On a certain type of universal behavior of coupled systems

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The dynamics of a Hamiltonian system consisting of two coupled identical oscillators excited by periodic pulses is investigated by the renormalization group method for the case in which the pulse intensities depend nonlinearly on the generalized coordinate. It is found that three parameters are needed for the description of the system in the vicinity of the order-disorder transition point (i.e., that this point is a tricritical one). Two of the parameters characterize the coupling between the oscillators, and are interpreted as the constants for two types of interaction between the Hamiltonian systems. The splitting of the parameter space into regions of different regimes in the vicinity of the tricritical point has a universal character, and does not change when the scales along the three coordinate axes are changed by factors of 8.7211, − 4.4039, and 1.8510 respectively. It is suggested that the only condition for the realization of this type of behavior is the existence of a sequence of period doublings for each of the coupled systems.

A chain of coupled nonlinear oscillators is a traditional object of investigation in the theory of vibrations and statistical mechanics. Such a chain serves, for example, as the simplest one-dimensional model of solids, with oscillator atoms at the crystal-lattice sites. In the majority of published investigations it is assumed that the individual oscillators are completely integrable Hamiltonian systems exhibiting just simple quasiperiodic motions.[1–3] Upon the introduction of sufficiently strong coupling, the chain may become a nonintegrable system and stochastic vibrations then become possible in it.[2,3]

Stochastic behavior is, however, exhibited by systems with a small number of degrees of freedom, e.g., nonlinear oscillators excited by an external periodic influence.[2–6] It is therefore of interest to investigate a chain of coupled systems, each of which can by itself exhibit stochastic vibrations. One of the systems for which such a chain serves as a model is a crystal under the action of a high-intensity optical or acoustic signal.

We shall assume that the individual cells that are the building blocks of the chain are each a Hamiltonian system that undergoes, upon the variation of some parameter λ, a transition from the state of regular behavior into a state of stochastic behavior via an infinite sequency of period-doubling bifurcations of the stable cycles. This type of order-disorder transition is widespread (it is, in particular, realized in the case of the above-indicated nonlinear oscillator when the intensity of the external influence is increased), and has been relatively well studied.[5,7,8] It is, for example, well known that the sequency of bifurcation values for the parameter λ_N converges to a limit (the critical point) according to the law of geometric progression:

$$\lambda_N = \lambda_c - K\delta_H^{-N}. \tag{1}$$

There exist directly at the critical point (unstable) cycles with all possible periods of 2^N units of the initial period. In the formula (1) $\delta_H = 8.7210972$ is a universal constant, while λ_c and K are constants that depend on the specific system. It is convenient to use in place of λ the parameter $\Lambda = (\lambda - \lambda_c)/K$, the bifurcation values of which are universal.

A major role in the understanding of the mechanisms underlying the order-disorder transition via doubling bifurcations has been played by the renormalization group (RG) method,[5,7–12] which is similar in its formal content to the well-known method used in the theory of phase transitions.[13,14] At the critical point the mapping describing the changes that occur in the state of the system over the period 2^N is invariant (in the case of sufficiently large N) under RG transformations, and depends neither on N, nor on the specific form of the basic equations of the system. This is the basis of a number of remarkable scaling properties, among which is the formula (1).

Thus, a chain of uncoupled systems is described by a set of mappings of universal form. Therefore, some scaling laws should also appear upon the introduction of coupling between the systems. To find these laws, it is sufficient to consider two coupled systems and analyze the RG-transformation properties of the terms describing the coupling.[1)]

Generally speaking, the correction, characterizing the coupling, to the Hamiltonian may depend in an aroitrary fashion on the coordinates and momenta of the coupled systems. It however follows from our results that any coupling can be represented as a combination of two fundamental types of interaction governing the critical effects at the threshold for the onset of stochasticity.

1. THE EQUATIONS OF MOTION AND POINT MAPPINGS

Let us consider a system of two oscillators with the Hamiltonian

$$H = \frac{1}{2}(p_1^2+p_2^2) + \frac{\omega^2}{2}(q_1^2+q_2^2) + \frac{\varepsilon}{2}(q_1-q_2)^2 + \sum_{n=-\infty}^{\infty}[F(q_1)+F(q_2)]\delta(t-n),$$

where q_1, q_2, p_1, and p_2 are the coordinates and momenta, ω is the eigenfrequency of the partial oscillator, ε is the coupling constant, and the function F characterizes the coordinate

