Dynamics of small stars and chains of spin-torque oscillators coupled via magnetic fields

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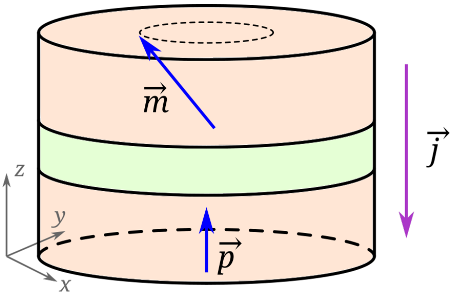
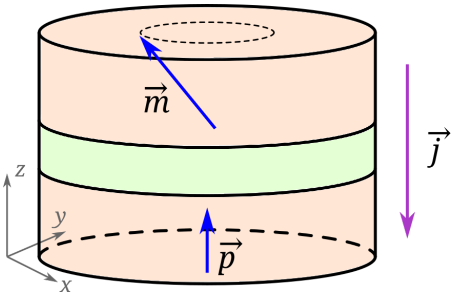
Spin-torque oscilator

Spin-torque oscillator is a nanoscale spintronic device generating periodic microwave (in the frequency range of several GHz) oscillations



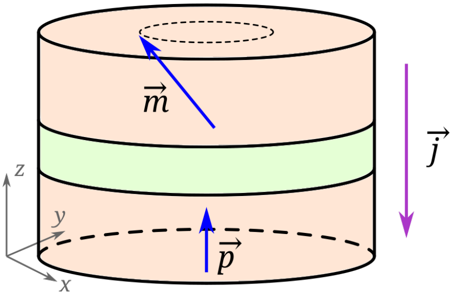
Spin-torque oscilator

- Three-layers: two ferromagnets with a diamagnet in between
- The lower layer is fixed: it has constant magnetization $\vec{p}$
- The upper layer is free, its magnetization $\vec{m}$ can be changed
- Current flows downward so electrons run upward



Spin-torque oscilator

- When moving through the lower layer electrons are polarized: align their spins along $\vec{p}$
- Then they come to the upper layer through the diamagnet and loses their polarization
- Due to conservation of angular momentum the magnetization $\vec{m}$ of the upper layer starts to oscillate



Landau–Lifshitz–Gilbert–Slonczewski equation

Model of dynamics of the spin-torque oscillator:

$$\dot{\vec{m}} = -\vec{m} \times \vec{h}_{\text{heff}} - \alpha \cdot \vec{m} \times \left(\vec{m} \times \vec{h}_{\text{heff}}\right) - \\ - \beta \cdot \vec{m} \times (\vec{m} \times \vec{p}) + \alpha \cdot \beta \cdot \vec{m} \times \vec{p}$$

Magnetizations are normalized: $\|\vec{p}\| = 1$, $\|\vec{m}\| = 1$.

Unit norm of $\vec{m}$ is preserved (dynamics on sphere).

α is damping constant, β is proportional to the current $\vec{j}$ and contains material parameters, $\vec{h}_{\text{heff}}$ is effective magnetic field.

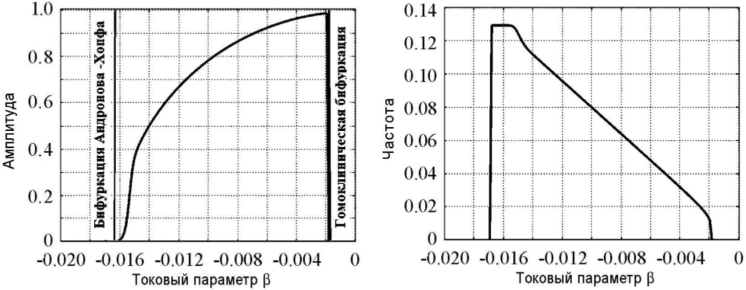
Landau–Lifshitz–Gilbert–Slonczewski equation

$$\vec{h}_{\mathrm{heff}} = K_u(\vec{m}, \vec{u})\vec{u} - N\vec{m} + \vec{h}_{\mathrm{ext}}$$

- $K_u(\vec{m}, \vec{u})\vec{u}$ is an easy-axis filed (anisotropy field), where $\vec{u}$ is a unit vector of anisotropy (the easy-axis) and K_u is an anisotropy constant.
- $N\vec{m}$ is a demagnetizing field, where N is a diagonal matrix with elements $N_x + N_y + N_z = 1$ (geometry-dependent demagnetization coefficients).
- $\vec{h}_{\mathrm{ext}}$ is an external magnetic field.

Single oscillator

6 fixed points, limit cycle



Taken from *Я.В. Туркин, П.В. Купцов* Изв. вузов «ПНД», т. 22, №6, 2014

Coupling schemes

- Oscillators are stacked into a linear array and share the common current $\vec{j}$. The current depends on the stack resistance and the resistance depends on the oscillator magnetizations $\vec{m}_i$. [*Zaks, M.*, *Pikovsky, A.* Sci Rep 7, 4648 (2017)]

- Oscillators are located in plane (in space) and interact via magnetic fields of each other. To avoid global coupling via common filed unnecessary influences are blocked with the help of diamagnet isolators. [*Я.В. Туркин*, *П.В. Купцов* Изв. вузов «ПНД», т. 22, №6, 2014]

Coupling via magnetic fields

As known from electrodynamics, in linear approximation magnetic filed is proportional to magnetization:

$$\vec{H} = \chi_m \vec{H}$$

where χ_m is the volume magnetic susceptibility.

The coupling via magnetic fields can be introduced via adding terms to the effective filed:

$$\vec{h}_{\text{heff}} \to \vec{h}_{\text{heff}} + \epsilon \sum_i \vec{m}_i$$

where ϵ is coupling strength.

