DYNAMICS OF FEIGENBAUM SYSTEMS WITH UNIDIRECTIONAL COUPLING NEAR ONSET OF CHAOS. THE BICRITICAL ATTRACTOR

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It is proposed that a bicritical point exists in the plane of the control parameters of two logistic maps with unidirectional coupling where the lines of transition to chaos in both subsystems converge. The global scaling properties (σ-functions, $f(\alpha)$ spectra, and generalized dimensions) of bicritical dynamics are examined. It is shown that bicriticality can also be observed in chains of more than two cells.

1. Introduction. After the most-fundamental concepts of the chaotic dynamics of small nonlinear systems were established, a promising approach to the analysis of more-complex, distributed systems was outlined, which consists of the examination of lattices of coupled maps constructed from cells with known dynamic properties [1-7]. In most cases, these are lattices of maps of the Feigenbaum type, which demonstrate the transition to chaos through period-doubling bifurcations. A number of interesting results have been obtained – for example, in the case of lattices with unidirectional coupling, which have been proposed for the modeling of turbulence in flow systems [6, 7]. These are spatial period doublings, the saturation of attractor dimensionality down stream, spatio-temporal chaos, the generation of moving domain walls in the presence of a small random perturbation, etc. Kaneko [1] has discovered a new, non-Feigenbaum type of critical behavior that can be realized in as few as two unidirectionally coupled systems that demonstrate period doublings. A suitable example is the model map

$$x_{n+1} = 1 - \lambda x_n^2, \qquad y_{n+1} = 1 - Ay_n^2 - Bx_n^2, \tag{1}$$

where x and y are the dynamic variables of the two coupled systems, λ and A are the control parameters, and B is the coupling constant. The indicated critical behavior is realized if the first and second systems are brought precisely to the onset of chaos simultaneously by means of λ and A, respectively. (For system (1), $\lambda_c = 1.40115519$ and $A_c = 1.12498140$ when $B = 0.375$.) The corresponding point on the plane λ, A has been called [11] the bicritical point, because of the natural analogy with phase transitions. This terminology usually denotes the point at which two different lines of phase transitions of the second kind converge.

Bicritical behavior obviously corresponds to the onset of dynamic conditions with two positive Lyapunov exponents – so-called hyperchaos [12]. A renormalization-group analysis of a bicritical situation was developed and the corresponding properties of universality and scaling were revealed earlier [13]. We shall examine the geometrical structure of the bicritical attractor, introduce σ-functions that describe its global scaling properties, examine the $f(\alpha)$ spectra and generalized dimensions, and show the possibility of realization of bicriticality in more-complex lattice systems.

2. Geometry of Bicritical Attractor. As we shall now see, an attractor at a bicritical point is a very interesting example of a multifractal set embedded in a two-dimensional phase space (x, y). In order to understand its geometrical nature, we recall first of all the well-known for construction of the Feigenbaum attactor [10, 15]. For this, we iterate the map $x_{n+1} = 1 - \lambda_c x_n^2, x_0 = 0$, obtaining a sequence $x_1, x_2, x_3, \ldots$. Then, the zeroth level of construction is a segment $[x_1, x_2]$, the first level is a union of two segments $[x_1, x_3]$ and $[x_2, x_4]$, the second is a union of four segments $[x_1, x_5]$, $[x_2, x_6]$, $[x_3, x_7]$, and $[x_4, x_8]$, etc. Each set obtained in a construction step contains all subsequent sets. The object that appears at the limit is the Feigenbaum attractor.

In order to construct the bicritical attractor, we take the initial point $x_0 = 0, y_0 = 0$ and find by iterations (1) a sequence of pairs (x_1, y_1), (x_2, y_2), (x_3, y_3), ... The zeroth level of construction not corresponds to a rectangle with opposite vertices at points (x_1, y_1) and (x_2, y_2). We denote it as

