ACCEPTED MANUSCRIPT
Chaos
AIP Publishing
This is the author's peer reviewed, accepted manuscript. However, the online version of record will be different from this version once it has been copyedited and typeset.
PLEASE CITE THIS ARTICLE AS DOI: 10.1063/5.0055579

FEATURES OF A CHAOTIC ATTRACTOR IN A QUASIPERIODICALLY DRIVEN NONLINEAR OSCILLATOR

V. P. Kruglov,[1, a)] D. A. Krylosova,[2, b)] I. R. Sataev,[1, c)] E. P. Seleznev,[1, d)] and N.V. Stankevich[3, e)]
[1)] *Kotelnikov Institute of Radioengineering and Electronics of RAS, Saratov Branch, Zelenaya str., 38, Saratov, 410019, Russia*
[2)] *Chernyshevsky Saratov State University, Astrakhanskaya str., 83, Saratov, 410012, Russia*
[3)] *HSE University, Bolshaya Pecherskaya str., 25/12, Nizhny Novgorod, 603155, Russia*

(Dated: 23 June 2021)

Transition to chaos via the destruction of a two-dimensional torus is studied numerically using the example of the Hénon map and the Toda oscillator under the quasiperiodic forcing, and also experimentally using the example of a quasiperiodically excited RL-diode circuit. A feature of chaotic dynamics in these systems is the fact that the chaotic attractor in them has additional zero Lyapunov exponent, which strictly follows from the structure of mathematical models. In the process of research, the influence of feedback is studied, in which the frequency of one of the harmonics of the external forcing becomes dependent on a dynamic variable. Charts of dynamic regimes were constructed, examples of typical oscillation modes were given, and the spectrum of Lyapunov exponents was analyzed. Numerical simulations confirm that chaos resulting from the cascade of torus doubling has close to zero Lyapunov exponent, beside the trivial zero exponent.

Transition to chaos via the destruction of quasiperiodic oscillations is one of the main scenarios for the birth of a chaotic attractor. The most well studied is the transition to chaos via the breakdown of a two-dimensional torus[1–10]. A huge contribution to the study of this scenario was made by Professor V.S. Anischenko. The particular way this scenario can evolve is a cascade of torus-doubling bifurcations[11–13]. Quasiperiodically driven oscillators are commonly considered as basic models demonstrating torus-doublings. One of the open problems is the type of the attractor, arising as a result of such cascades. The particular case is the situation when the external quasiperiodic type of driving is broken via feedback introduced. The studying of the peculiarities of the chaotic attractor arising as a result of torus-doubling scenario is the aim of our work.

I. INTRODUCTION

Torus doubling scenario attracts great attention of researchers in particular due to its prevalence and the place it takes in the destruction of high-dimensional tori. One of the first to study the role the torus doublings may play in the scenario of destruction of a three-frequency torus was V.S. Anischenko[10,14,15]. And it is largely thanks to him that many researchers turn to the study of the evolution of two-dimensional and multidimensional quasiperiodic oscillations[16–21]. The variety of problems associated with the study of the transition to chaos via the destruction of quasiperiodic oscillations is very wide. Among the whole set, the work[22] should be noted, in which the breakdown of three-dimensional torus was studied and as a result of a long sequence of tori doubling, the birth of a chaotic attractor with two zero Lyapunov exponents was observed. This is a very interesting fact from the point of view of the theory of dynamical systems. The authors of[22] show the presence of a chaotic attractor with two zero Lyapunov exponents based on numerical calculations with high accuracy.

The aim of this work is a numerical and experimental study of examples of dynamical systems in which as a result of the destruction of a two-dimensional torus a chaotic attractor with two zero Lyapunov exponents is born.

The studies of quasiperiodical dynamics of nonlinear dissipative systems is very complicated largely due to the fact, that it is sometimes in principle impossible to carry out rigorous analytical proofs. Many results in this case are obtained via numerical studies. Therefore, the choice of models for research is an important point. We propose to start from dynamical systems, that have two-dimensional tori and demonstrate their destruction and transition to chaos, and which a priori have chaotic attractors with two zero Lyapunov exponents. These include nonlinear systems under quasiperiodic forcing.

It should be noted, that the interest in the study of quasiperiodically driven dynamical systems was largely associated with the study of the scenarios of the birth of a strange non-chaotic attractor (SNA), its properties and diagnostics[23–26]. According to[25], SNA is characterized by non-regular fractal structure but non-positive Lyapunov exponent. At the same time, much less attention was paid to the chaotic attractor, which was born from a strange non-chaotic attractor. Due to the specificity of quasiperiodically driven systems, the chaotic attractor in them possesses two zero Lyapunov exponents. In this paper,

[a)] Also at Yuri Gagarin State Technical University of Saratov, Politehnicheskaya str., 77, Saratov, 410054, Russia; Electronic mail: kruglovyacheslav@gmail.com
[b)] Electronic mail: krylosovadarina@gmail.com
[c)] Electronic mail: sataevir@gmail.com
[d)] Electronic mail: evgenii_seleznev@mail.ru
[e)] Also at Yuri Gagarin State Technical University of Saratov, Politehnicheskaya str., 77, Saratov, 410054, Russia; Also at St.Petersburg State University, Universitetskiy proezd, 28, Peterhof, Saint-Petersburg, 198504, Russia; Electronic mail: stankevichnv@mail.ru

ACCEPTED MANUSCRIPT
Chaos
AIP Publishing
This is the author's peer reviewed, accepted manuscript. However, the online version of record will be different from this version once it has been copyedited and typeset.
PLEASE CITE THIS ARTICLE AS DOI: 10.1063/5.0055579

the main attention will be paid to it. The fact, that in systems with quasiperiodic forcing a chaotic attractor has two zero Lyapunov exponents, should be considered as trivial, and this oscillation mode may be regarded as a test one. Of greatest interest in this study is the situation, when a feedback is introduced to a source of external forcing. For example, the phase of the second harmonic component of the external forcing could depend on the dynamic variable. In this case, the situation with one zero Lyapunov exponent may be regarded as trivial. It is difficult to predict how the second zero Lyapunov exponent will behave. In the present work, these studies are carried out with the examples of the Hénon map[27], the quasiperiodically driven Toda oscillator[28], and experimentally with an RL-diode circuit under quasi-periodic forcing[26]. First, quasiperiodic dynamics is investigated in all the systems under study, and then feedback is introduced and its influence on the behavior of the systems under study is investigated.

II. DYNAMICS OF THE HÉNON MAP UNDER QUASIPERIODIC FORCING.

Quasiperiodically forced Hénon map may be considered as a representative model for quasiperiodically forced period-doubling systems:

$$\begin{aligned} x' &= a - x^2 + u + \varepsilon \sin(2\pi y),\\ u' &= bx,\\ y' &= y + c, \quad \mod 1. \end{aligned} \tag{1}$$

It consists of two coupled mappings, each of them being a representative specie of the related class — the first is the Hénon map which demonstrates transition to chaos via period doubling cascade, the other is a circle map, in this case a simple rotation with an irrational rotation number.

It is easy to see that for ε equal to zero, we have a trivial mapping composed of two uncoupled maps. Chaotic attractor in this case is the direct product of the supercritical Feigenbaum attractor and that of the circle map. Such attractor is called the Hénon quasiperiodic attractor[29,30]. The zero Lyapunov exponent is provided here by the second mapping.