Network of coupled spin-torque oscillators

$$\dot{\vec{m}}_n = -\vec{m}_n \times (\vec{h}_{\text{heff}})_n - \alpha_n \cdot \vec{m}_n \times \left(\vec{m}_n \times (\vec{h}_{\text{heff}})_n\right) -$$
$$- \beta_n \cdot \vec{m}_n \times (\vec{m}_n \times \vec{p}_n) + \alpha_n \cdot \beta_n \cdot \vec{m}_n \times \vec{p}_n$$

$$(\vec{h}_{\text{heff}})_n = (K_u)_n(\vec{m}, \vec{u}_n)\vec{u}_n - N_n\vec{m}_n + \vec{h}_{\text{ext}} + \epsilon_n \sum_{i=1, i\neq n}^{S} a_{ni}\vec{m}_i / k_n$$

where a_{ni} is coupling matrix and k_n is its row sum.

The external field $\vec{h}_{\text{ext}}$ can be assumed to be the same for all oscillators.

$a_{nn} = 0$ because $\vec{m}_n \times \vec{m}_n = 0$.

Remark on normalization issues

The oscillators are different but magnetizations $\vec{m}_n$ are normalized so that $\|\vec{m}_n\| = 1$.

What if the magnetizations have actually different magnitudes?

In this case our coupling via magnetic fields via $\epsilon_n \sum_{i=1,i\neq n}^{S} a_{ni}\vec{m}_i/k_n$ must be rewritten as $\sum_{i=1,i\neq n}^{S} \epsilon_{ni}a_{ni}\vec{m}_i/k_n$ where ϵ_{ni} takes into account different magnitudes.

But this is not the case, since the original magnitudes of the magnetization depend only on the material, since this is so called saturation magnetization:

$$\vec{m}_n = \vec{M}_n/\vec{M}_s$$

For example cobalt: $\vec{M}_s = 1.4 \cdot 10^6$ A m^{-1}

Kuramoto order parameter

Kuramoto order parameter (measure of phase synchronization):

$$p[\phi(t),\psi(t)] = \left|\left\langle \mathrm{e}^{\mathrm{i}[\phi(t)-\psi(t)]} \right\rangle_t\right|$$

Works well only if $\phi(t) - \psi(t) = \mathrm{const}$

- fails if $\phi(t) - \psi(t)$ oscillates periodically, however this is still phase synchronization
- fails if $n\phi(t) - m\psi(t) = \mathrm{const}$ (or periodic), however this is $n\colon m$ synchronization

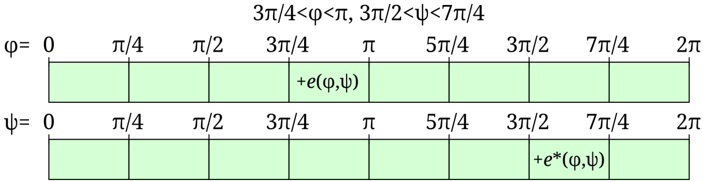
Generalized Kuramoto synch criterion

Prepare two arrays of N elements: for ϕ and for ψ.

Cells correspond to ranges $[2i\pi/N, 2(i+1)\pi/N)$, $i = 0, \ldots, N-1$ (just like in histogram computation).

Given new ϕ and ψ find cell indexes i_ϕ and i_ψ.

Compute $e(\phi, \psi) = \mathrm{e}^{\mathrm{i}[\phi - \psi]}$ and add it to the corresponding cells.



Generalized Kuramoto synch criterion

When computations are over, divide values collected in each cell by the corresponding count (averaging).

The result p is the largest value from both arrays.

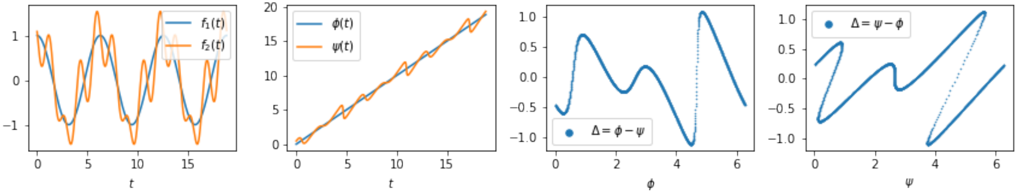
Synchronization when $p \approx 1$.

Generalized Kuramoto synch criterion

Why it works betters then simple Kuramoto order parameter:

When oscillations are synchronized $\Delta = \phi - \psi$ can oscillate periodically. If Δ is computed it at fixed ϕ or ψ it is constant.

$\Delta(\phi)$ or $\Delta(\psi)$ can be unambiguous. This is avoided by taking the largest value in two arrays.



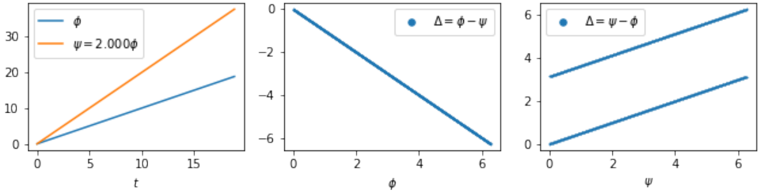
Generalized Kuramoto synch criterion

We can also reveal $n{:}m$ synchronization.

Let $\psi = 2\phi$ (1:2 synchronization).

Function $\Delta(\psi)$ is unambiguous so that $\langle e(\phi, \psi) \rangle$ is small.

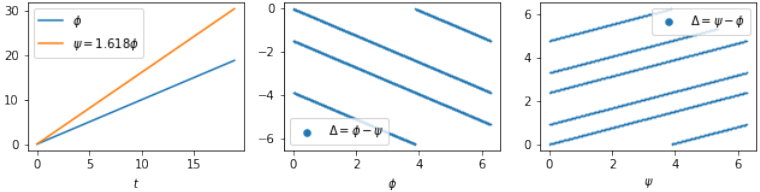
But $\Delta(\phi)$ have constant Δ for each particular ϕ so that $\langle e(\phi, \psi) \rangle = 1$.