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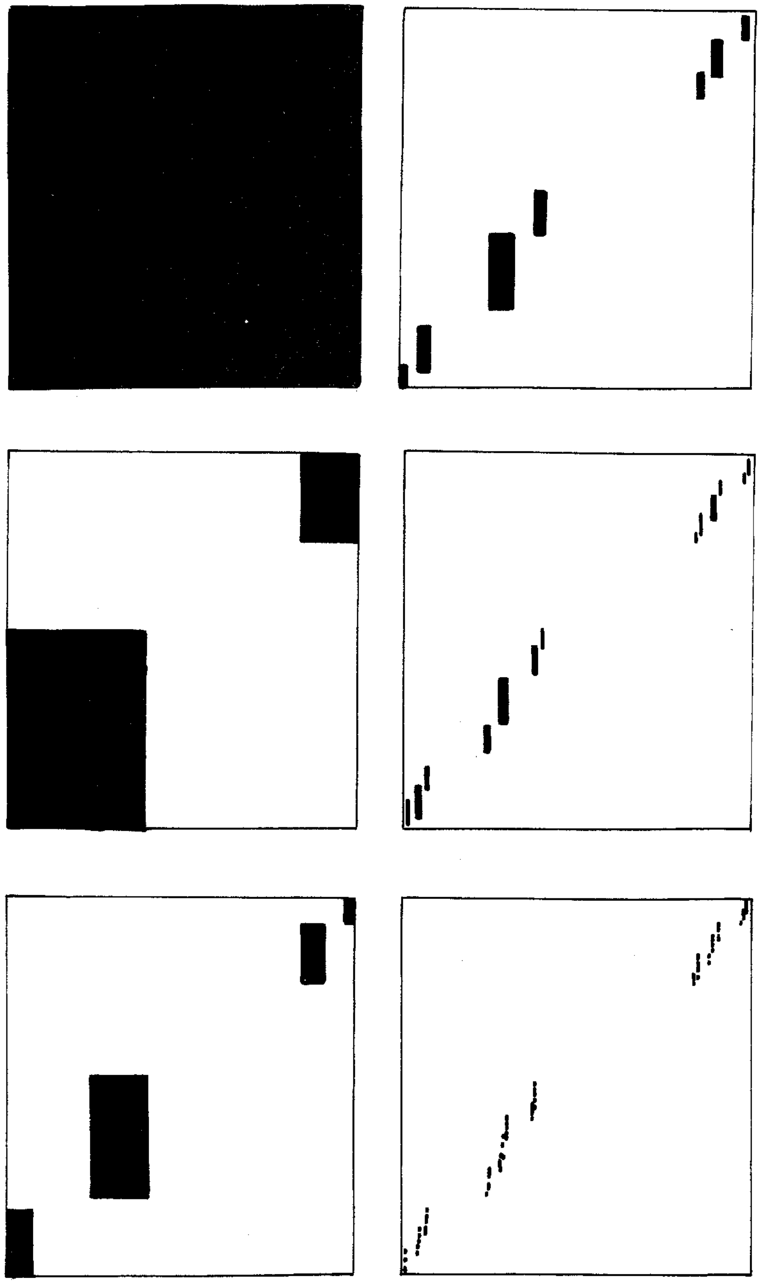

Fig. 1

$$\Pi_0 = [(x_1, y_1), (x_2, y_2)]. \tag{2}$$

The first level will be represented by two rectangles with, respectively, opposite vertices (x_1, y_1), (x_3, y_3) and (x_2, y_2), (x_4, y_4):

$$\Pi_1 = [(x_1, y_1), (x_3, y_3)] \cup [(x_2, y_2), (x_4, y_4)], \tag{3}$$

We shall continue the construction, as shown in Fig. 1. At the n-th level, we have

$$\Pi_n = \bigcup_{i=1}^{2^n} [(x_i, y_i), (x_{i+2^n}, y_{i+2^n})]. \tag{4}$$

The bicritical attractor is obtained at the limit $n \to \infty$.