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dependence of the intensity of the external influence. The period of the external influence is taken to be unity. Let us set

$$x=\frac{q_1+q_2}{2^{1/2}},\quad y=\frac{q_1-q_2}{2^{1/2}},\quad \xi=\frac{p_1+p_2}{2^{1/2}},\quad \eta=\frac{p_1-p_2}{2^{1/2}}$$

and write the equations of motion in the new variables:

$$\dot{x}=\xi,\ \dot{\xi}=-\omega^2 x-\frac{1}{2^{1/2}}\sum_n\left[F'\left(\frac{x+y}{2^{1/2}}\right)+F'\left(\frac{x-y}{2^{1/2}}\right)\right]\delta(t-n),$$
$$\dot{y}=\eta,\tag{2}$$
$$\dot{\eta}=-(\omega^2+2\varepsilon)y-\frac{1}{2^{1/2}}\sum_n\left[F'\left(\frac{x+y}{2^{1/2}}\right)-F'\left(\frac{x-y}{2^{1/2}}\right)\right]\delta(t-n).$$

We shall be interested only in the x, y, ξ, and η values assumed immediately after each n-th pulse, and shall label them by the index n. Solving the system (2), we can find the following recursion formulas:

$$x_{n+1}=cx_n+\frac{s}{\omega}\xi_n+\frac{s}{2^{1/2}\omega}\left[F'\left(\frac{x_n+y_n}{2^{1/2}}\right)+F'\left(\frac{x_n-y_n}{2^{1/2}}\right)\right],$$
$$\xi_{n+1}=-\omega s x_n+c\xi_n+\frac{c}{2^{1/2}}\left[F'\left(\frac{x_n+y_n}{2^{1/2}}\right)+F'\left(\frac{x_n-y_n}{2^{1/2}}\right)\right],$$
$$y_{n+1}=\bar{c}y_n+\frac{\bar{s}\eta_n}{(\omega^2+2\varepsilon)^{1/2}}\tag{3}$$
$$+\frac{\bar{s}}{(2\omega^2+4\varepsilon)^{1/2}}\left[F'\left(\frac{x_n+y_n}{2^{1/2}}\right)-F'\left(\frac{x_n-y_n}{2^{1/2}}\right)\right],$$
$$\eta_{n+1}=-(\omega^2+2\varepsilon)^{1/2}\bar{s}y_n+\bar{c}\eta_n$$
$$+\frac{\bar{c}}{2^{1/2}}\left[F'\left(\frac{x_n+y_n}{2^{1/2}}\right)-F'\left(\frac{x_n-y_n}{2^{1/2}}\right)\right],$$

where

$$s=\sin\omega,\ c=\cos\omega,\ \bar{s}=\sin(\omega^2+2\varepsilon)^{1/2},\ \bar{c}=\cos(\omega^2+2\varepsilon)^{1/2}.$$

Let us introduce the notation $u_n=cx_n-s\xi_n$, $v_n=\bar{c}y_n-\bar{s}\eta_n$, and concretize the form of the influence function F in such a way that

$$cq+\frac{sF'(q)}{2^{1/2}\omega}=\lambda-q^2,$$

where λ is a parameter. Then the system (3) becomes significantly simpler:

$$x_{n+1}=\lambda-x_n^2-y_n^2-u_n,\quad u_{n+1}=x_n,$$
$$y_{n+1}=-2x_ny_n-v_n+(\alpha+\beta x_n)y_n,\quad v_{n+1}=y_n,\tag{4}$$

where $\alpha=\bar{c}-\bar{s}c\omega/s(\omega^2+2\varepsilon)^{1/2}$ and $\beta=1-\bar{s}\omega/s(\omega^2+2\varepsilon)^{1/2}$. The mapping (4) can also be rewritten in the form

$$x_{n+1}+x_{n-1}=\lambda-x_n^2-y_n^2,\ y_{n+1}+y_{n-1}=-2x_ny_n+(\alpha+\beta x_n)y_n,\tag{5}$$

from which it is evident that the system is invariant under time reversal.

The parameters α and β vanish at $\varepsilon=0$, and should therefore be regarded as coupling constants. We shall call the expression in the brackets in the equation for y a coupling function, and denote if by $\varepsilon\varphi(x)$. In the example under consideration the coupling function is $\alpha+\beta x$, but it may have a more complicated form in the case of a different interaction Hamiltonian or a different $F(q)$ function.