Generalized Kuramoto synch criterion

How it works for quasiperiodicity: $\psi = \phi(\sqrt{5}+1)/2$.

The resulting criterion will be small because both $\Delta(\phi)$ and $\Delta(\psi)$ are unambiguous.



Generalized Kuramoto synch criterion

Detection of fixed points: at fixed point $p = 1$ and only one cell in array of phases is filled (provided that phase still can be defined).

In rare cases two cells can be filled if a constant phase fits exactly the boundary between cells, e.g. at $\phi = \pi$. Due to numerical errors two neighboring cells will be filled. It must be checked separately.

Thus if one cell in both phase arrays are filled we conclude that this is a fixed point and set $p = 0$.

Angle coordinates

Spherical coordinates ($r = 1$)

$$\phi = \text{atan2}(m_y, m_x),\ \theta = \text{atan2}(\sqrt{m_x^2 + m_y^2}, m_z)$$

Compute separately sync. criterion for ϕ and θ, p_ϕ and p_θ.

$$p = \begin{cases} 0 & \text{if } p_\phi = 0 \text{ and } p_\theta = 0 \\ (p_\phi + p_\theta)/2 & \text{if } p_\phi > 0 \text{ and } p_\theta > 0 \\ p_\phi & \text{if } p_\theta = 0 \\ p_\theta & \text{if } p_\phi = 0 \end{cases}$$

Multistability

Network of spin-torque oscillators demonstrates multistability.

In this study we avoid it and fix constant initial conditions.

Remote synchronization

Starlike network: peripheral oscillators get synchronized while the central one is not.

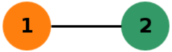
Oscillators must be different.

To observe remote synchronization in networks of spin-torque oscillators we set eigen frequency of the central one being strongly different from the others.

The frequency is controlled by β.

We are going to consider charts of Lyapunov exponents and Kuramoto synchronization parameters vs. ϵ (coupling strength) and β_1 (controls frequency of the central oscillator).