Quasiperiodic Hénon attractor arises in the general case as a result of a homoclinic bifurcation of a saddle torus (which corresponds to the saddle closed invariant curve in descrete-time mapping), a particular case is the bifurcation of the merging of the attractor bands in the supercritical region of the torus doubling cascade[29,30]. If ε is not equal to zero, then the number of observed doublings of the torus is finite. However, a zero exponent is still present.

Figure 1a shows a chart of Lyapunov exponents for the system (1). One can see the region of nonchaotic regimes, represented here by quasiperiodic attractors and also by the SNA. Regions of SNA are located along the border of the transition to chaos. Next to them is the domain of actually chaotic regimes, where one of the Lyapunov exponents becomes positive. Figure 2a shows a one-parameter graph of the dependence of the Lyapunov exponents on the a parameter for $\varepsilon = 0.1$.

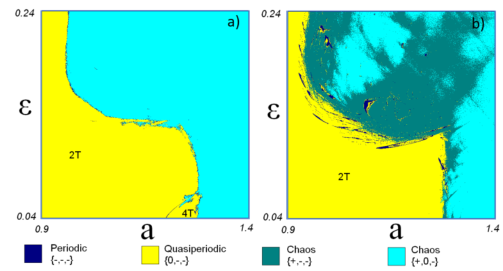

FIG. 1. Lyapunov exponents chart of the system (1) for b=0.05, $c = (\sqrt{5}-1)/2$. a) k=0; b) k=0.01

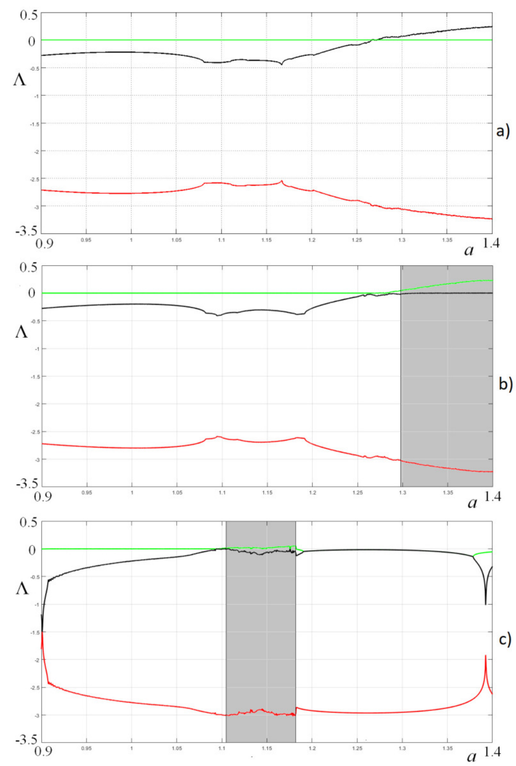

FIG. 2. One-parameter graph of the dependence of the Lyapunov exponents on the a parameter for ε=0.1. a) k=0; b) k=0.01; c) k=0.1

Consider a modification of the mapping (1) with the feedback introduced:

$$\begin{aligned} x' &= a - x^2 + u + \varepsilon \sin(2\pi y),\\ u' &= bx,\\ y' &= y + c + kx, \quad \mod 1. \end{aligned} \tag{2}$$

According to[25], introduction of the feedback typically destroys the SNA, it probably exists only over a set of measure zero in parameter space. At the same time, the introduction of feedback does not lead to the destruction of the homoclinic

ACCEPTED MANUSCRIPT
Chaos
This is the author's peer reviewed, accepted manuscript. However, the online version of record will be different from this version once it has been copyedited and typeset.
PLEASE CITE THIS ARTICLE AS DOI: 10.1063/5.0055579
AIP Publishing

structure of the attractor, since the intersection of manifolds is a robust property. For sufficiently small values of the feedback parameter, as a result of the torus doubling cascade and the subsequent reverse cascade of absorption of saddle tori by the attractor, an attractor appears, the characteristic property of which is the Lyapunov exponent close to zero[29,30]. Due to the feedback involved, the attractor may contain not only unstable tori, but unstable cycles as well, hence the Lyapunov exponent may be not zero exactly. Nonetheless we show that even in the case of only one torus doubling in the cascade, the value of the exponent after the absorption of the saddle torus by attractor is very close to zero.

Figure 1b shows a chart of Lyapunov exponents for the system (2) at k = 0.01, on which the region of existence of a chaotic attractor with a close to zero Lyapunov exponent is highlighted (here Lyapunov exponent was regarded as zero if it was smaller in magnitude than 10^{-3}). Figure 2b shows a one-parameter graph of the dependence of the Lyapunov exponents on the a parameter for $\varepsilon = 0.1$, $k = 0.01$. On this graph one can see an interval in the parameter on which a quasiperiodic Hénon attractor with a close to zero Lyapunov exponent is realized (highlighted in grey).

Figure 3 shows phase portraits of the attractor for different parameter a values along the path ε = 0.1. For each portrait a corresponding spectrum of Lyapunov exponents is presented. While the Figure 3b shows presumably an SNA portrait (SNAs are distinguished from other regimes via application of the rational approximations criteria[31]), the Figure 3e corresponds to chaotic attractor in the system (2) with feedback involved, the zero exponent becomes positive. Figures 3c and 3f show the portraits of quasiperiodic Hénon-like attractors.

At large values of the feedback parameter, the quasiperiodic Hénon attractor does not arise (see Figure 2c), zero Lyapunov exponent is absent, the related interval is highlighted in grey. Figures 3h and 3i show the portraits of chaotic attractors for k=0.1. Judging by these portraits it can be assumed that the doubled two-dimensional torus fractalizes in accordance with the standard Afraimovich-Shilnikov scenario[9].

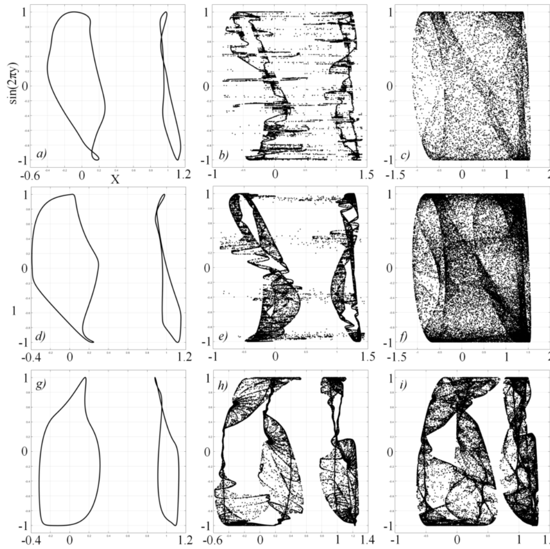

FIG. 3. Phase portraits of the attractors for different parameter a values along the path ε = 0.1. a) a=1, k=0.0, (0.0, -0.22, -2.77); b) a=1.2657, k=0.0, (0.0, -0.0139±0.0001, -2.98); c) a=1.4, k=0.0, (0.24, 0.0, -3.23); d) a=1, k=0.01, (-0.0000001±0.000001, -0.198, -2.79); e) a=1.29, k=0.01, (0.0278, -0.0119±0.0001, -3.01); f) a=1.4, k=0.01, (0.229, 0.000053±0.000005, -3.22); g) a=1.0, k=0.1, (0, -0.220, -2.775); h) a=1.12, k=0.1, (0.0157, -0.053, -2.96); i) a=1.16, k=0.1, (0.030, -0.033, -2.99)