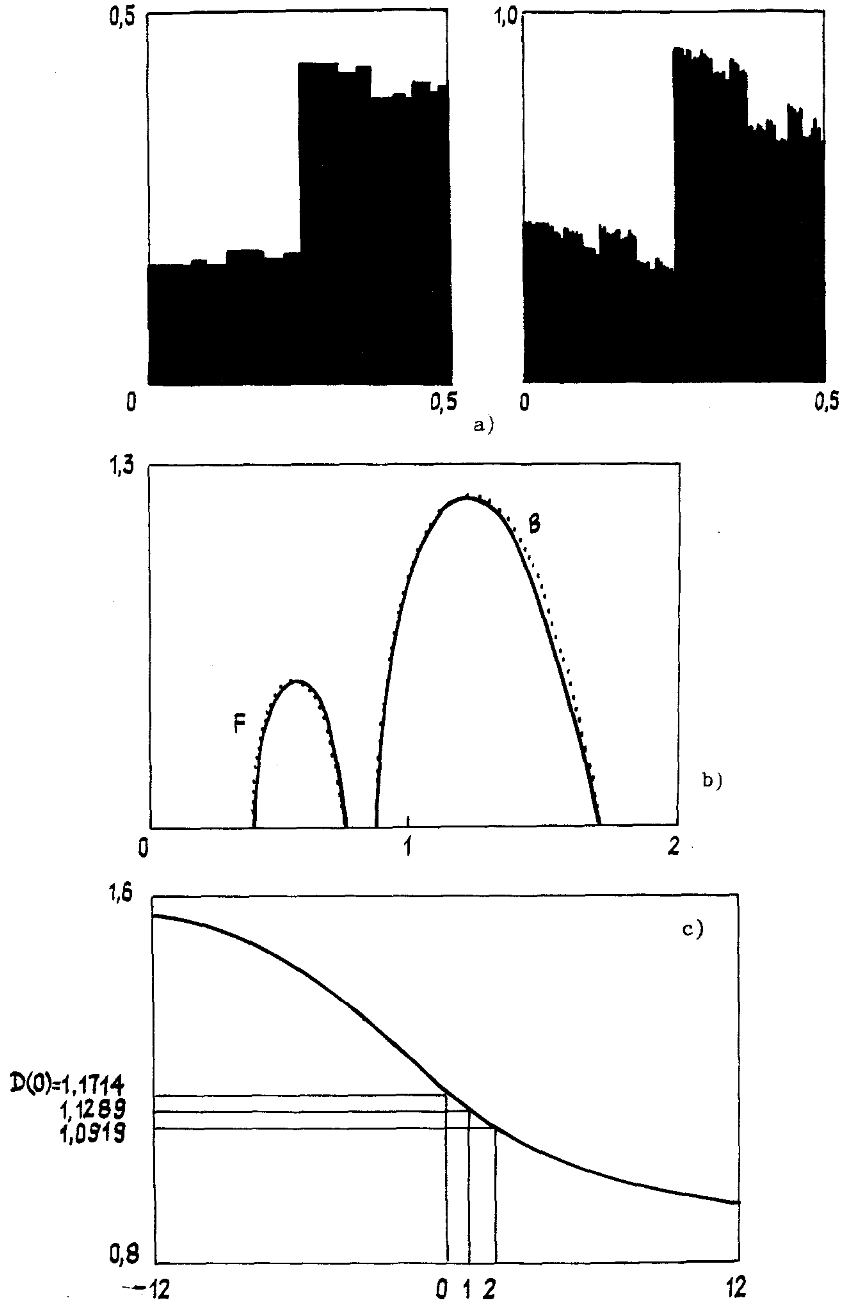

Fig. 2

Let us evaluate the Hausdorff dimension of this set. At the n-th level, it is approximately 2^n rectangles of size $l_i \times L_i$. For large n, these rectangles extend along the y axis so that $L_i >> l_i$ (see Fig. 1 and subsequent discussion of the σ_x and σ_y functions). The number of small squares of size l_i required to cover the i-th rectangle is estimated as L_i/l_i. Following the well-known definition of the Hausdorff dimension D [14], we shall examine the sum l_i^D over all small squares and rectangles

$$S_n = \sum_{i=1}^{2^n} (L_i/l_i)/l_i^D \tag{5}$$

and select a D value that ensures the finiteness of S_n when $n \to \infty$. Numerical calculation gives $D = 1.0794$.