Two oscillators

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eps: [*]
alf: [0.02]
bet: [*, -0.0015]
Ku: [0.001]
u:
  - [0, 1, 0]
p:
  - [0, 0, 1]
N:
  - [0.0490, 0.0777, 0.8733]
  - [0.0645, 0.0774, 0.8581]
```

Oscillator 1 is controlled

Two oscillators

There are $2 \times 3 = 6$ Lyapunov exponents, 2 zeros due to $\|\vec{m}\| = 1$

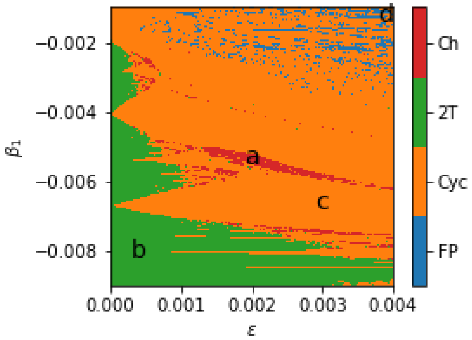
- Ch: Chaos, $\lambda_1 > 0$
- FP: Fixed point, $\lambda_{1,2} = 0$, $\lambda_{3,4,5,6} < 0$
- Cyc: Limit cycle (synchronization), $\lambda_{1,2,3} = 0$, $\lambda_{4,5,6} < 0$
- 2T: Two freq. torus (no synchronization), $\lambda_{1,2,3,4} = 0$, $\lambda_{5,6} < 0$

Synchronization criterion p

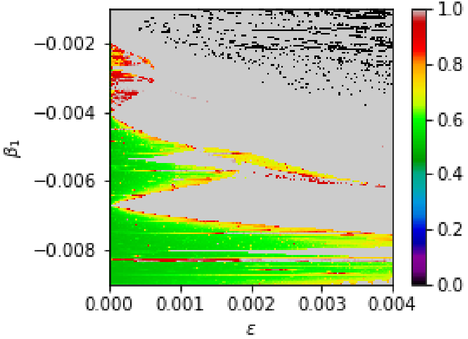
- $p \approx 1$ - synchronization
- $p = 0$ - fixed point

Two oscillators, charts

Lyapunov exps. chart



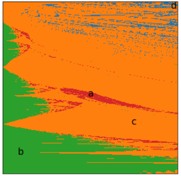
Sync. chart

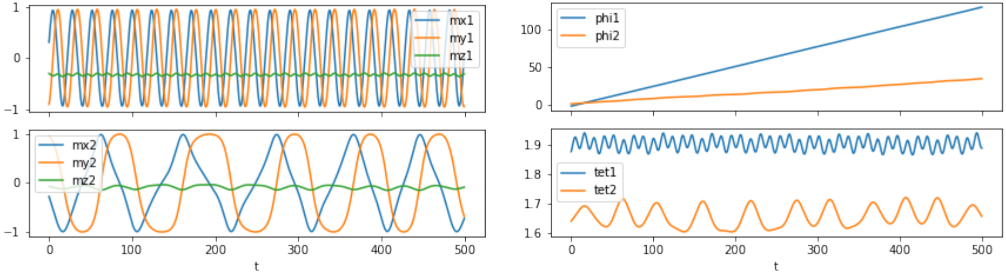


(a) Chaos, (b) 2-Torus, (c) Cycle, (d) Cycle

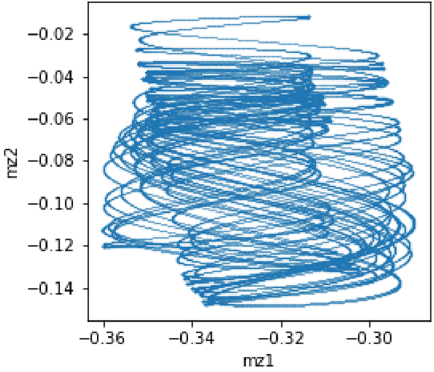
Two oscillators: (a) Chaos

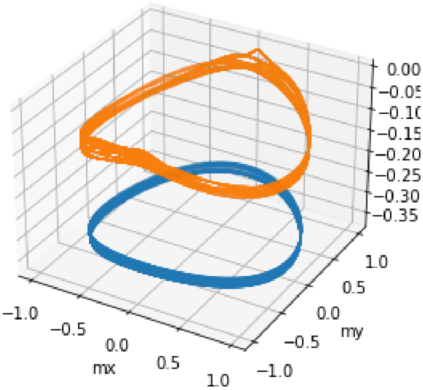
$\epsilon = 0.002$, $\beta_1 = -0.0053$: $\lambda_1 = 0.0148$, $\lambda_{2,3,4} = 0$,
$\lambda_5 = -0.0145$, $\lambda_6 = -0.0303$, $p = 0.71$



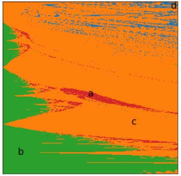


Two oscillators: (a) Chaos

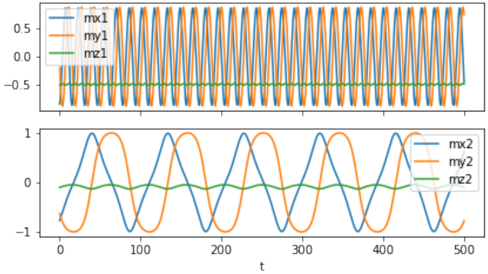


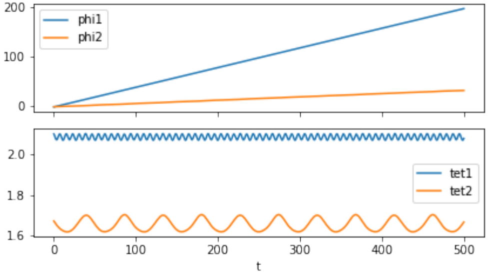


Two oscillators: (b) 2-Torus

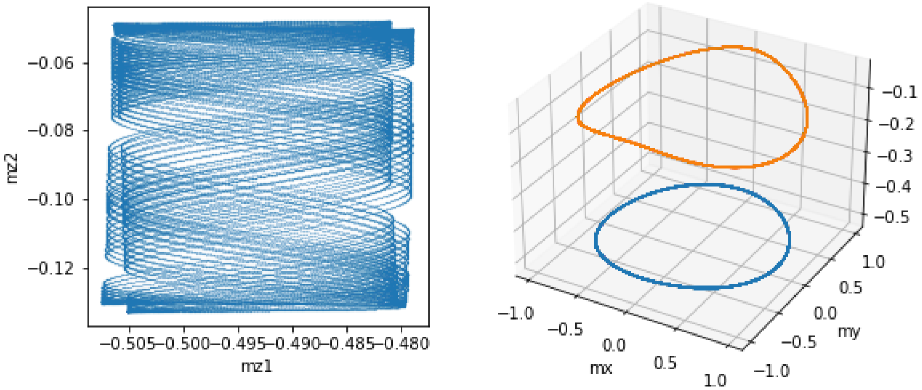


$\epsilon = 0.0004$, $\beta_1 = -0.008$ (no sync): $\lambda_{1,2,3,4} = 0$,
$\lambda_5 = -0.0123$, $\lambda_6 = -0.0155$, $p = 0.96$



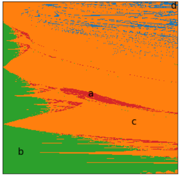


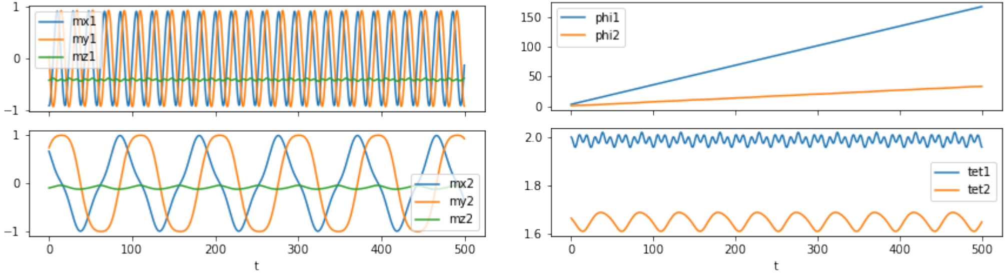
Two oscillators: (b) 2-Torus



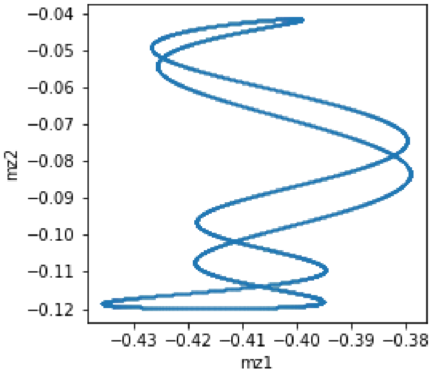
Two oscillators: (c) Cycle

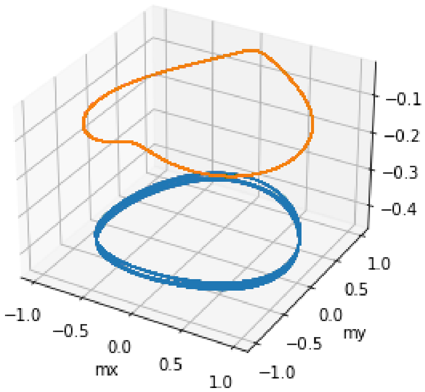
$\epsilon = 0.003$, $\beta_1 = -0.0066$ (sync): $\lambda_{1,2,3} = 0$,
$\lambda_4 = -0.0077$, $\lambda_5 = -0.00775$, $\lambda_6 = -0.0135$,
$p = 0.9996$





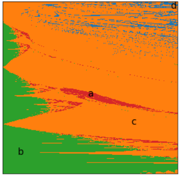
Two oscillators: (c) Cycle

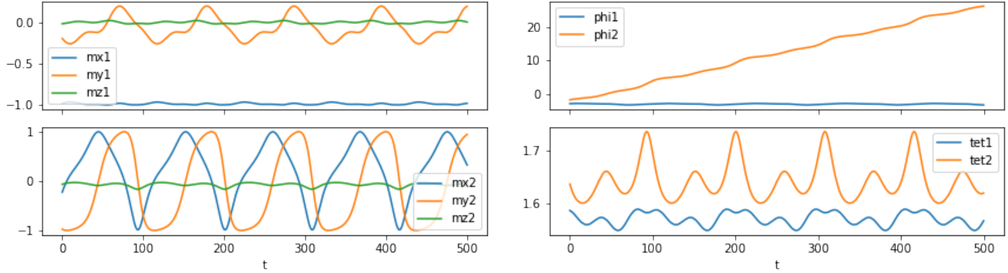




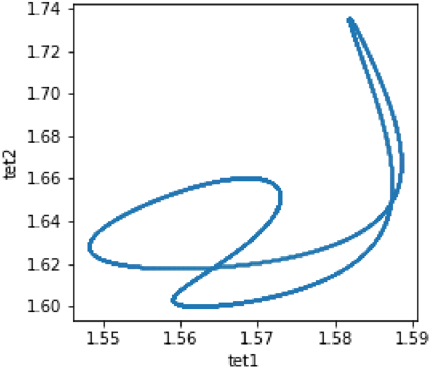
Two oscillators: (d) Cycle

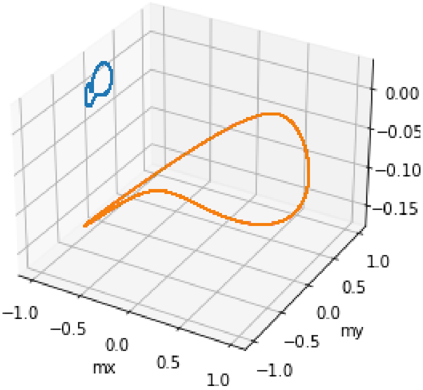
$\epsilon = 0.0039$, $\beta_1 = -0.0011$ (sync): $\lambda_{1,2,3} = 0$,
$\lambda_4 = -0.00862$, $\lambda_5 = -0.00862$, $\lambda_6 = -0.0153$,
$p = 0.99992$



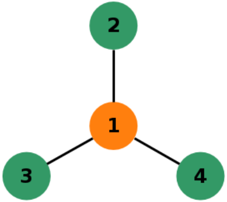


Two oscillators: (d) Cycle





3-Star



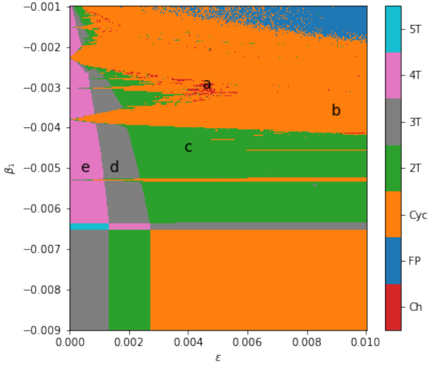