2. THE RENORMALIZATION GROUP METHOD

Let us apply to the analysis of the mapping (4) an approach based on the approximate construction of the renormalization-group equations.[2)] The main idea consists in the following. Carrying out the original mapping twice, we express the quantities x, y, u, and v pertaining to the moment of time $n+2$ in terms of their values at the moment n. Then, we try by making a change of variables, to reduce the mapping obtained to the form (4), with, generally speaking, different λ, α, and β. In the process, if we are able to do it, we shall find the RG equations: expressions connecting the new and old values of the parameters. These equations are a powerful tool for the study of the dynamics of the system over long periods of time. Indeed, the new mapping is similar in form to the original one; therefore, the same procedure can again be applied to it. Carrying out this procedure N times, we reduce the problem of finding the state of the system after 2^N units of time to a single application of the mapping (4), the values of the parameters in which are determined with the aid of the RG equations.

The actual derivation of the RG equations is made easier by the use of the invariance property of the mapping under time reversal. Let us limit ourselves to the consideration of the class of solutions satisfying the condition $x_n=x_{-n}$, $y_n=y_{-n}$. Then, as can be seen from (5),

$$x_1=1/2(\lambda-x_0^2-y_0^2),\quad y_1=1/2(-2x_0+\alpha+\beta x_0)y_0,\tag{6}$$
$$x_2=\lambda-x_1^2-y_1^2-x_0,\quad y_2=(-2x_1+\alpha+\beta x_1)y_1-y_0.\tag{7}$$

Let us substitute (6) into (7), and try to reduce the resulting expressions for x_2 and y_2 to the form (6). Let us consider only the region of small x_0 and y_0, retaining in the expressions for x_2 and y_2 the terms of order in x_0 and y_0 not higher than two. Further, making the change of variables

$$x=\frac{1-X}{\lambda},\quad y=\frac{Y}{\lambda(1-\alpha^2/2\lambda)^{1/2}},\tag{8}$$

we finally have

$$X_2=1/2(\lambda'-X_0^2-Y_0^2),\quad Y_2=1/2(-2X_0+\alpha'+\beta'X_0)Y_0,$$

where

$$\lambda'=3-2\lambda^2+\lambda^3/2,\tag{9}$$
$$\alpha'=-(\lambda+2/\lambda)\alpha-2\beta+\alpha^2+\beta^2/2+\alpha\beta\lambda/2+\alpha\beta/\lambda,\tag{10}$$
$$\beta'=2\beta+2\alpha/\lambda-\beta^2/2-\alpha\beta/\lambda.\tag{11}$$

The relations (9)–(11) are sought RG equations.

3. THE STATIONARY POINT OF THE RG EQUATIONS. TWO TYPES OF COUPLING

The system of equations (9)–(11) possesses the stationary point

$$\lambda=\lambda_c=4.13264;\ \alpha=0;\ \beta=0.\tag{12}$$

Let $\lambda=\lambda_c+K\Lambda$, $\Lambda\ll 1$, $\alpha\ll 1$, $\beta\ll 1$. Then from (9)–(11) we

find in the linear approximation that

$$\begin{pmatrix}\Lambda\\ \alpha\\ \beta\end{pmatrix}=\hat{M}\begin{pmatrix}\Lambda\\ \alpha\\ \beta\end{pmatrix}=\begin{pmatrix}-4\lambda_c+{}^3/_2\lambda_c^2 & 0 & 0\\ 0 & -\lambda_c-2/\lambda_c & -2\\ 0 & 2/\lambda_c & 2\end{pmatrix}\begin{pmatrix}\Lambda\\ \alpha\\ \beta\end{pmatrix}$$
$$=\begin{pmatrix}-4\lambda_c+{}^3/_2\lambda_c^2 & 0 & 0\\ 0 & & \\ 0 & & \hat{C}\end{pmatrix}\begin{pmatrix}\Lambda\\ \alpha\\ \beta\end{pmatrix}. \qquad (13)$$

The matrix $\hat{M}$ naturally breaks up into blocks, the block $\hat{C}$ effecting the transformation of the coupling function represented by the parameter vector $\binom{\alpha}{\beta}$. The matrix $\hat{M}$ has three eigenvalues, each of which has a modulus greater than unity, specifically, the eigenvalues $\delta_H = 9.08768$, $\nu_1 = 4.6692$, and $\nu_2 = 1.85033$. The eigenvalues ν_1 and ν_2 pertain to the block $\hat{C}$. Thus, our analysis indicates the existence of three unstable directions in the parameter space near the stationary point (12).

