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Scaling in the Presence of an External Noise in Discrete Systems with Period-Doublings

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Fundamental scaling properties in the absence and in the presence of noise for different maps which exhibit transition to chaos through period-doubling cascade are considered. Numerical method of determination of noise constants for one-dimensional and two-dimensional maps has been presented which allows to illustrate scaling properties under random noise influence.

Key words: chaos, scaling, noise, period-doubling cascade

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1 Logistic map

In this work the influence of an external noise and properties of universality and scaling are considered in the presence of random fluctuations for systems, which exhibit transition to chaos through the cascade of period-doubling bifurcations. Computer illustrations are obtained that show smearing of the bifurcation tree structure and of the Lyapunov exponent plot for logistic map in the presence of the random noise (fig.1a,2a). Let us consider the equation

$$X_{n+1} = 1 - \lambda X_n^2 + \varepsilon\xi_n, \tag{1}$$

where ε is a noise amplitude and ξ_n represents discrete-time white noise.

As shown by Cruthfield et al. [1] and Schraiman et al. [2] by renormalization of the noise intensity ε by a constant universal factor $\mu_F = 6.61903$

$$\varepsilon \to \varepsilon/\mu_F \tag{2}$$

the property of self-similarity is reproduced on successive levels of the structure (fig.1b,2b). The renormalization constant μ_F is valid only for systems that belong to the class of Feigenbaum universality (this class includes 1D maps with quadratic extremum).

2 Unimodal maps. 1D method for determination of noise constants

The noise constants for unimodal maps with higher order of extremum

$$X_{n+1} = 1 - \lambda|X_n|^p + \varepsilon\xi_n, p > 1 \tag{3}$$

appear to be different from μ_F. In this report we suggest new universal, sufficiently accurate method for determination of the noise rescaling constants. It is worth noting that the method allows generalization onto a multidimensional case. Let us consider one-dimensional map of a general form with the additive noise

$$X_{n+1} = f(X_n) + \varepsilon\xi_n. \tag{4}$$

Differentiation of this equation with respect to ε yields

$$(X_{n+1})'_\varepsilon = f'_X(X_n)(X_n)'_\varepsilon + \xi_n. \tag{5}$$

Then we square both parts of the equation (5)

$$\begin{aligned}(X'_{n+1})^2_\varepsilon = f'^2_X(X_n)(X_n)'^2_\varepsilon \\ +2f'_X(X_n)(X_n)'_\varepsilon\xi_n + \xi_n^2\end{aligned} \tag{6}$$