```
eps: [*]
alf: [0.008, 0.010, 0.011, 0.012]
bet: [*, -0.0010, -0.0011, -0.0012]
Ku: [0.08]
u:
  - [0, 1, 0]
p:
  - [0, 0, 1]
N:
  - [0.0318, 0.1269, 0.8413]
  - [0.0320, 0.1269, 0.8411]
  - [0.0319, 0.1270, 0.8411]
  - [0.0317, 0.1271, 0.8412]
```

Oscillator 1 is controlled

3-Star: Lyapunov exps. chart

There are $4 \times 3 = 12$ Lyapunov exponents, 4 zeros due to $\|\vec{m}\| = 1$

- Ch: Chaos, $\lambda_1 > 0$
- FP: Fixed point, $\lambda_{1-4} = 0$
- Cyc: Full sync, $\lambda_{1-5} = 0$
- 2T: 2-Torus, $\lambda_{1-6} = 0$
- 3T: 3-Torus, $\lambda_{1-7} = 0$
- 4T: 4-Torus, $\lambda_{1-8} = 0$
- 5T: 5-Torus, $\lambda_{1-9} = 0$

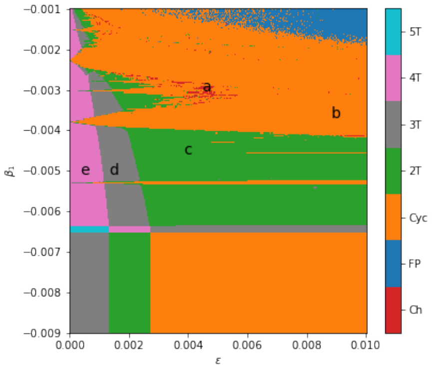


Horizontal stripe at $\beta_1 = -0.0065$ is a numerical artifact.

No 5-Torus observed.

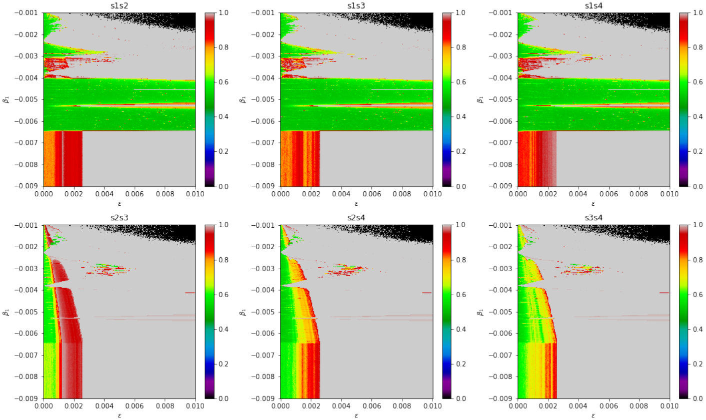
3-Star: Lyapunov exps. chart

- (a) Chaos
- (b) Cycle
- (c) 2-Torus (remote sync)
- (d) 3-Torus
- (e) 4-Torus



3-Star: Pairwise sync. charts

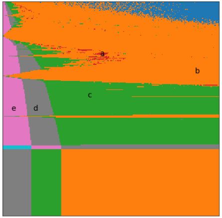
Gray areas indicate phase synchronization.

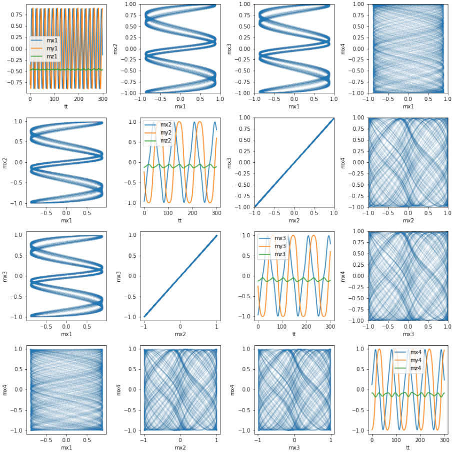


3-Star: (a) Chaos

$\lambda_1 = 0.00530, \lambda_{2,3,4,5,6} = 0, \lambda_{7-12} < 0,$
$p_{12} = p_{13} = 0.995, p_{23} = 1$

Extra zero Lyapunov exponent λ_6 due to synchronization of oscs. 2 and 3

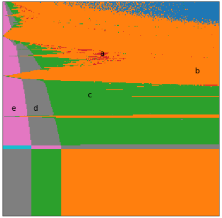


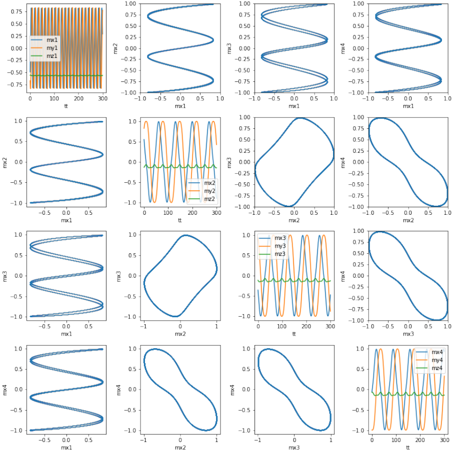


3-Star: (b) Cycle

$$\lambda_{1,2,3,4,5} = 0,\ \lambda_{6-12} < 0,\ p_{ij} = 1$$

Full synchronization: center with rays 5:1, rays 1:1



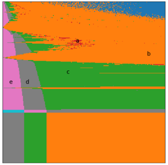


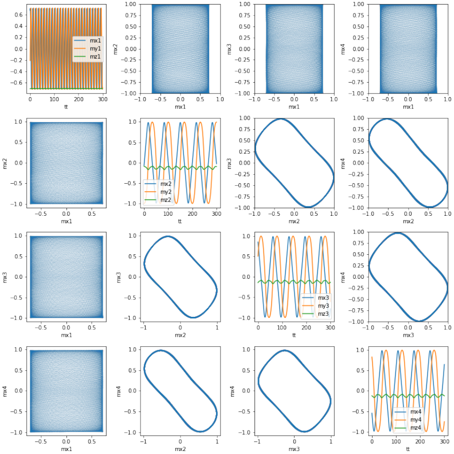
3-Star: (c) 2-Torus

$\lambda_{1-6} = 0$, $\lambda_{7-12} < 0$, $p_{12,13,14} = 0.53$, $p_{23,24,34} = 0.9998$

Rays are synchronized while center is not: remote synchronization

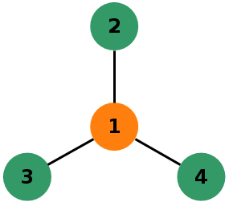
Amplitude of the center approx. two times smaller then the amplitude of the rays. Its frequency is approx. one order higher.





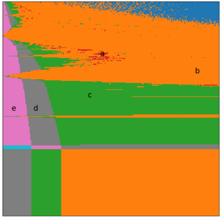
3-Star: Remote synchronization

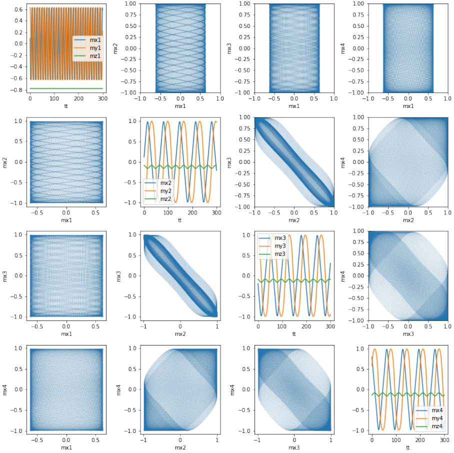
Oscillators 2, 3, and 4 get synchronized



3-Star: (d) 3-Torus

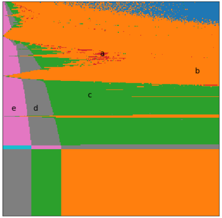
$\lambda_{1,2,3,4,5,6,7} = 0, \lambda_{8-12} < 0,$
$p_{12,13,14} = 0.51, p_{23} = 0.99, p_{24} = 0.83,$
$p_{34} = 0.92$

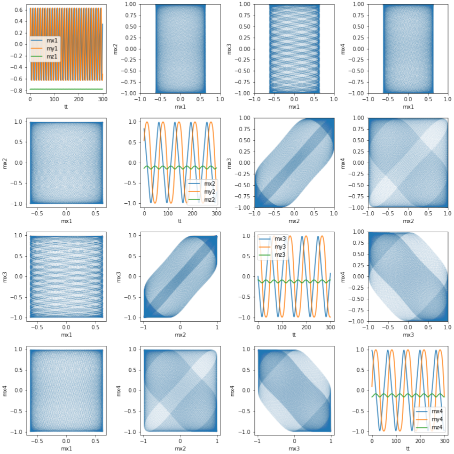




3-Star: (e) 4-Torus

$\lambda_{1,2,3,4,5,6,7,8} = 0,\ \lambda_{9-12} < 0$
$p_{12,13,14} = 0.51,\ p_{23} = 0.98,\ p_{24} = 0.75,$
$p_{34} = 0.91$

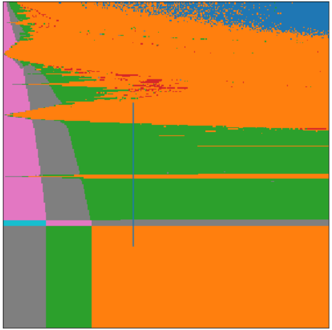


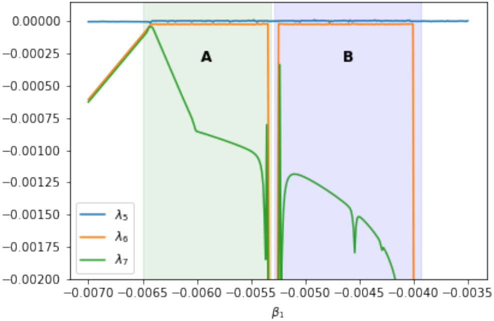


3-Star: Transition to remote synchronization

Area A: Transition to remote synchronization occurs via Neimark–Sacker bifurcation at the left edge and via hard excitation at the right edge.

Area B: Hard excitation.





3-Star: Transition to remote synchronization

All computations are done with the same initial conditions.

Hard excitation: multistability, subcritical bifurcation.

2-Star (chain of three nodes)

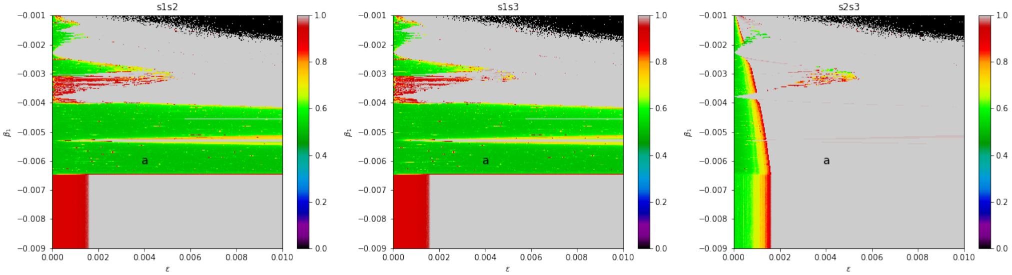

