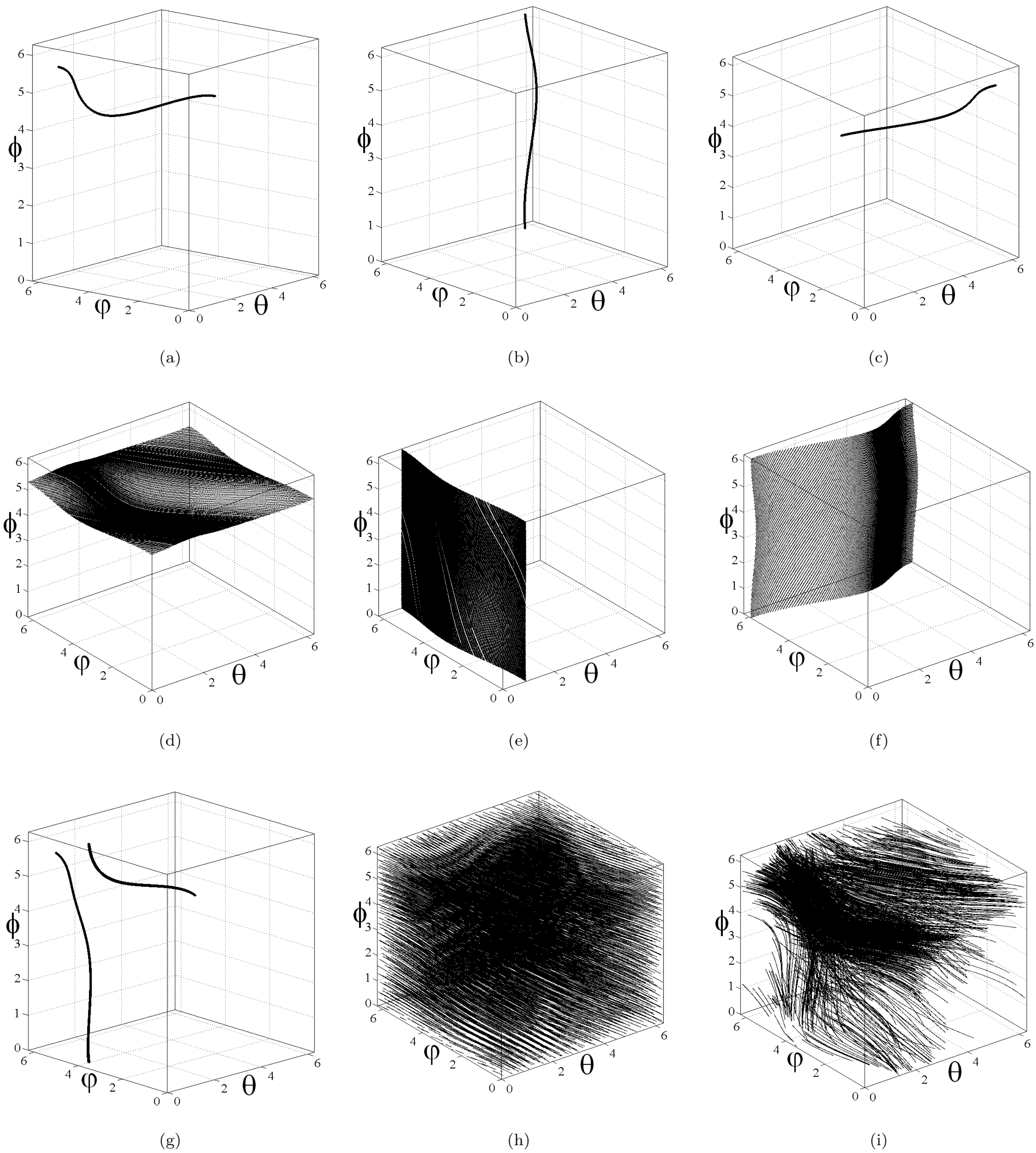


Fig. 2. (a)–(i) Attractors of the system (5) plotted at the parameter plane points, which are indicated in Fig. 1(a).