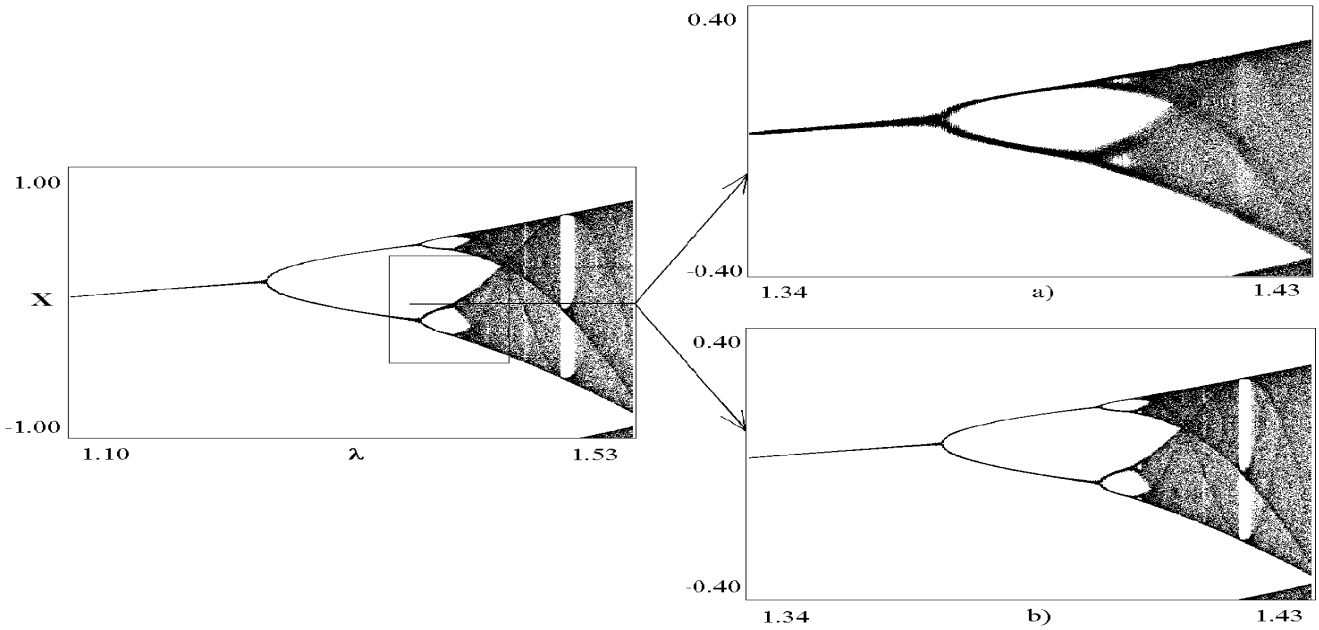

FIG. 1. The bifurcation tree. The influence of the external noise in the case of the logistic map. The noise amplitude $\varepsilon = 0.0005$. The noise amplitude is fixed (a) and decreased by factor μ_F (b).

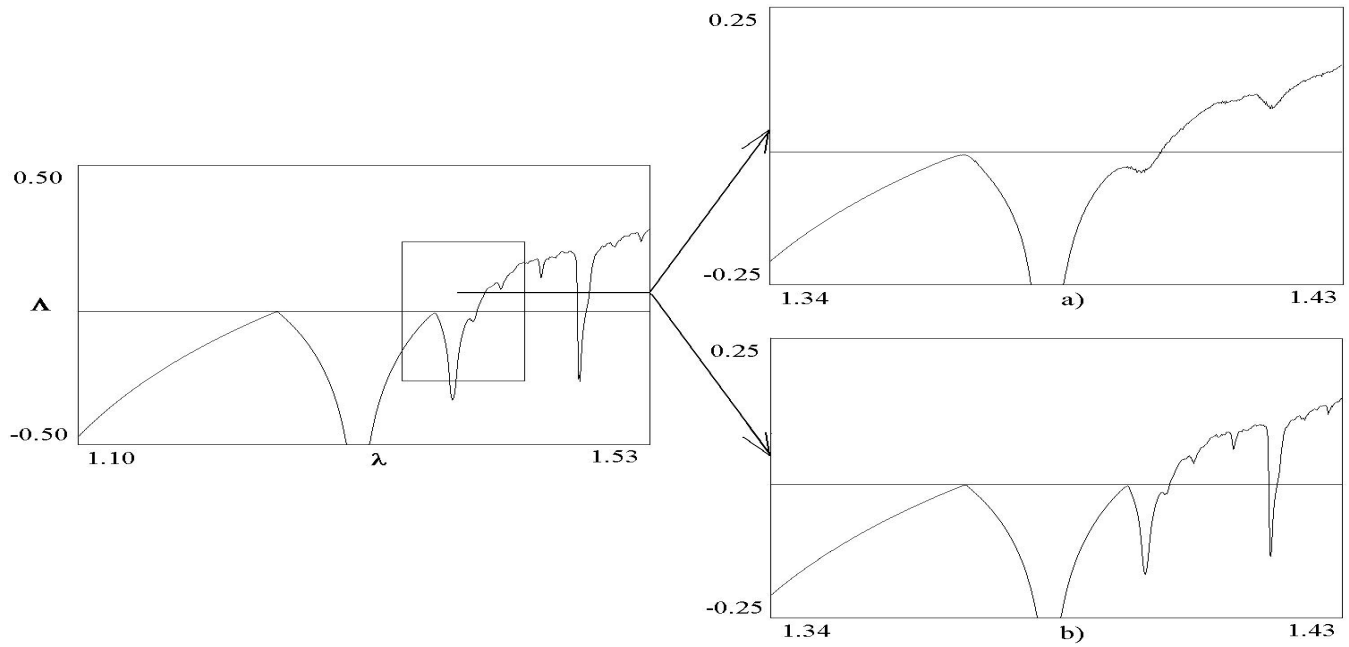

FIG. 2. The Lyapunov exponent plots, illustrating the influence of the external noise in the case of the logistic map. The noise amplitude $\varepsilon = 0.0005$. The noise amplitude is fixed (a) and decreased by factor μ_F (b).