```
eps: [*]
alf: [0.008, 0.010, 0.012]
bet: [*, -0.0010, -0.0012]
Ku: [0.08]
u:
  - [0, 1, 0]
p:
  - [0, 0, 1]
N:
  - [0.0318, 0.1269, 0.8413]
  - [0.0320, 0.1269, 0.8411]
  - [0.0317, 0.1271, 0.8412]
```


There are $3 \times 3 = 9$ Lyapunov exponents, 3 zeros due to $\|\vec{m}\| = 1$

Oscillator 1 is controlled

2-Star: Synchronization charts

Observe similarity of charts with 3-star. Letter (a) marks remote synchronization

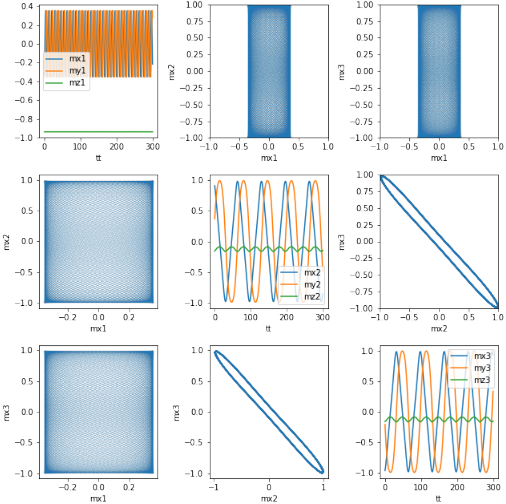


2-Star: (a) Remote synchronization

$\lambda_{1-5} = 0, \lambda_{6-9} < 0,$
$p_{1,2} = 0.512,$
$p_{1,3} = 0.510, p_{2,3} = 1$

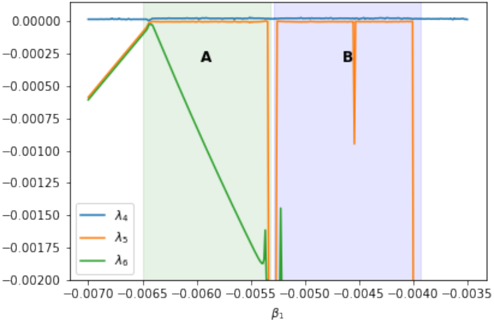
2-Torus: Remote synchronization

Observe again small amplitude of the center and high frequency.

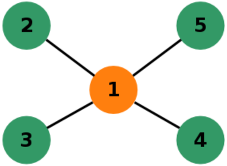


2-Star: Transition to remote synchronization

The same scenario as for 3-Star



4-Star

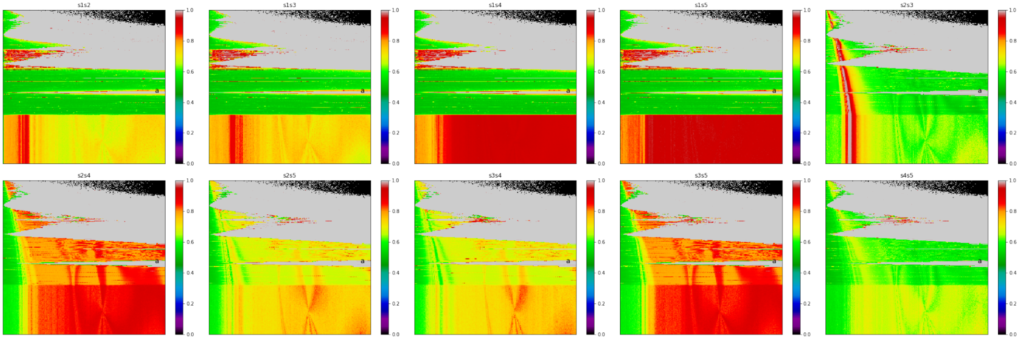