If, in working out the RG procedure, we gave the coupling function in the form $\varphi(x)=\alpha+\beta x+\ldots+\gamma x^m$, then the matrix block $\hat{C}$ would have been an $m\times m$ matrix, and would have accordingly had m eigenvalues. The question how many of them have a modulus greater than unity is nontrivial. But the results of numerical calculations performed with different smooth coupling functions confirm the conclusion that there exist precisely three unstable directions (see the Appendix). The numerical calculations yield for the parameter λ_c the critical value $\lambda_c = 4.1361669$ and the following eigenvalues: $\delta_H = 8.7210972$ (Ref. 5), $\nu_1 = -4.4039$, and $\nu_2 = 1.8510$, which are in good agreement with our approximate results.

Let us write out the eigenvectors of the matrix $\hat{M}$, choosing the normalization for reasons of convenience (see the Appendix):

$$\mathbf{a}_0=\begin{pmatrix}1\\ 0\\ 0\end{pmatrix},\quad \mathbf{a}_{1,2}=\begin{pmatrix}0\\ \alpha_{1,2}\\ \beta_{1,2}\end{pmatrix}=\begin{pmatrix}0\\ \lambda_c(\nu_{1,2}-2)/d\\ 2/d\end{pmatrix}, \qquad (14)$$
$$d=(1+\lambda_c)^{1/2}-1-\lambda_c+\lambda_c\nu_{1,2}/2.$$

Here $\alpha_1 = 2.20923$; $\beta_1 = -0.16533$; $\alpha_2 = -0.64682$; $\beta_2 = 2.09147$.

Let us now represent the result of the transformation that the parameters Λ,α, and β undergo when the RG transformation is applied N times in the form of a linear combination of the eigenvectors of the matrix $\hat{M}$:

$$\begin{pmatrix}\Lambda\\ \alpha\\ \beta\end{pmatrix}_N=\Lambda\delta_H{}^N\mathbf{a}_0+c_1\nu_1{}^N\mathbf{a}_1+c_2\nu_2{}^N\mathbf{a}_2. \qquad (15)$$

The expansion coefficients are determined from the known parameters Λ,α, and β of the original (bare) mapping. Setting $N = 0$, we find from (15) that

$$c_1=0.46338\alpha+0.14330\beta,\quad c_2=0.03663\alpha+0.48946\beta. \qquad (16)$$

Thus, after an N-tuple RG transformation, in the case when N is sufficiently large, any coupling function can be represented in the form of a combination of two components, which we shall interpret as two types of coupling (interaction). The RG transformation corresponds to the multipication of one of these components by ν_1 and the multiplication of the second component by ν_2. Thus, the complete characterization of the bound systems in the large-N asymptotic region requires the specification of the two coefficients c_1 and c_2 determining the coupling and the quantity Λ, i.e., the specification of three parameters in all. The stationary point (12) should, in accordance with the terminology of the theory of phase transitions, be classified as a tricritical point.[13]

4. UNIVERSALITY AND SCALING RELATIONSHIPS IN THE NEIGHBORHOOD OF THE TRICRITICAL POINT

The analysis performed allows us to make the following assertions.

1) The structure of the (Λ, c_1, c_2)-parameter space in the vicinity of the tricritical point possesses the property of scaling invariance, i.e., goes over into itself when the scales along the coordinate axes are changed by factors of δ_H, ν_1, and ν_2 respectively. In other words, there occur at the point with coordinates $(\Lambda/\delta_H, c_1/\nu_1, c_2/\nu_2)$ regimes of the motion that are similar to those realized at the point (Λ, c_1, c_2), but with the time scale increased by a factor of two. The regimes are similar in respect of the character of the time dependence of the state (periodic, quasiperiodic or stochastic) and the character of the stability of the motion.

2) As it often happens when the RG method is used, the results obtained turn out to be applicable to a broad class of systems; for these results are determined not by the specific bare system, but by the properties of the RG procedure itself.[5,7-14] Evidently, our results are also applicable to a broad class of coupled Hamiltonian systems; we suggest that the only condition for this is the occurrence of an order-disorder transition via period doubling for each individual coupled system. Thus, the structure of the (Λ, c_1, c_2)-parameter space in the vicinity of the tricritical point should be universal. If to two chains of coupled systems corresponds one and the same point in this space, then the two chains will exhibit similar motions when the proper initial conditions are chosen for them. This allows us to propose the following procedure for analyzing coupled systems.

Let there exist a chain described by complicated differential equations. We should, after performing a limited amount of numerical computations, be able to find the parameters Λ, c_1, and c_2 for it. Then, the subsequent investigation of the possible regimes of motion can be carried out, using a fairly simple mapping of the type (5).