the whole volume of the phase cube, as shown in Fig. 2(h). Note that chaotic attractors can be observed in some ranges of parameters. The parameter plane has a very involved structure in such regions due to the possibility of multistability and existence of hidden attractors [Dudkowski *et al.*, 2016; Kuznetsov, 2020]. But these phenomena are beyond the scope of our study. An example of chaotic attractor is shown in Fig. 2(i). Visually, the four-frequency mode and chaotic attractors are

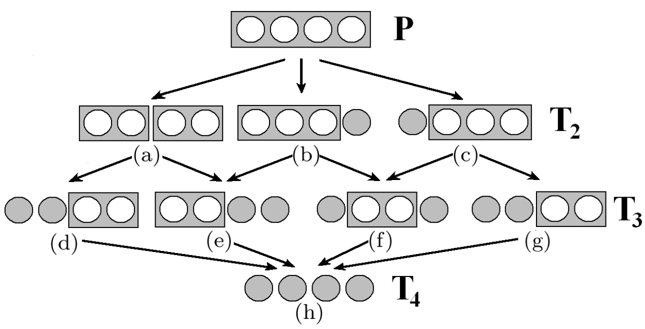


Fig. 3. Schematic "clustering tree" of four phase oscillators. The "tree" illustrates the emergence of clusters with decreasing coupling. The letters correspond to the points in Fig. 1. [The configuration in the fourth figure in the third line is equivalent to the case in Fig. 2(d).]

difficult to distinguish. But in the case of a quasi-periodic mode, the phase trajectories fill the phase cube more uniformly.

4. Bifurcation Scenarios of the Destruction of Complete Synchronization

Let us discuss the bifurcations responsible for the destruction of the regime of complete synchronization. Assuming $\dot{\theta} = \dot{\varphi} = \dot{\phi} = 0$ from Eqs. (5), we can obtain:

$$\sin\theta = \frac{\Delta_1 + \Delta_2 + \Delta_3}{4\mu},$$

$$\sin\varphi = \frac{\Delta_1 - \Delta_2 - \Delta_3}{2\mu},$$

$$\sin\phi = \frac{\Delta_1 + \Delta_2 - 3\Delta_3}{4\mu}. \tag{6}$$

Solutions to each of Eqs. (6) appear in pairs: (θ_1, θ_2), (φ_1, φ_2) and (ϕ_1, ϕ_2). Therefore, system (5) has eight equilibrium states located in the phase space (θ, φ, ϕ) at the vertices of the parallelepiped [Emelianova *et al.*, 2014]. One of these solutions is always stable, and the rest are saddles and an unstable node. Moreover, all eight equilibrium states are interconnected by common invariant manifolds, which determine the evolution of attractors.

When we vary any of the three combinations of parameters on the right-hand sides of expressions (6), the two sides of the parallelepiped will approach each other and merge. In this case, as soon as the sine of one of the phase variables turns to unity in (6), the eight equilibrium states merge pairwise and simultaneously disappear via a nonrobust state consisting of four neutral equilibrium states. Herewith, complete synchronization is destroyed. From the above considerations, we obtain the following expressions that determine the saddle-node bifurcations of the mentioned type:

$$\mu = \frac{\Delta_1 + \Delta_2 + \Delta_3}{4},$$

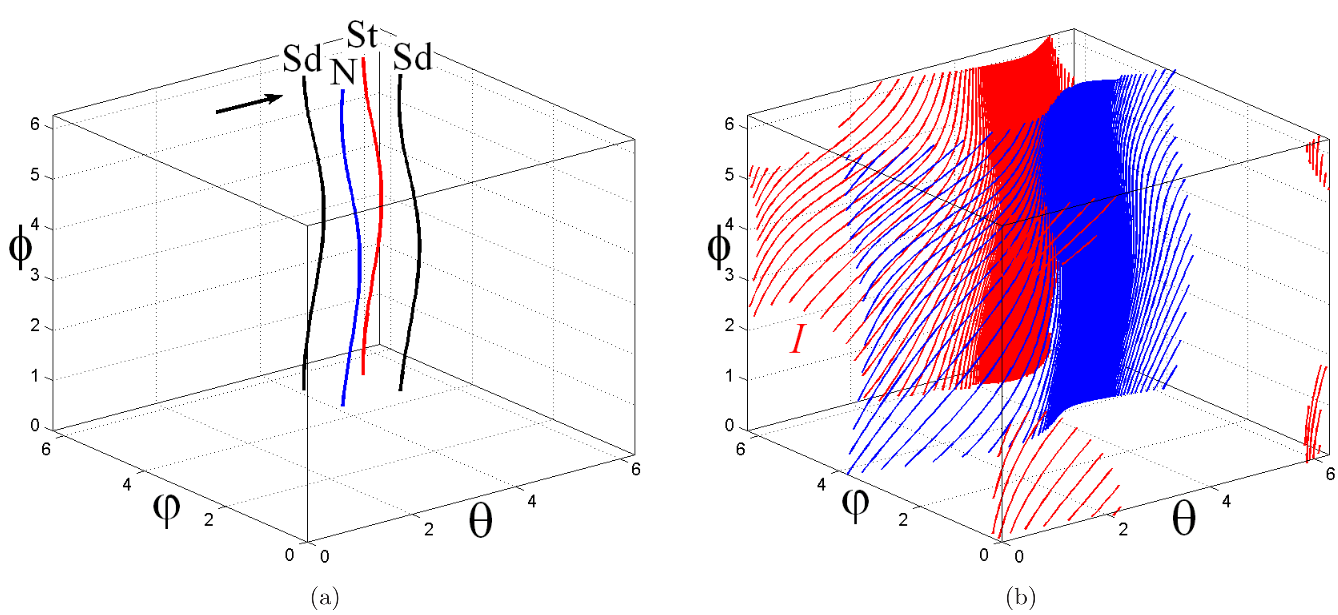


Fig. 4. Saddle-node bifurcations of the invariant curves. This bifurcation is responsible for the emergence of three-frequency quasi-periodicity. $\mu = 0.2$, (a) $\Delta_1 = 0.16$ and (b) $\Delta_1 = 0.26$.

$$\mu = \frac{\Delta_1 - \Delta_2 - \Delta_3}{2},$$
$$\mu = \frac{\Delta_1 + \Delta_2 - 3\Delta_3}{4}. \tag{7}$$

Thus, three variants of such a bifurcation arise. Each of them corresponds to the merging of the faces of the parallelepiped along one of the three phase axes.

Segments of lines (7) limit the domain of complete synchronization shown in Fig. 1 by black color. Their intersections define points of codimension two, which lead to the appearance of characteristic "kinks" at the domain's boundary, they are marked with numbers 1 and 2. Moreover, one of them defines the threshold of the region of complete synchronization.

As a result of the above described bifurcation, four invariant curves arise from the closed invariant manifolds of neutral equilibrium states. They are shown in Fig. 4. One of them will be stable (it is marked by *St*). The second of them will be unstable (it is marked by *N*). And the other two