III. DYNAMICS OF THE QUASIPERIODICALLY DRIVEN TODA OSCILLATOR.

In this section we study Toda oscillator (with exponential restoring force $1-e^x$) under quasiperiodic forcing:

$$\begin{aligned} \dot{x} &= u, \\ \dot{u} &= 1-e^x-2\alpha u+A_1\sin\omega_1 t+A_2\sin\omega_2 t, \end{aligned} \tag{3}$$

where ω_1 and ω_2 are two incommensurate frequencies, A_1 and A_2 are two forcing amplitudes, α is dumping factor. Toda oscillator under quasiperiodic forcing manifests very similar behaviour to the forced Hénon map[28], in particular, they both have strange nonchaotic attractors. One can formally rewrite the equations (3) as four-dimensional autonomous system by introducing two independent phases:

$$\begin{aligned} \dot{x} &= u, \\ \dot{u} &= 1-e^x-2\alpha u+A_1\sin\theta+A_2\sin\varphi, \\ \dot{\theta} &= \omega_1, \\ \dot{\varphi} &= \omega_2. \end{aligned} \tag{4}$$

Perturbations of phases $\theta+\delta\theta$ and $\varphi+\delta\varphi$ do not grow or decay with time: $\delta\dot{\theta}=0$, $\delta\dot{\varphi}=0$. Therefore system (4) has two trivial zero LEs. The sum of two non-trivial LEs is always equal to -2α — the divergence of the vector field, provided by the right hand sides of (4). The possible regimes of (4) are quasiperiodic two-dimensional tori (with two negative non-trivial LEs), strange nonchaotic attractors, and chaotic attractors (with one positive non-trivial LE and one negative).

Fig. 4 shows the chart of LEs for system (4) on the plane of parameters A_1 and A_2. Frequencies have been fixed at golden ratio: $\omega_1 = 1$, $\omega_2 = (\sqrt{5}+1)/2$. LEs have been calculated with the usual and well-known procedure[32,33], while the equations have been solved with Runge – Kutta 4^{th} order method. Only the non-trivial exponents have been evaluated. Yellow color refers to parameter values at which both non-trivial exponents are negative. With two trivial LEs the corresponding attractors are two-dimensional tori or strange non-chaotic attractors close to the boundary of chaos onset. The blue color

ACCEPTED MANUSCRIPT
Chaos
AIP Publishing
This is the author's peer reviewed, accepted manuscript. However, the online version of record will be different from this version once it has been copyedited and typeset.
PLEASE CITE THIS ARTICLE AS DOI: 10.1063/5.0055579

marks situations, where the first non-trivial LE is close to zero. Exact zeroes of the first non-trivial LE correspond to bifurcation lines. One can see such lines for torus doubling bifurcations. Cyan dots are chaotic regimes with one positive and one negative exponent. With two trivial zero LEs the chaotic attractors are based on destroyed tori. Extra panels show examples of attractors of Poincaré stroboscopic map (with section $\varphi = 2\pi$) at four parameters value sets. Figures 4 (a) and (c) demonstrate fractal tori, possibly SNA. Remarkably, the (c) panel shows the attractor from the stability window. Figures 4 (b) and (d) are chaotic attractors. Well-developed attractor (d) consists of one "piece". Attractors (a), (b) and (c) are of two "parts", since they are based on doubled torus.

We have calculated one-parameter dependencies of the non-trivial LEs vs. force amplitude A_1 at fixed value $A_2 = 0.5$. Figure 5 shows these dependencies (a) and the enlarged part for the largest LE (b) at the onset and development of chaos. The examples of attractors on Figure 4 correspond to Figure 5 also. One can see that the plot of the largest exponent is zero at $A_1 \approx 1.42$. This is the torus doubling bifurcation, we clearly observe exactly one such bifurcation before the torus breakdown and chaos onset (at parameters fixed as in caption to Fig. 5). The torus fractalization occurs in the interval $A_1 \in [1.806; 1.812]$ approximately, see Fig 4 (a) as an example of fractalized torus with non-trivial LEs $\lambda_1 = -0.002137 \pm 5\cdot 10^{-6}$, $\lambda_2 = -0.197863 \pm 5\cdot 10^{-6}$. There is a short comeback of stability at the interval $A_1 \in [1.974; 2.014]$ approximately. The example of the produced attractor is in Figure 4 (c), looking very unregular, while the corresponding non-trivial LEs are negative: $\lambda_1 = -0.004618 \pm 5\cdot 10^{-6}$, $\lambda_2 = -0.195382 \pm 5\cdot 10^{-6}$.

All of the chaotic attractors of the quasiperiodically forced Toda oscillator (4) have two trivial zero LEs. We regard the Toda system (4) as the base which we supplement with feedback on external forcing (introduced into the equation for phase φ):

$$\begin{aligned} \dot{x} &= u, \\ \dot{u} &= 1 - e^x - 2\alpha u + A_1 \sin\theta + A_2 \sin\varphi, \\ \dot{\theta} &= \omega_1, \\ \dot{\varphi} &= \omega_2 + kx, \end{aligned} \tag{5}$$

here k is small parameter. System (4) is its particular case with $k = 0$. System (5) has only one trivial zero LE. The occurence of a chaotic attractor with an extra non-trivial zero LE is a curious phenomena.

For system (5) the sum of LEs is equal to -2α. We have calculated numerically the non-trivial exponents by the standard procedure. Figure 6 shows the chart of LEs for system (5) with $k = 0.001$. Compare it with Fig. 4. Yellow regions correspond to two-dimensional tori, black lines mark torus doubling bifurcations. Here SNA are absent due to the feedback on frequency $\dot{\varphi}$. Dark-cyan regions correspond to usual chaos (one non-trivial LE is positive, two other are negative). Cyan regions correspond to chaos with extra zero LE (one positive, one is close to zero, one is negative).