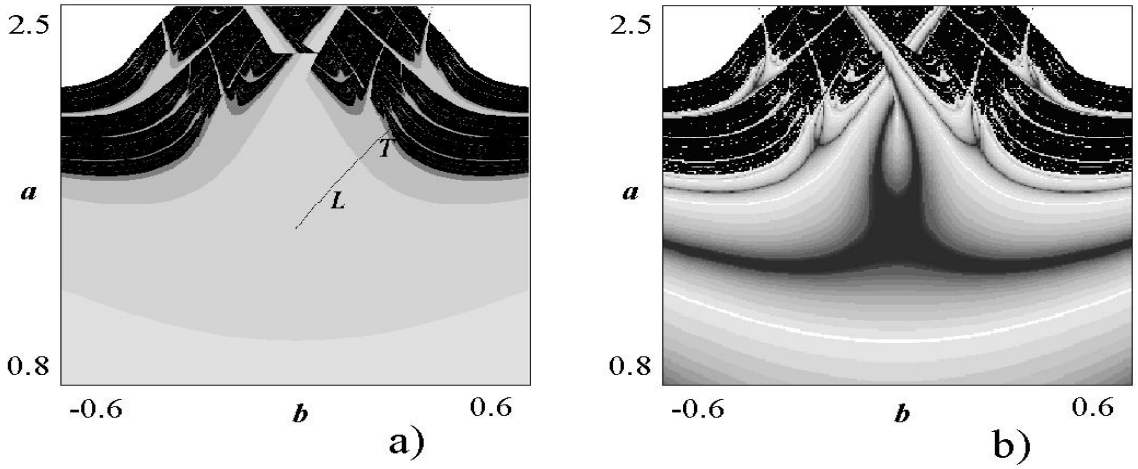

FIG. 3. The chart of dynamical regimes (a) and the Lyapunov chart (b) for the cubic map (10). L designates the curve (11), T is the tricritical point.

