Self-acceleration of Chaplygin sleigh

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A recognized paradigmatic model in non-holonomic mechanics is the Chaplygin sleigh, that is a platform moving on a horizontal plane with a constraint that translational velocity at some point is oriented always along a direction fixed relative to the platform. This non-holonomic constraint can be arranged with the help of a knife-edge fixed on the sleigh, or with a wheel pair.

In a frame of robotics, it is of fundamental importance to consider possibilities of self-acceleration of the sleigh caused by motions of internal masses.

The present report is devoted to a study of the Chaplygin sleigh motions provided by oscillations of a single internal mass. We suppose that the attached moving particle of relative mass μ performs a periodic motion orthogonal to the direction of the knife-edge at distance *s* from it, around a point located on a straight line connecting the constraint location and the center of mass of the platform [1]. Normalized equations for the generalized momenta *p* and *q* for the translational and rotational motions of the platform are

$$\dot{p} = (Dw + \mu\varepsilon\cos\tau)w - \nu_1 u,\ \dot{q} = -(Dw + \mu\varepsilon\cos\tau)u - \nu_3 w, \tag{1}$$

being supplemented by expressions of the generalized velocities via the momenta

$$u - w\mu\varepsilon\sin\tau = p,\ -u\mu\varepsilon\sin\tau + (J + \mu\varepsilon^2\sin^2\tau)w = q - \mu s\varepsilon\cos\tau. \tag{2}$$

Here *a* is a distance from the constraint location point to the center of mass of the platform, ε is amplitude of the particle oscillations, $\nu_{1,3}$ are coefficients of friction for the translational and rotational motions, $D = a - \mu a + \mu s$, $J = I_0 + a^2 + \mu(s^2 - a^2)$, and I_0 is the normalized moment of inertia of the platform.

Being given some values of p_n, q_n at $\tau = 2\pi n$, one can solve numerically Eqs.(7) to obtain

$$(p_{n+1},\ q_{n+1}) = \mathbf{f}(p_n,\ q_n) \tag{3}$$

that defines the two-dimensional stroboscopic Poincaré map.

Equations (1) and (2) and the map (3), respectively, in continuous and discrete time, define the reduced system, which can be considered independently of the other variables (coordinates in the laboratory frame and the angle of rotation of the platform), which obey the equations

$$\dot{x} = u\cos\varphi,\ \dot{y} = u\sin\varphi,\ \dot{\varphi} = \omega \tag{4}$$

We distinguish two mechanisms of the sleigh self-acceleration. One mechanism, observed with small oscillations of the internal mass, leads to a regular unidirectional motion. Another mechanism is effect of parametric instability observed when the moving internal mass is of value comparable with the mass of the platform; it leads to oscillatory motions of the sleigh.

To consider the first mechanism, we write down an equation averaged over a period of particle oscillations neglecting terms of order ε^2. It reads

$$\dot{p} = -\nu_1 p + \tfrac{1}{2}\mu^2\varepsilon^2(s - \nu_1 p)\frac{Ds - J - (2Dp + \nu_3)\nu_1}{J^2 + (Dp + \nu_3)^2} \tag{5}$$

In the absence of friction unbounded acceleration takes place under conditions

$$D = a - \mu a + \mu s > 0,\ s\bigl(Ds - J\bigr) = s\bigl(a(s-a)(1-\mu) - I_0\bigr) > 0 \tag{6}$$

In this case, we get $\tfrac{1}{3}p^3D^2 + J^2p = \tfrac{1}{2}\mu^2 s\varepsilon^2(Ds - J)\tau + \mathrm{const}$. So, the momentum asymptotically follows the power law $t^{1/3}$, and the same is true for the translational velocity of the sleigh.

In the presence of friction the growth of the momentum appears to be bounded on some level depending on the friction coefficient. Computations show a remarkably good agreement between the description with (5) and the results of numerical simulation using (1).

To consider the second mechanism we remark, first of all, that at D=0 the equations (1) become a set of linear equations with periodic coefficients, similar to those studied in theory of parametric oscillations, like the Mathieu equation. Varying parameters, one can observe zones of parametric instability. There the solutions of (1) manifest oscillations of growing amplitude as illustrated in Fig.1a. It is unbounded because of linear nature of the equations in this case.

If the line of the moving mass oscillations is shifted from the center of mass, the equations become nonlinear, and it leads to saturation of the parametric instability. In such a case the trajectory in the state space of the reduced equations evolves to attractor, which may be regular or chaotic; the last is the case illustrated in Fig.1b. In laboratory frame it corresponds to a random-like walk as illustrated in Fig.1c. Such a motion gives rise to the Rayleigh distribution for distances passed for a fixed long time interval and to uniform distribution for the azimuth angles.

The existence of chaotic attractors exhibited by the reduced equations of the Chaplygin sleigh with moving internal mass makes it possible to apply chaos control technique for the sleigh motion using small variations of characteristics of the oscillations of the internal mass because of sensitivity of chaotic orbits to small perturbations.

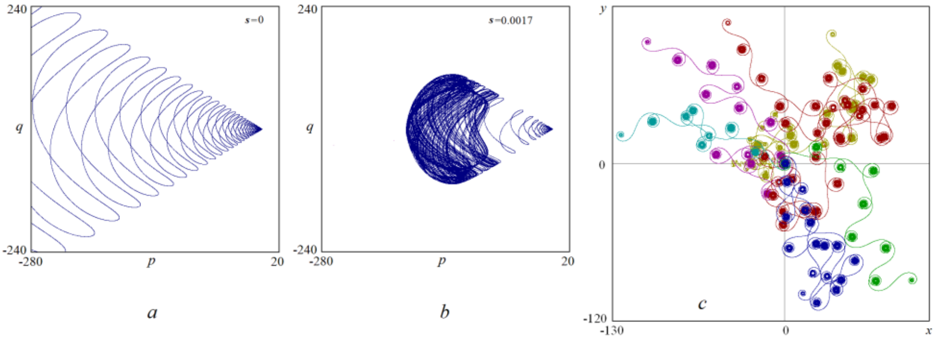

Fig.1. Unbounded acceleration of the sleigh (a) and bounded acceleration where the trajectory approaches a chaotic attractor (*b*) accompanied by random-like walk in the laboratory frame (c). Parameters are a=0, $I_0 = 0.15$, $\mu = 0.85, \nu_1 = 0.35, \nu_3 = 0.1$, $\varepsilon = 1$

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References

[1] Bizyaev I.A., Borisov A.V., Kuznetsov S.P. // EPL, 2017, vol. 119, no. 6, 60008.