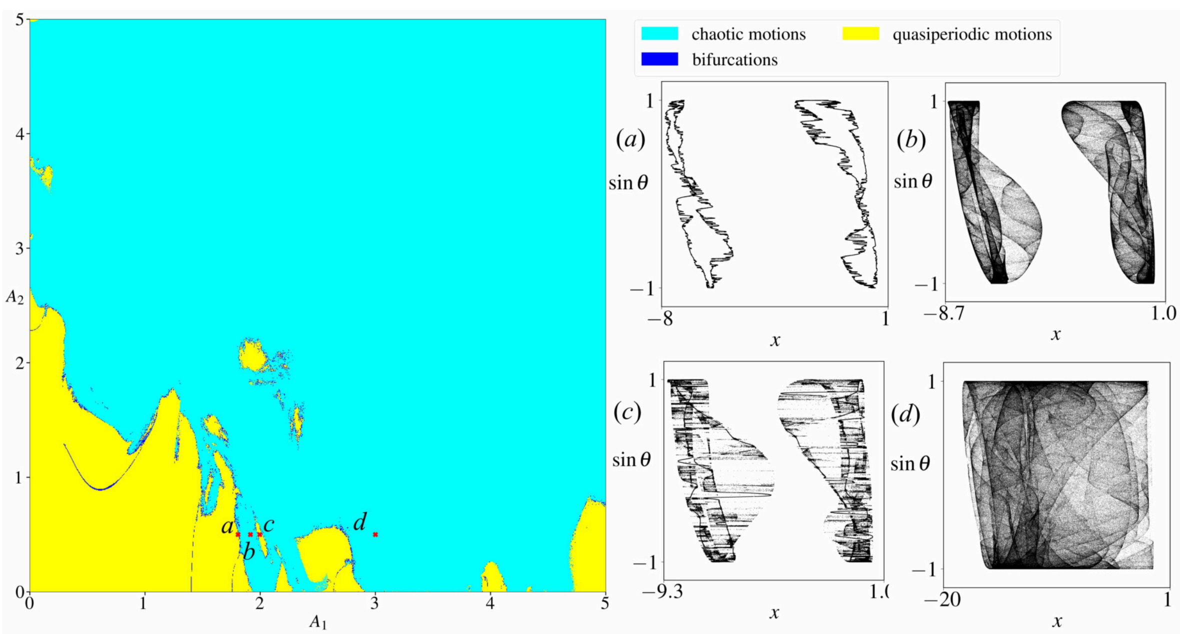

FIG. 4. Chart of Lyapunov exponents for system (4), varying amplitudes A_1 and A_2, while $\alpha = 0.1$, $\omega_1 = 1$ and $\omega_2 = (\sqrt{5}+1)/2$ are fixed. Yellow dots mark quasiperiodic regimes and SNA, cyan dots mark chaotic regimes. Blue dots mark situations close to bifurcation. Red crosses tagged by letters mark parameter values for phase portraits on panels (a) – (d). (a): $A_1 = 1.807$; (b): $A_1 = 1.915$; (c): $A_1 = 1.995$; (d): $A_1 = 3$.

ACCEPTED MANUSCRIPT
Chaos
AIP Publishing
This is the author's peer reviewed, accepted manuscript. However, the online version of record will be different from this version once it has been copyedited and typeset.
PLEASE CITE THIS ARTICLE AS DOI: 10.1063/5.0055579

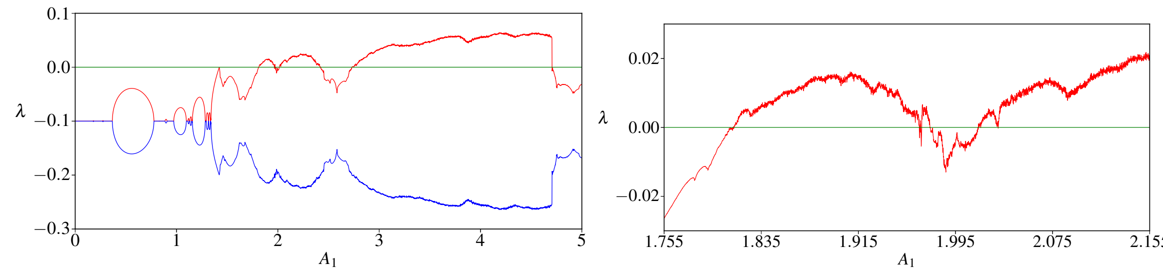

FIG. 5. Lyapunov exponents vs. amplitude of forcing A_1 for system (4) at fixed values $A_2 = 0.5$, $\alpha = 0.1$, $\omega_1 = 1$, $\omega_2 = (\sqrt{5}+1)/2$. Green line shows trivial zero exponents. Enlarged plot shows the onset of chaos (the largest exponent is plotted).

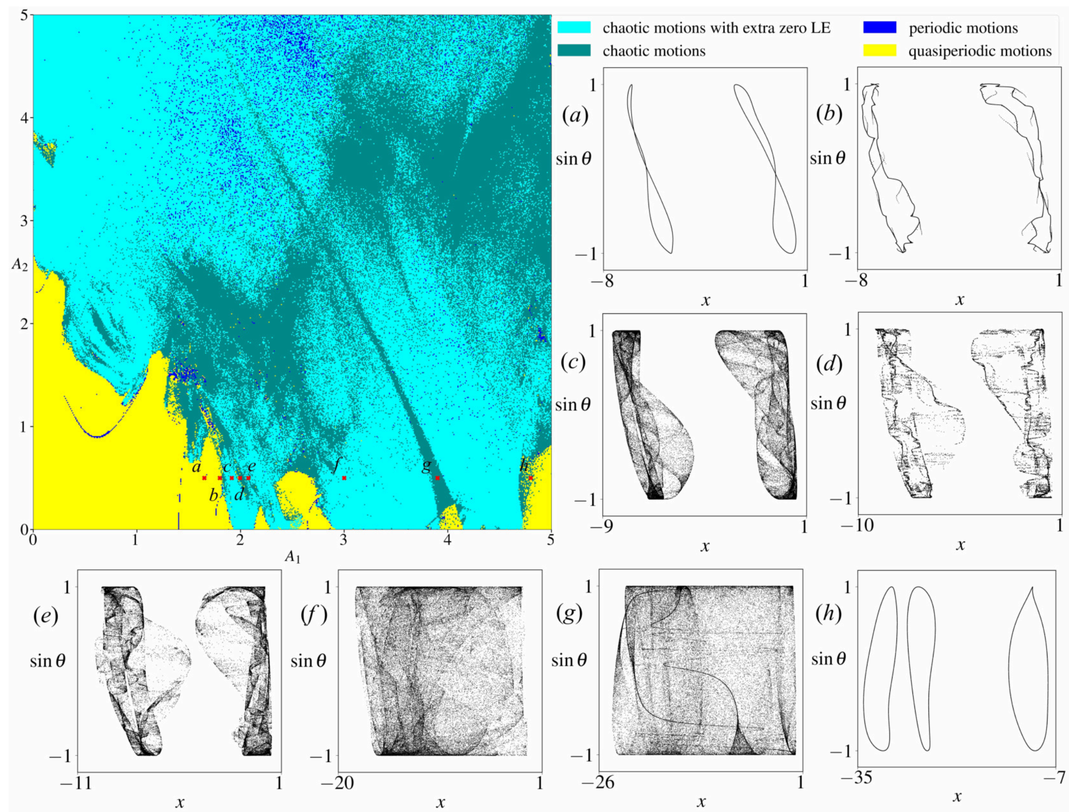

FIG. 6. Chart of Lyapunov exponents for system (5), varying amplitudes A_1 and A_2, while $k = 0.001$, $\alpha = 0.1$, $\omega_1 = 1$ and $\omega_2 = (\sqrt{5}+1)/2$ are fixed. Yellow dots mark quasiperiodic regimes, dark-cyan dots mark chaotic regimes, cyan dots correspond to chaotic regimes with an extra zero LE. Blue dots mark situations close to bifurcation. Red crosses tagged by letters mark parameter values for phase portraits on panels (a) – (h). (a): $A_1 = 1.65$; (b): $A_1 = 1.8$; (c): $A_1 = 1.915$; (d): $A_1 = 1.995$; (e): $A_1 = 2.075$; (f): $A_1 = 3$; (g): $A_1 = 3.9$; (h): $A_1 = 4.8$.