```
eps: [*]
alf: [0.008, 0.010, 0.011, 0.012, 0.013]
bet: [-0.005, -0.0010, -0.0011, -0.0012, -0.0013]
Ku: [0.08]
u:
  - [0, 1, 0]
p:
  - [0, 0, 1]
N:
  - [0.0318, 0.1269, 0.8413]
  - [0.0320, 0.1269, 0.8411]
  - [0.0319, 0.1270, 0.8411]
  - [0.0317, 0.1271, 0.8412]
  - [0.0322, 0.1268, 0.8410]
```

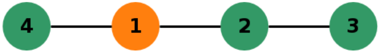
Oscillator 1 is controlled

4-Star: Synchronization chart

No area of remote synchronization



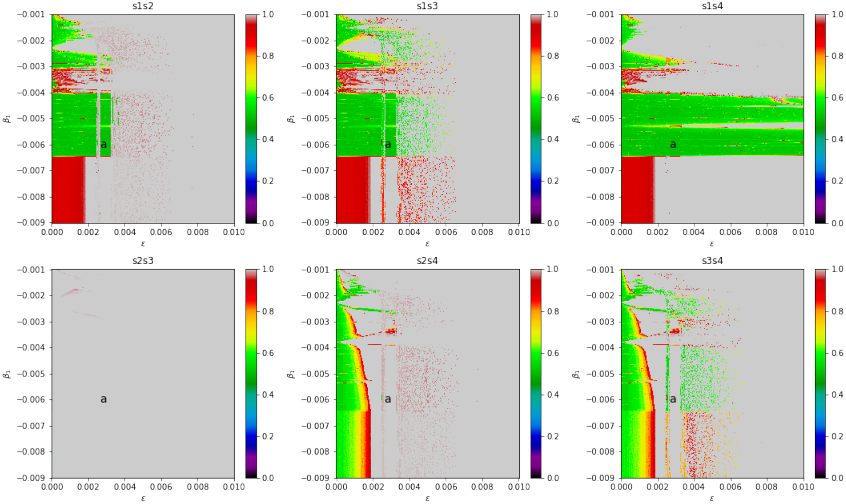
4-Chain