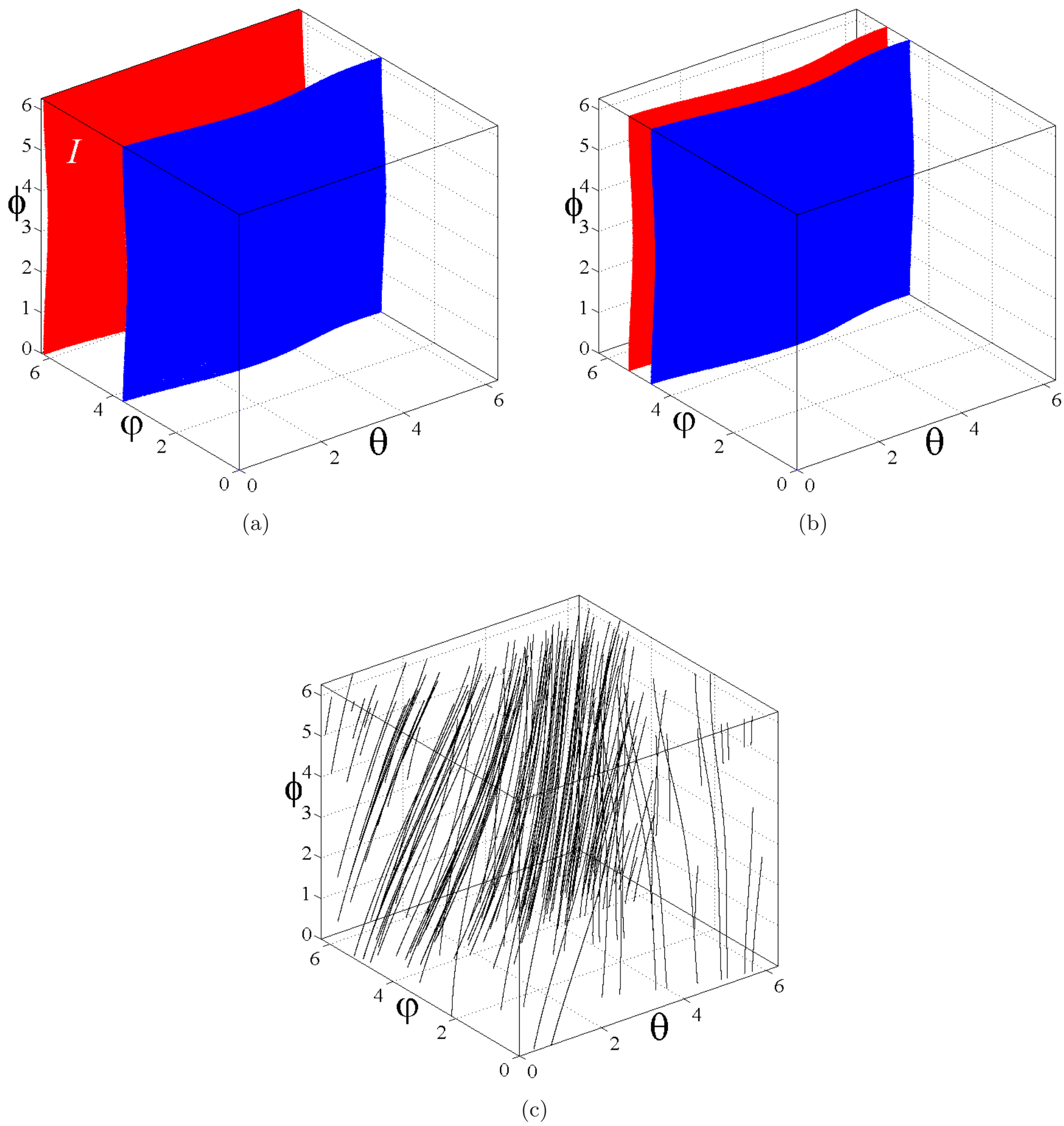


Fig. 5. Saddle-node bifurcation of the invariant surfaces. This bifurcation is responsible for the emergence of four-frequency quasi-periodicity. I is the stable invariant surface. $\mu = 0.05$, (a) $\Delta_1 = 0.28$, (b) $\Delta_1 = 0.22$ and (c) $\Delta_1 = 0.2$.

will be saddles (they are marked by Sd). The stable invariant curve of the system (5) will be responsible for the two-frequency quasi-periodic mode of the system (1). Let us now discuss the bifurcations corresponding to the destruction of two-frequency quasi-periodicity.