Inserted panels on Fig. 6 show examples of attractors in stroboscopic Poincaré section $\varphi = 2\pi$ at different values of parameter A_1, with other parameters fixed. All of the examples of attractors belong to the line $A_2 = 0.5$, thus we supplement the chart of LEs with one-parameter dependencies of LEs on A_1, see Fig. 7, with $k = 0.001$. There is also Table I with LEs for attractors on inserted Panels of Fig. 6. One can see a part of the plot with one non-trivial LE equal to zero and the other negative, corresponding to quasiperiodic regimes. At $A_1 \approx 1.42$ the second non-trivial exponent comes close to zero, marking the torus doubling bifurcation. The transition to chaos goes without the appearance of SNA, but through the

ACCEPTED MANUSCRIPT

Chaos

AIP Publishing

This is the author's peer reviewed, accepted manuscript. However, the online version of record will be different from this version once it has been copyedited and typeset.

PLEASE CITE THIS ARTICLE AS DOI: 10.1063/5.0055579

emergence of cycles of very high periods. Chaotic attractors at very wide parameter ranges have zero or very close to zero non-trivial LE.

The panel (a) of Fig. 6 shows the doubled torus at $A_1 = 1.65$. Fig. 6 (b) shows an unsmooth torus at $A_1 = 1.8$. The panel (c) demonstrates a chaotic attractor with extra zero non-trivial LE at $A_1 = 1.915$. The panel (d) shows an attractor from the stability window at $A_1 = 1.995$. The attractor looks like fractalized torus, while it is really a long-periodic cycle with all non-trivial LEs negative. The panel (e) demonstrates a chaotic attractor without non-trivial zero LE at $A_1 = 2.075$. The panel (f) contains a portrait of chaotic attractor at $A_1 = 3$ with one of the LEs very close to zero. From the physical point of view the second LE is zero. The panel (g) shows a chaotic attractor at $A_1 = 3.9$ without non-trivial LE close to zero. The panel (h) demonstrates a torus from the stability window at $A_1 = 4.8$.

Parameter values	LEs	Panel on Fig. 6			
$A_1 = 1.65$	$\lambda_1 = -8\cdot10^{-8} \pm 1.5\cdot10^{-7}$, $\lambda_2 = -0.0553 \pm 0.0001$, $\lambda_3 = -0.1447 \pm 0.0001$	(a)	$A_1 = 2.075$	$\lambda_1 = 0.0112 \pm 0.0002$, $\lambda_2 = -0.0003 \pm 4\cdot10^{-5}$, $\lambda_3 = -0.2109 \pm 0.0002$	(e)
$A_1 = 1.8$	$\lambda_1 = -6\cdot10^{-7} \pm 1.4\cdot10^{-6}$, $\lambda_2 = -0.00620 \pm 2\cdot10^{-5}$, $\lambda_3 = -0.19379 \pm 2\cdot10^{-5}$	(b)	$A_1 = 3$	$\lambda_1 = 0.0326 \pm 0.0002$, $\lambda_2 = -6\cdot10^{-5} \pm 3\cdot10^{-5}$, $\lambda_3 = -0.2325 \pm 0.0002$	(f)
$A_1 = 1.915$	$\lambda_1 = 0.0143 \pm 0.0001$, $\lambda_2 = -6\cdot10^{-6} \pm 10^{-5}$, $\lambda_3 = -0.2143 \pm 0.0001$	(c)	$A_1 = 3.9$	$\lambda_1 = 0.0425 \pm 0.0004$, $\lambda_2 = -0.0066 \pm 0.0001$, $\lambda_3 = -0.2359 \pm 0.0005$	(g)
$A_1 = 1.995$	$\lambda_1 = -0.0006 \pm 0.0003$, $\lambda_2 = -0.006 \pm 0.001$, $\lambda_3 = -0.194 \pm 0.001$	(d)	$A_1 = 4.8$	$\lambda_1 = 3\cdot10^{-6} \pm 1.5\cdot10^{-5}$, $\lambda_2 = -0.0359 \pm 0.0001$, $\lambda_3 = -0.1641 \pm 0.0001$	(h)

TABLE I. Non-trivial LEs for attractors on Panels (a) – (h) on Fig. 6.

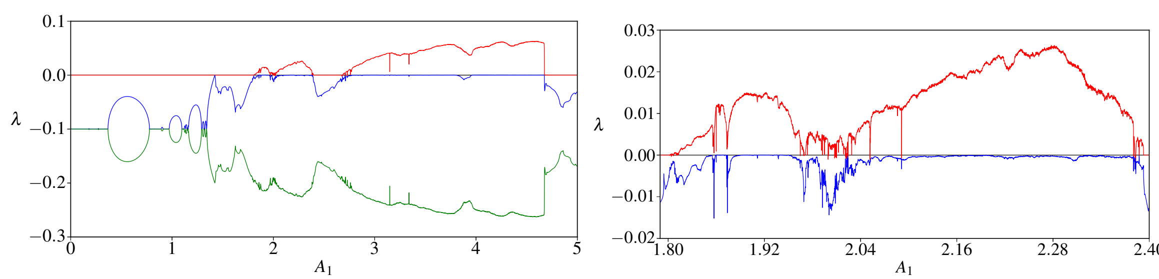

FIG. 7. Lyapunov exponents vs. amplitude of forcing A_1 for system (5) at fixed values $k = 0.001$, $A_2 = 0.5$, $\alpha = 0.1$, $\omega_1 = 1$, $\omega_2 = (\sqrt{5}+1)/2$. Enlarged plot shows the onset of chaos (the largest exponent is plotted).

IV. RL–DIODE CIRCUIT UNDER QUASIPERIODIC FORCING.

The object of research in a physical experiment is a ***RL***–diode circuit (Fig. 8), excited by a signal representing the sum of two harmonic components $A_1 Sin2\pi f_1 t$ and $A_2 Sin2\pi f_2 t$. The linear resonant frequency of the circuit was 50 kHz. The nature of the oscillations was analyzed using various projections of the phase portrait, stroboscopic maps and power spectra of oscillations. The values of the external forcing frequencies are set as follows: f_1 was chosen close to the linear resonance frequency of the oscillatory circuit and was equal to 55 kHz, f_2=kf_1, where: $k = (\sqrt{5}-1)/2$, in the experiment the frequency f_2 =33.991869381 kHz. The experimental data: the diode voltage U, the diode current I, the charge on the diode Q, and the external force $A_2 Sin2\pi f_2 t$ were entered into a personal computer using the NI USB–6251 analog input/output device (Fig. 8).