3. Characteristics of Bicritical Attractor. For a global description of the scaling properties of the bicritical attractor, we introduce by analogy with Feigenbaum [8, 9] the functions σ_x and σ_y:

$$\sigma_x(m/2N) = \frac{x'_m - x'_{m+N}}{x_m - x_{m+N/2}}, \qquad \sigma_y(m/2N) = \frac{y'_m - y'_{m+N}}{y_m - y_{m+N/2}}, \tag{6}$$

where m and $N = 2^n$ approach infinity, and x_m, y_m and x'_m, y'_m are cycle elements of periods N and $2N$ for map (1) at the bicritical point. These functions, which were found numerically, are shown in Fig. 2a (σ_x on the left and σ_y on the right). Owing to the independence of the dynamics of the first subsystem, the function σ_x coincides with the well-known Feigenbaum function. On the other hand, σ_y is a new universal function, which describes the scaling properties of the y-projection of the bicritical attractor. The fact that $|\sigma_x| > |\sigma_y|$ indicates that trajectory splitting in the y-direction is reduced more slowly than in the x-direction when the resolution of the attractor structure is increased. The function σ_y, just as σ_x, has discontinuities at all binary rational points $m/2^n$. Note that the nonuniformity of the fine structure for σ_y is expressed more strongly than for σ_x.

It is known that the following relations are valid for σ_x:

$$\sigma_x(+0) \to 1/a^2, \qquad \sigma_x(1/2-0) \to 1/|a|, \qquad a = -2,50291. \tag{7}$$

Similarly, for σ_y we find that

$$\sigma_y(+0) \to 1/b^2, \qquad \sigma_y(1/2-0) \to 1/|b|, \qquad b = -1,50532. \tag{8}$$

Roughly, σ_x and σ_y can be represented by two level stems (7) and (8). This corresponds to approximation of the bicritical attractor by a multifractal that gives, when projected onto the coordinate axes, two-scale Cantor sets [15] with parameters $1/a$, $1/a^2$ and $1/b$, $1/b^2$. For the Feigenbaum attractor, such an approximation provides, in particular, useful approximation formulas for the power spectrum, dimensions, $f(\alpha)$ spectrum, etc., which are difficult to generalize as applied to the y-projection of the attractor.

Now let us evaluate the spectrum of scale indices $f(\alpha)$. We prefer to see this spectrum as an attribute of a signal generated by a "black box" than as a characteristic of the bicritical attractor. Having two signals, x and y, we shall limit ourselves to their independent processing. Using the procedure of Halsey et al. [15], we shall determine the partition function at the n-th resolution level

$$\Gamma_n = \sum_{i=1}^{2^n} p_i^q / l_i^\tau, \tag{9}$$

where $p_i = 1/2^n$, $l_i = |x_i - x_{i+2^n}|$ or $|y_i - y_{i+2^n}|$, x_i, y_i are sequences generated by map (1) at the bicritical point with zero initial conditions. Further, we shall require that Γ_n remain finite for $n \to \infty$ and let

$$q = \frac{1}{n} \log_2 \Big| \sum_{i=1}^{2^n} l_i^{-\tau} \Big|. \tag{10}$$

Following Halsey et al. [15], we obtain the $f(\alpha)$ spectrum in parametric form

$$\alpha = d\tau/dq, \qquad f = \alpha q - \tau. \tag{11}$$

In a two-scale approximation, the sum Γ_{n+1} can be obtained from Γ_n by replacing each term p_i^q/l_i^τ by two:

$$\frac{|a|^\tau}{2^q}\frac{p^q}{l^\tau} + \frac{|a|^{2\tau}}{2^q}\frac{p^q}{l^\tau} \ \text{for } x \quad \text{and} \quad \frac{|b|^\tau}{2^q}\frac{p^q}{l^\tau} + \frac{|b|^{2\tau}}{2^q}\frac{p^q}{l^\tau} \ \text{for } y.$$

For example, we have $\Gamma_{n+1} = (|a|^\tau + |a|^{2\tau})2^{-q}\Gamma_n$ for x and $\Gamma_{n+1} = (|b|^\tau + |b|^{2\tau})2^{-q}\Gamma_n$ for y. Hence, we find, respectively, $q = \log_2(|a|^\tau + |a|^{2\tau})$ and $q = \log_2(|b|^\tau + |b|^{2\tau})$ and obtain α and f from (11) analytically.