APPENDIX

Let us consider the problem of the dynamics of a weak perturbation that is in phase with the vibration regime of oscillators with $\lambda = \lambda_c$. As will become clear, this allows us to: 1) verify the existence of exactly three unstable directions in the vicinity of the stationary point of the RG equations that is being studied, 2) determine more accurately the values of the constants ν_1 and ν_2, and 3) give a general procedure for computing the coupling constants c_1 and c_2.

Let $x_n = x_{-n}, y_n = y_{-n}$, and $y_0 \ll 1$; as x_0 we specify the element of the 2^Nth cycle of the isolated system. Let us

expand the quantity y_{2^N} in a series in powers of the coupling constant. Assuming the coupling to be sufficiently weak, we limit ourselves to two terms of the expansion: $y_{2^N}(\varepsilon) \approx y_{2^N}(0) + \varepsilon y_0 \varkappa_2{}^N$. The values of

$$\varkappa_{2^N} = \frac{1}{y_0}\left[\frac{\partial y_{2^N}}{\partial \varepsilon}\right]_{\varepsilon=0}$$

can be found a) approximately by the RG method and b) through exact numerical calculations.

a) Let us write down the N-tuply renormalized equation (6):

$$X_{2^N} = {}^1/_2(\lambda_c - X_0{}^2), \quad Y_{2^N} = {}^1/_2[-2X_0 + \varepsilon(\alpha_N + \beta_N X_0)]\, Y_0.$$

Setting $X_{2^N} = X_0$, we find from the first equation that $X_0 = -1 + (1+\lambda_c)^{1/2}$. Differentiating the second equation with respect to ε, and using (14)–(16), we have

$$\varkappa_{2^N} = \frac{1}{2}(\alpha_N + X_0\beta_N) = \frac{1}{2}(1, X_0)\begin{pmatrix}\alpha_N \\ \beta_N\end{pmatrix} = c_1\nu_1{}^N + c_2\nu_2{}^N. \quad (A.1)$$

b) Let us turn to the original mapping (5). Let us differentiate the equation for y with respect to ε and write out the recursion formulas for the three variables x_n, $\zeta_n = [y_n/y_0]_{\varepsilon=0}$, and $\varkappa_n = y_0{}^{-1}[\partial y_n/\partial\varepsilon]_{\varepsilon=0}$, which formulas have the form

$$x_{n+1} = \lambda_c - x_n{}^2 - x_{n-1},\ \zeta_{n+1} = -2x_n\zeta_n - \zeta_{n-1},\ \varkappa_{n+1} = -2x_n\varkappa_n - \varkappa_{n-1} + \varphi(x_n)\zeta_n,$$

where $\varphi(x)$ is the coupling function. By taking as x_0 the element of the 2^Nth cycle, and assuming that $\zeta_0 = 1$, $\varkappa_0 = 0$, $\zeta_1 = \zeta_{-1}$, $\varkappa_1 = \varkappa_{-1}$, $x_1 = x_{-1}$, we can compute $\varkappa_{2^N}$.

If the conclusion that the stationary point (12) has a tricritical character is correct, then the formula (A.1) should give the results of the numerical computations performed with formally different coupling functions. We have computed $\varkappa_{2^N}$ with N ranging from 1 to 6 for the coupling functions $\varphi(x) = 1, x, \ldots, x^5$. In all the cases we obtained agreement with the formula (A.1), the best agreement being found in the case in which we used the constants $\nu_1 = -4.4039$ and $\nu_2 = 1.8510$. Thus, the possibility of representing the functions $\varphi(x)$ considered (and, hence, their linear combinations) in the form of superpositions of two types of coupling even in the case when $N \gtrsim 2$ is confirmed.

Having the set of quantities $\varkappa_{2^N}$, the constants ν_1 and ν_2, and the formula (A.1), we can easily compute coefficients c_1 and c_2 for any coupling function. In particular, we obtain good agreement with the formula (16) in the $\varphi(x) = 1$ and $\varphi(x) = x$ cases. This method of computing the coupling constants can be generalized and used in those cases when the explicit form of the point transformation is unknow, and we have to proceed from differential equations.

[1)]A similar investigation is carried out in Ref. 15 for a chain of dissipative systems.
[2)]A similar method has previously been used to analyze an isolated system described by the mapping $x_{n+1} + x_{n-1} = f(x_n)$ (Ref. 5).

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