ACCEPTED MANUSCRIPT
Chaos
AIP Publishing
This is the author's peer reviewed, accepted manuscript. However, the online version of record will be different from this version once it has been copyedited and typeset.
PLEASE CITE THIS ARTICLE AS DOI: 10.1063/5.0055579

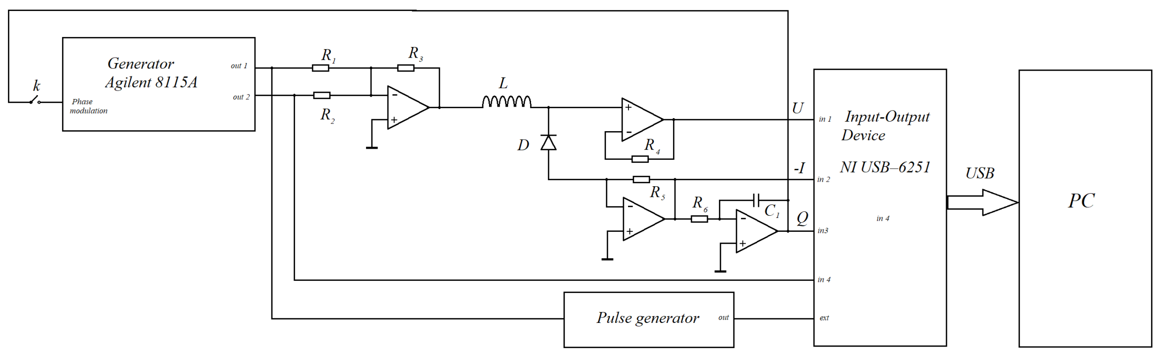

FIG. 8. Scheme of the system under study, RL-diode circuit with two harmonic influences.

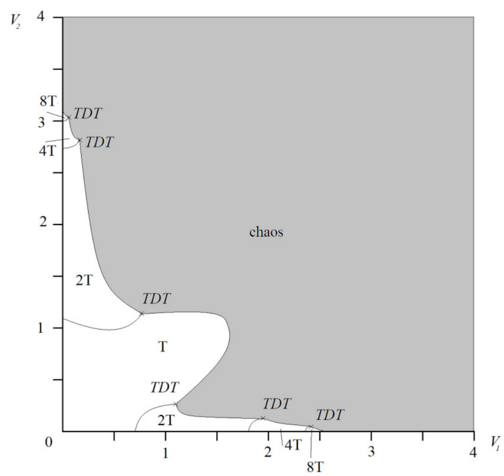

FIG. 9. Experimental chart of dynamic regimes on the plane of the external driving amplitude parameters (A_1, A_2).

Fig. 9 shows an experimental chart of dynamic regimes in the parameter plane (A_1,A_2). Light areas correspond to regular oscillations, areas of chaos existence are marked in gray tone, the transition to chaos was recorded based on the analysis of the power spectra of the oscillations. Solid lines indicate the lines of torus doubling bifurcations, these lines end at the point ***period doubling terminal, TDT***, indicated by crosses, the area marked **T** corresponds to a two-dimensional torus, **2T** - to a doubled (period-two) two-dimensional torus, in the regions marked **4T** and **8T**, there are period-four and period-eight two-dimensional tori, respectively. The transition to chaos in this system occurs via the birth of a strange non-chaotic attractor; however, the regions of its existence are not shown, but they were studied in detail in[26].

Fig. 10 shows the phase portraits of attractors of the stroboscopic maps corresponding to f_1 external force frequency as well as the oscillation power spectra illustrating the transition to chaos via the destruction of a two-dimensional torus. Projections onto the plane $(sin(2\pi f_2), Q)$ are shown, where $sin(2\pi f_2)$ reflects the phase of the corresponding harmonic component of the driving force, Q is the charge on the diode. The use of the projection onto the plane $(sin(2\pi f_2), Q)$ is explained by the fact that the phase portrait reflects the dependence of the behavior of the dynamic variable on the phase of the external action. Fig. 10a corresponds to a smooth torus, in these modes a smooth closed curve is observed in the stroboscopic section, and the oscillation demonstrates a discrete form of the power spectrum. With an increase in parameter A_1, distortion of the attractor is observed in the Poincare section, and the oscillation power spectrum is enriched with combination harmonics (Fig. 10b). While approaching the boundary of existence of regular oscillation modes, a strange non-chaotic attractor is born in the system under study. The transition to a strange non-chaotic attractor was registered using the method of rational approximations, in which the irrational ratio of exposure frequencies is replaced by a rational one from the sequence of Fibonacci numbers (in the case of gold rotation number). For this, in the experiment, the ratio of frequencies was set from the sequence of Fibonacci numbers f_1/f_2=34/55, and then the phase shift between the harmonics of the effect was changed. At A_1=1.545V period doubling bifurcation of the limit cycle was observed, which in the irrational limit corresponds to a strange non-chaotic attractor, the phase portrait and spectrum of which is shown in Fig. 10c. Fig. 10d illustrates the developed chaos in the system with an increase in the amplitudes of the harmonic components of the external forcing.

The transition to chaos via the destruction of a doubled two-dimensional torus seems to be interesting. A feature of systems under quasiperiodic action is that usually a finite number of doublings are observed, and then the invariant curve loses its smoothness and a transition to chaos occurs via the loss of smoothness of the invariant curve. Fig. 11 illustrates the transition to chaos from a doubled torus. For a doubled torus, a doubled invariant curve is observed in the stroboscopic section (Fig. 11a). With an increase in parameter A_1, a deformation of the invariant curve and appearance of a kinks is observed, indicating the presence of local instability. At a certain value of A_1, the kinks are merging, which is possibly due to the con-

ACCEPTED MANUSCRIPT
Chaos
AIP Publishing
This is the author's peer reviewed, accepted manuscript. However, the online version of record will be different from this version once it has been copyedited and typeset.
PLEASE CITE THIS ARTICLE AS DOI: 10.1063/5.0055579

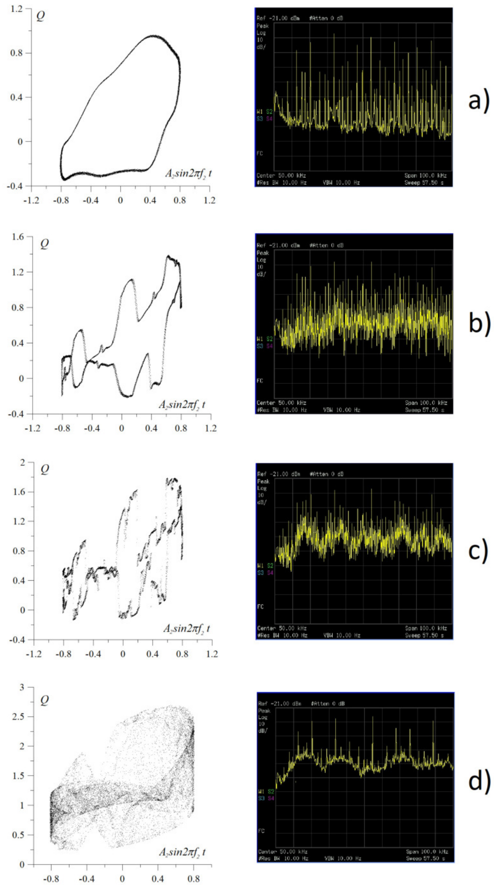

FIG. 10. Attractors in the stroboscopic section and the power spectra of oscillations illustrating the transition to chaos from a two-dimensional torus via the birth of an SNA: a) A_1=0.5 V, A_2=0.8 V, b) A_1=1.0 V, A_2=0.8 V, c) A_1=1.545 V, A_2=0.8 V, d) A_1=3.05 V, A_2=0.8 V.