The vicinity of the threshold point of two-frequency quasi-periodicity is shown in an enlarged form in Fig. 1(b). It should be noted that the observed structure in the parameter space is similar to the so-called "*saddle node fan*" point, which was discussed for the case of three coupled oscillators in [Emelianova *et al.*, 2013; Pikovsky *et al.*, 2001]. (Saddle-node fan is a codimension two bifurcation point at which two different saddle-node bifurcation lines of the resonant cycles merge. It is located at the border of the complete synchronization region.) The difference consists of an increase by one in the dimension of the observed quasiperiodic regimes: i.e. instead of the region of complete synchronization, there is a region of two-frequency quasiperiodicity. Instead of a region of two-frequency quasiperiodicity we have a region of three-frequency quasi-periodicity, and instead of a system of fan-shaped tongues of three-frequency modes, a similar system of four-frequency tongues exists. Therefore, we will call such a point "*the quasi-periodic point saddle node fan*" and denote it *QSNF*. We also note a certain difference. There is a new system of narrow tongues of two-frequency modes inside the regions T_3 [Fig. 1(b)]. These resonance regimes correspond to the situation when invariant curves of different types appear on the corresponding invariant surface.

When going beyond the limits of the corresponding tongue, the two-frequency quasiperiodicity is destroyed. Figure 4 illustrates the bifurcation scenario corresponding to the exit from the region T_2 [Fig. 1(b)] through its right border. In this case, the stable, unstable, and two saddle invariant curves approach each other in pairs [in the direction shown by the arrow in Fig. 4(a)] as the parameter Δ_1 increases. As a result, they simultaneously merge and disappear in pairs. Stable I and an unstable invariant surface, shown in Fig. 4(b), arise from their common closed invariant manifolds.

If we leave the region T_2 [Fig. 1(b)] through its left boundary, the direction in which the curves approach and, hence, the content of merging pairs, changes. Now the saddle curve marked with an arrow in Fig. 4 merges with the unstable invariant curve. As a result, invariant surfaces oriented in a different way arise.

Directly at the point *QSNF*, all four invariant curves shown in Fig. 4(a) simultaneously contract into one curve. This is a codimension two degenerate situation.

Figure 5 illustrates further evolution of the quasi-periodic regime, when the destruction of the three-frequency quasi-periodicity is observed. This occurs as a result of the merging of stable and unstable invariant surfaces. Such a scenario is observed when we cross the left border of the tongue of three-frequency modes T_3 [Fig. 1(b)].

In Fig. 5, one can see that when we cross the bifurcation point (transition from fragment *b* to fragment *c*) the invariant surfaces disappear abruptly with the appearance of trajectories filling the entire phase cube. This is a characteristic feature of the discussed bifurcation.

5. Conclusion

For a chain of four phase oscillators with dissipative coupling, the pattern of domains in the parameter space (frequency detuning and parameter of coupling) includes hierarchically organized regions of multifrequency regimes with different numbers of incommensurate frequencies. The regions of four-frequency quasi-periodicity are dominating at small values of the parameter coupling. And only in the vicinity of the zero value of the frequency detuning, small tongues of three-frequency quasi-periodicity are observed. The regions of complete synchronization with two-frequency quasi-periodicity have a threshold in terms of the coupling parameter at the points "saddle node fan" and "the quasi-periodic point saddle node fan", respectively.

The bifurcation mechanism of the destruction of two-frequency quasi-periodicity consists in the pairwise merging of four invariant curves: stable, unstable, and two saddle curves. As a result, stable and unstable invariant surfaces appear in the phase space, which are the images of three-frequency quasi-periodicity. The kink point of the border of the corresponding domain is a point of codimension two bifurcation at which all four curves merge simultaneously. In turn, the destruction of three-frequency regime is due to the merging of stable and unstable invariant surfaces. It can be assumed that the features presented in the work (both the structure of the parameters plane and the bifurcations responsible for the destruction of quasi-periodic regimes) have a high degree of universality and will be observed in chains of higher dimension.

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