```
eps: [*]
alf: [0.008, 0.010, 0.011, 0.012]
bet: [*, -0.0010, -0.0011, -0.0012]
Ku: [0.08]
u:
  - [0, 1, 0]
p:
  - [0, 0, 1]
N:
  - [0.0318, 0.1269, 0.8413]
  - [0.0320, 0.1269, 0.8411]
  - [0.0319, 0.1270, 0.8411]
  - [0.0317, 0.1271, 0.8412]
```

Oscillator 1 is controlled

4-Chain: Synchronization chart

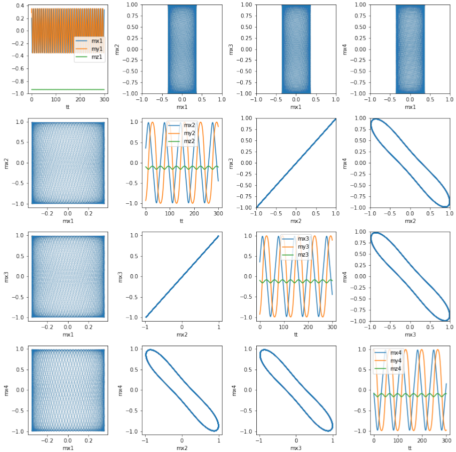


4-Chain: (a) Asymmetric remote synchronization

$\lambda_{1-6} = 0, \lambda_{7-12} < 0,$
$p_{12,13,14} = 0.51,$
$p_{23,24,34} = 1$

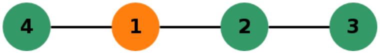
2-Torus: Remote synchronization

Small amplitude of the center and high frequency



4-Chain: Asymmetric remote synchronization

Oscillators 2, 3, and 4 get synchronized



Outline

- Coupling via magnetic fields allows building various network structures
- The model is closely related to a physical system and have very rich dynamics. Interesting to analyze and classify its regimes
- When the remote synchronization occurs the frequency of the central oscillator is at least one order higher then the synchronization frequency of the peripherals. Its amplitude is at least two times smaller.
- Asymmetric remote synchronization is observed in a chain of four oscillators when the middle oscillator is controlled
- Since frequency of the central oscillator is very sensitive to the current through it, is can be use for control the whole network regime
- Increasing number of rays result in vanishing of the remote synchronization

Outline

- Synchronization effects can potentially be useful for development new spintronic devices
    - Remote synchronization and its control: switching device. Current through the hub is input, average of ray-oscillators is output. High output if rays are synchronized and low if not