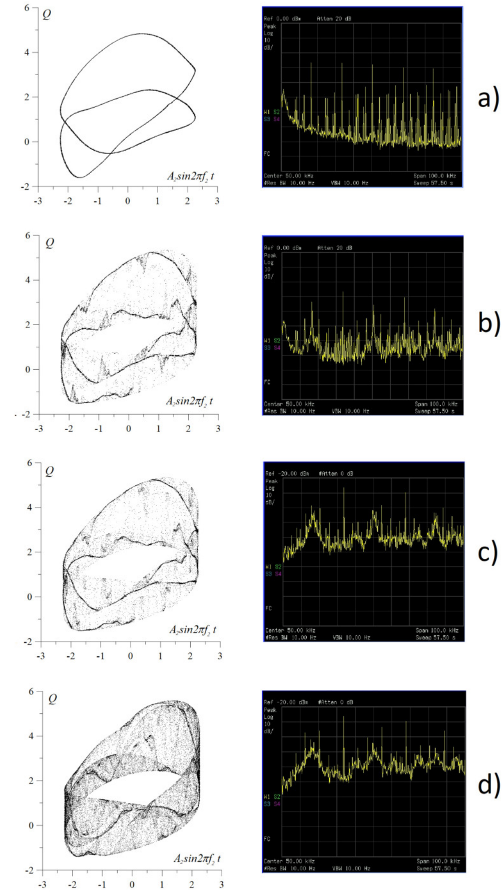

FIG. 11. Attractors in the stroboscopic section and the power spectra of oscillations illustrating the transition to chaos from a doubled two-dimensional torus: a) A_1=0.6 V, A_2=2.25 V, b) A_1=0.7985 V, A_2=2.25 V, c) A_1=0.802 V, A_2=2.25 V, d) A_1=1V, A_2=2.25 V.

tact of the doubled torus with the unstable torus (Fig. 11b), and as a result, with a further increase in A_1 a chaotic attractor is formed (Fig. 11c,d).

Now we introduce feedback. This is realized by switching the key k on (see Fig. 8), as a result of which the variable Q is fed to the input of the phase modulation of the generator and changes the phase of the harmonic component f_2.

Fig. 12 illustrates the dynamics of the experimental system in the presence of feedback. The values of the parameters are chosen in such a way that they repeat the analogous ones for Fig. 10. Fig. 12a corresponds to a smooth torus, but unlike Fig. 10a, the attractor in the stroboscopic section is somewhat blurred due to the influence of feedback. An increase in parameter A_1 leads to deformation of the attractor in the stroboscopic section, and at the boundary of the order-chaos transition the attractor takes the form shown in Fig. 12b. In this case, it is impossible to speak about the birth of a strange non-chaotic attractor, since the frequency f_2 depends on a dynamic variable. A further increase in parameter A_1 leads to the transition and development of chaos in the system, which is illustrated in Fig. 12c.

Fig. 13 illustrates the transition to chaos via the destruction of a doubled two-dimensional torus in the presence of feedback, the parameter values correspond to those in Fig. 11. Fig. 13a illustrates a doubled torus, the attractor in the stroboscopic section, as in the previous case, is somewhat blurry, due to the control of the phase of the action of the harmonic component f_2. With an increase in the parameter A_1, a deformation of the invariant curve is observed (Fig. 13b). A further increase in the parameter A_1 leads to the merging of the bands of the doubled torus and to absorbing of the unstable torus by the attractor, and as a result, with a further increase in A_1, chaotic quasiperiodic Hénon attractor is formed (Fig. 13c).

ACCEPTED MANUSCRIPT
Chaos
AIP Publishing
This is the author's peer reviewed, accepted manuscript. However, the online version of record will be different from this version once it has been copyedited and typeset.
PLEASE CITE THIS ARTICLE AS DOI: 10.1063/5.0055579

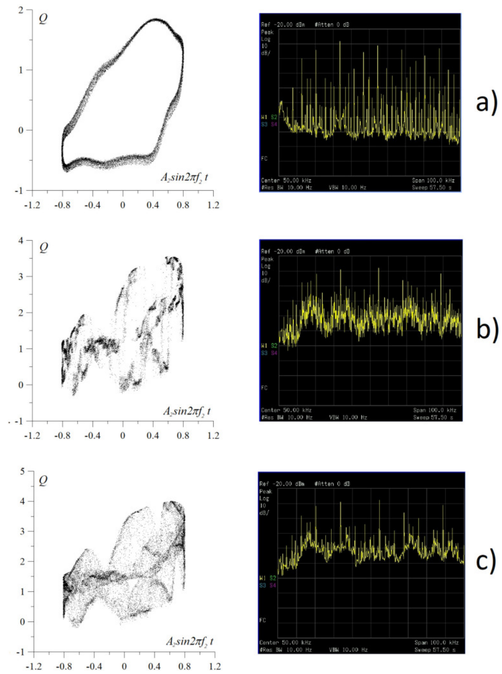

FIG. 12. Attractors in the stroboscopic section and the power spectra of oscillations illustrating the transition to chaos from a two-dimensional torus in the system with feedback: a) A_1=0.5 V, A_2=0.8 V , b) A_1=1.65 V, A_2=0.8 V , c) A_1=2.0 V, A_2=0.8 V.

Thus, the introduction of feedback in a system with quasiperiodic forcing, with the help of which the phase of action of one of the harmonics is controlled, does not qualitatively change the dynamics of the system. It can be assumed that the chaotic attractor, as in the case without feedback, has two zero Lyapunov exponents.

V. CONCLUSION

As a result, a study was carried out of dynamical systems with quasiperiodic driving in which there is a two-dimensional torus, undergoing destruction and transition to chaos, and in which there is a chaotic attractor with two zero Lyapunov exponents. The introduction of feedback changes the dynamics of the system; however, in the case of a weak perturbation of the system, that is, for small values of the feedback parameter, the properties of the chaotic attractor do not qualitatively change; it also has two zero Lyapunov exponents. An increase in the feedback parameter leads to a situation where a chaotic attractor with one zero Lyapunov exponent is born, when crossing the boundary between order and chaos, but with an increase in the amplitude of the action, another Lyapunov exponent turns to zero. With a strong feedback, the birth of a chaotic attractor with one Lyapunov exponent

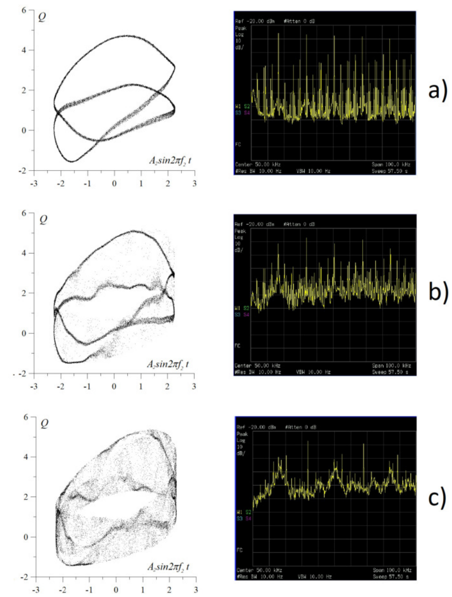

FIG. 13. Attractors in the stroboscopic section and the power spectra of oscillations illustrating the transition to chaos from a doubled two-dimensional torus in the system with feedback: a) A_1=0.6 V, A_2=2.25 V, b) A_1=0.795 V, A_2=2.25 V, c) A_1=1 V, A_2=2.25 V.

is observed. Altogether, the situation can be characterized as robust, that is, weak perturbations of the system do not qualitatively change the properties of the chaotic attractor. Comparison of the results of numerical and experimental studies suggests that the attractors shown in Fig. 12c,d in the stroboscopic section also have two zero Lyapunov exponents.