In Fig. 2b are $f(\alpha)$ spectra generated by both variables x and y. The dotted curves correspond to the two-scale approximation; the solid curves represent exact numerical calculation. For the first variable x we obtain, finally, the traditional Feigenbaum form of $f(\alpha)$ [15], while the second spectrum is entirely different. It is located in the interval of α of from $1/\log_2 b^2 = 0.843736$ to $1/\log_2|b| = 1.69472$ and has extreme $f_{max} = D_0^{(y)} \approx 1.1714$. It could have been assumed that this value would give the Hausdorff dimension of the y-projection of the bicritical attractor. This is not so, however, due to overlapping of the y-projections of different elements of the set construction (see Fig. 1). This is expressed in the fact that $D_0^{(y)} > 1$. In spite of this,

$D_0^{(y)}$ can be considered a dimension that reflects the scaling properties of the variable y. With the same stipulations, we can introduce an entire family of generalized dimensions $D_q^{(y)} = \tau/(1 - q)$ [14]. Their graph is shown in Fig. 2c, in which some important points are indicated by corresponding numerical values.

4. Lattice Systems. Bicriticality is a prevalent type of universal behavior of this class of systems, which is typically realized in the presence of not fewer than two control parameters. It can be encountered not only in systems of two coupled cells but also in chains of a larger number of cells. For example, in a system of three cells

$$x_{n+1} = 1 - \lambda x_n^2, \quad y_{n+1} = 1 - Ay_n^2 - Bx_n^2, \quad z_{n+1} = 1 - Cz_n^2 - Dy_n^2 \tag{12}$$

we can observe the following bicritical situations.

1) Let $\lambda < \lambda_c$, which ensures a stable cycle in the first system. Then we select an A such that a Feigenbaum critical situation is realized in the second system and a C such that bicriticality is obtained in the third system. This situation can be described by means of the diagram

$$P \longrightarrow F \longrightarrow B,$$

where P represents periodic motion, F Feigenbaum critical dynamics, and B bicritical dynamics. Calculations have shown that such a lattice state can be realized, for example, with the following parameter values: $\lambda = 1$, $A = 1.272008$, $B = 0.25$, $C = 1.128102$, and $D = 0.375$.

2) Let $\lambda = \lambda_c$ and $A = A_c$ for a given B and small C and D. Then, the second system will be in a bicritical state while the third accomplishes forced (synchronized) motion with the scaling properties that are inherent in bicriticality:

$$F \longrightarrow B \longrightarrow B.$$

This situation is observed, for example, when $\lambda = 1.401155$, $A = 1.124981$, $B = 0.375$, $C = 0.6$, and $D = 0.375$.

3) For $\lambda = \lambda_c$ and small A and B, we have a Feigenbaum bicritical state in the first system and synchronized motion of the second system with the same scaling properties. With suitable selection of C for a given D, bicriticality can be obtained in the third system:

$$F \longrightarrow F \longrightarrow B.$$

This situation is observed for $\lambda = 1.401155$, $A = 0.8$, $B = 0.375$, $C = 1.179791$, and $D = 0.375$.

Similar considerations are obviously applicable to lattices that consist of a larger number of cells. For example, in a lattice of four cells, the states characterized by the following diagrams can be realized:

$$\begin{array}{ll} P \longrightarrow P \longrightarrow F \longrightarrow B, & F \longrightarrow F \longrightarrow F \longrightarrow B \\ P \longrightarrow F \longrightarrow F \longrightarrow B, & F \longrightarrow F \longrightarrow B \longrightarrow B \\ P \longrightarrow F \longrightarrow B \longrightarrow B, & F \longrightarrow B \longrightarrow B \longrightarrow B \end{array}$$

It is clear that the degree of diversity of the possible situations increases with an increase in the number of cells in the lattice. They are all, however, easily described by similar diagrams.

The above examination shows that the joining into a lattice of even the simplest systems that demonstrate the transition to dynamic chaos leads to new versions of critical mechanisms. The gradual "connection" of cells to a lattice in a situation of unidirectional coupling is a possible method for study of these critical states.

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