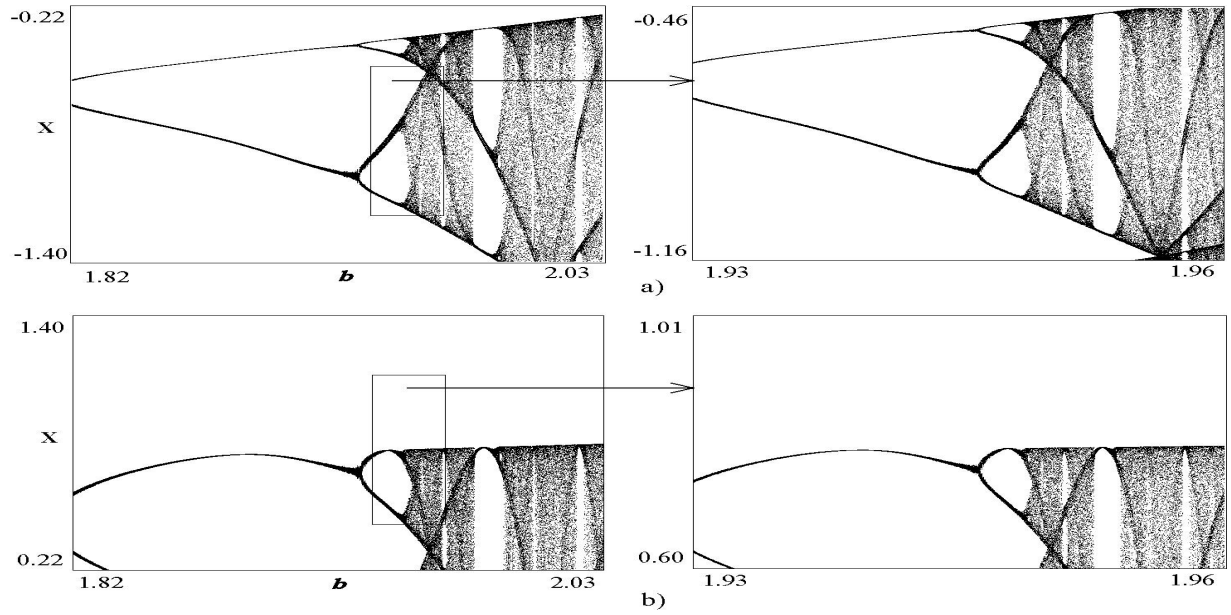

FIG. 4. Scaling on the bifurcation tree for the cubic map in the presence of the noise with the amplitude 0.001 near the maximum (a) and near the minimum(b).

and average the result over the ensemble of the random force realization:

$$< (X_{n+1})'^2_\varepsilon >= f'^2_X(X_n) < (X_n)'^2_\varepsilon > + < \xi_n^2 > . \quad (7)$$

Besides these equations we need another one

$$(X_{n+1})'_\lambda = f'_X(X_n)(X_n)'_\lambda. \quad (8)$$

In accordance with the logic of the RG analysis, the following relations will be valid asymptotically at

$N \to \infty$:

$$\begin{array}{l} \frac{X_N}{X_{2N}} \to \alpha, \\ \frac{(X_N)_\lambda}{(X_{2N})_\lambda} \to \frac{\delta}{\alpha}, \\ \frac{\langle (X_{2N})_\varepsilon^2 \rangle}{\langle (X_N)_\varepsilon^2 \rangle} \to \frac{\mu^2}{\alpha^2}, \end{array} \qquad (9)$$

where X_N and X_{2N} are the elements of the period-N and period-$2N$ cycles, respectively, α and δ are the well-known scaling factors. The depending on the exponent p in (3), the computer renormalization constants of noise for the unimodal maps (3) with an even order of the extremum are the following: $\mu_F = 6.61903648$ for $p = 2$, $\mu_T = 8.2439$ for $p = 4$, $\mu_S = 10.0378$ for $p = 6$ and $\mu_E = 11.5937$ for $p = 8$.

3 1D maps with three parameters

These results may be used to obtain illustrations of scaling in the cases of the higher codimension period-doubling onset of chaos (the Feigenbaum scenario has codimension equals to 1). Let us turn to a simple example, the system with two control parameters represented by the so-called bimodal cubic map

$$X_{n+1} = a - bX_n + X_n^3. \qquad (10)$$

For $b > 0$ the map (10) has two quadratic extrema. Fig.3a presents a chart of dynamical regimes, every tone of the grey colour on the parameter plane corresponds to the definite regime.

As we see, moving along "typical" routes one crosses the period-doubling bifurcation curves, accumulating to the Feigenbaum critical line. In the parameter plane a curve exists where the condition is fulfilled that one extremum is mapped precisely to another one

$$a = (b/3)^{1/2}(1 - 2b/3). \qquad (11)$$

Obviously, if we stay at this curve, the iterated map will have an extremum of the fourth order. If the period-doubling cascade occurs along the curve (11), the limit will be the tricritical point [3]: $a_T = 0.24269876, b_T = 1.95138576$. The scaling properties near this point may be studied to some extent with the help of the unimodal map (3) with $p = 4$. Therefore, scaling in the presence of noise may be illustrated by using the calculated constant μ_T (noise is introduced in the map by adding the term $\varepsilon\xi_n$, as usual). For cubic map the property of self-similarity for the bifurcation tree is demonstrated in fig.4 near both extrema in the the presence of the fluctuations. Analogous illustrations have been obtained for the Lyapunov exponent plot.