ACKNOWLEDGMENTS

The work was carried out with the financial support of the Russian Science Foundation, Grant No. 21-12-00121.

DATA AVAILABILITY

The data supporting numerical experiments presented in this paper are available from the corresponding author upon reasonable request.

[1] V. S. Anishchenko, *Dynamical Chaos: Models and Experiments: Appearance Routes and Structure of Chaos in Simple Dynamical Systems*, Vol. 8 (World Scientific, 1995).
[2] A. Pikovsky, J. Kurths, M. Rosenblum, and J. Kurths, *Synchronization: a universal concept in nonlinear sciences*, 12 (Cambridge university press, 2003).
[3] P. S. Landa, *Nonlinear oscillations and waves in dynamical systems*, Vol. 360 (Springer Science & Business Media, 2013).

ACCEPTED MANUSCRIPT
Chaos
AIP Publishing
This is the author's peer reviewed, accepted manuscript. However, the online version of record will be different from this version once it has been copyedited and typeset.
PLEASE CITE THIS ARTICLE AS DOI: 10.1063/5.0055579

[4] H. W. Broer, G. B. Huitema, F. Takens, and B. L. J. Braaksma, *Unfoldings and bifurcations of quasi-periodic tori*, 421 (American Mathematical Soc., 1990).

[5] V. S. Anishchenko, T. Vadivasova, and G. Strelkova, "Deterministic nonlinear systems: A short course," **294** (2014).

[6] L. D. Landau, "On the problem of turbulence," in *Dokl. Akad. Nauk USSR*, Vol. 44 (1944) p. 311.

[7] E. Hopf, "A mathematical example displaying features of turbulence," Communications on Pure and Applied Mathematics **1**, 303–322 (1948).

[8] D. Ruelle and F. Takens, "Note concerning our paper:"on the nature of turbulence"," Communications in Mathematical Physics **23**, 343–344 (1971).

[9] V. Afraimovich and L. Shilnikov, "Nonlinear dynamics and turbulence," Interaction Mech. Math. Ser.(Pitman, Boston, MA, 1983) pp , 1–34 (1983).

[10] V. Anishchenko, T. Letchford, and M. Safonova, "Critical phenomena in the harmonic modulation of two-frequency self-excited oscillations-transitions to chaos through a three-torus," Pisma v Zhurnal Tekhnischeskoi Fiziki **11**, 536–541 (1985).

[11] A. Arneodo, P. Coullet, and E. A. Spiegel, "Cascade of period doublings of tori," Physics Letters A **94**, 1–6 (1983).

[12] K. Kaneko, "Doubling of torus," Progress of theoretical physics **69**, 1806–1810 (1983).

[13] V. Franceschini, "Bifurcations of tori and phase locking in a dissipative system of differential equations," Physica D: Nonlinear Phenomena **6**, 285–304 (1983).

[14] V. S. Anishchenko and T. Letchford, "Destruction of trifrequent oscillations and the chaos in the generator during the biharmonic effect," Zhurnal Tekhnicheskoi Fiziki **56**, 2250–2253 (1986).

[15] V. Anishchenko, M. Safonova, U. Feudel, and J. Kurths, "Bifurcations and transition to chaos through three-dimensional tori," International Journal of Bifurcation and Chaos **4**, 595–607 (1994).

[16] N. Stankevich, A. Kuznetsov, E. Popova, and E. Seleznev, "Experimental diagnostics of multi-frequency quasiperiodic oscillations," Communications in Nonlinear Science and Numerical Simulation **43**, 200–210 (2017).

[17] U. Feudel, M. Safonova, J. Kurths, and V. Anishchenko, "On the destruction of three-dimensional tori," International Journal of Bifurcation and Chaos **6**, 1319–1332 (1996).

[18] A. P. Kuznetsov, S. P. Kuznetsov, I. R. Sataev, and L. V. Turukina, "About landau–hopf scenario in a system of coupled self-oscillators," Physics Letters A **377**, 3291–3295 (2013).

[19] N. Stankevich, O. Astakhov, A. Kuznetsov, and E. Seleznev, "Exciting chaotic and quasi-periodic oscillations in a multicircuit oscillator with a common control scheme," Technical Physics Letters **44**, 428–431 (2018).

[20] R. Vitolo, H. Broer, and C. Simó, "Quasi-periodic bifurcations of invariant circles in low-dimensional dissipative dynamical systems," Regular and Chaotic Dynamics **16**, 154–184 (2011).

[21] N. Stankevich, A. Kuznetsov, and E. Seleznev, "Quasi-periodic bifurcations of four-frequency tori in the ring of five coupled van der pol oscillators with different types of dissipative coupling," Technical Physics **62**, 971–974 (2017).

[22] N. V. Stankevich, N. A. Shchegoleva, I. R. Sataev, and A. P. Kuznetsov, "Three-dimensional torus breakdown and chaos with two zero lyapunov exponents in coupled radio-physical generators," Journal of Computational and Nonlinear Dynamics **15** (2020).

[23] C. Grebogi, E. Ott, and J. A. Yorke, "Are three-frequency quasiperiodic orbits to be expected in typical nonlinear dynamical systems?" Physical Review Letters **51**, 339 (1983).

[24] C. Grebogi, E. Ott, and J. A. Yorke, "Attractors on an n-torus: Quasiperiodicity versus chaos," Physica D: Nonlinear Phenomena **15**, 354–373 (1985).

[25] C. Grebogi, E. Ott, S. Pelikan, and J. A. Yorke, "Strange attractors that are not chaotic," Physica D: Nonlinear Phenomena **13**, 261–268 (1984).

[26] B. P. Bezruchko, S. P. Kuznetsov, and Y. P. Seleznev, "Experimental observation of dynamics near the torus-doubling terminal critical point," Physical Review E **62**, 7828 (2000).

[27] M. Hénon, "A two-dimensional mapping with a strange attractor," in *The Theory of Chaotic Attractors* (Springer, 1976) pp. 94–102.

[28] W. Lim and S. Kim, "Mechanism for the band-merging route to strange nonchaotic attractors in quasiperiodically forced systems," JOURNAL-KOREAN PHYSICAL SOCIETY **47**, 414 (2005).

[29] H. W. Broer, C. Simó, and R. Vitolo, "Quasi-periodic hénon-like attractors in 3d diffeomorphisms," ENOC-2005, Eindhoven , 7–12 (2005).

[30] H. W. Broer, C. Simó, and R. Vitolo, "Chaos and quasi-periodicity in diffeomorphisms of the solid torus," Discrete Contin. Dyn. Syst. Ser. B **14**, 871–905 (2010).

[31] A. S. Pikovsky and U. Feudel, "Characterizing strange nonchaotic attractors," Chaos: An Interdisciplinary Journal of Nonlinear Science **5**, 253–260 (1995).

[32] G. Benettin, L. Galgani, A. Giorgilli, and J.-M. Strelcyn, "Lyapunov characteristic exponents for smooth dynamical systems and for hamiltonian systems; a method for computing all of them. part 1: Theory," Meccanica **15**, 9–20 (1980).

[33] A. Pikovsky and A. Politi, *Lyapunov exponents: a tool to explore complex dynamics* (Cambridge University Press, 2016).