4 1D maps with four parameters

Now let us investigate the model three-parameter quartic map [4]

$$X_{n+1} = 1 - aX_n^2 - bX_n^4 - cX_n. \qquad (12)$$

As shown in Ref.[4] there are four typical situations in the parameter plane that deserve special consideration. Such situations are realized at the line t, where extremum of the fourth power exists; at the line s_1, where the condition of mapping of the extremum to the cubic inflection point is satisfied; at the line s_2, where the inflection point is mapped to the extremum and at the line e, where one quadratic extremum is mapped to another and the second is mapped to the third one (fig.5).

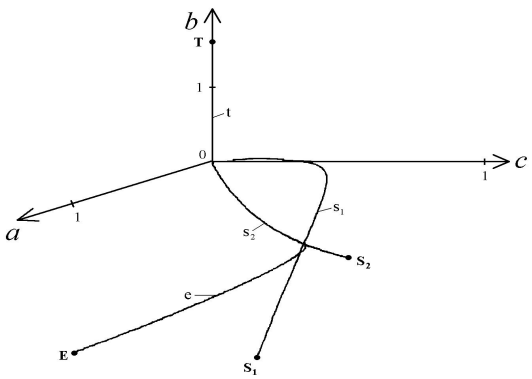

FIG. 5. Configuration of curves t, s_1, s_2, e and the respective critical points T, S_1, S_2, E in the parameter space of the map (12).

If a period-doubling cascade occurs along one of these curves, the map (12) will behave like the unimodal map of the power $p = 4$, 6 or 8. Graphic

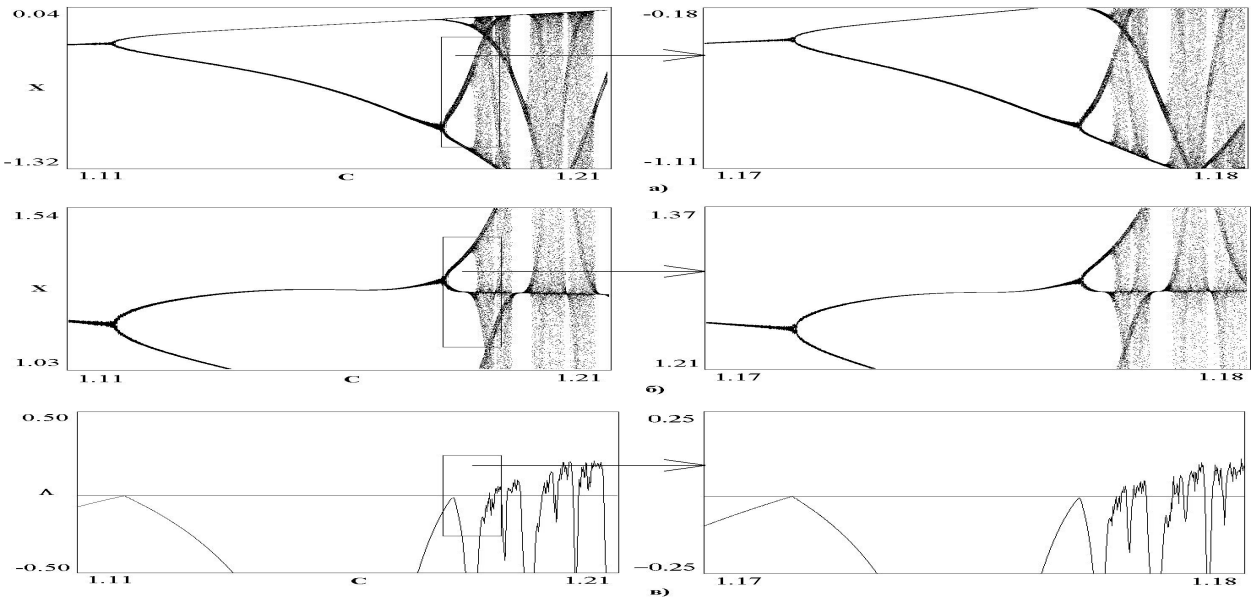

FIG. 6. Scaling for the three-parameter map (12) in presence of noise with the amplitude 0.001 along the critical line s_2: the bifurcation tree near the inflection point (a), the bifurcation tree near the extremum (b) and the Lyapunov exponent plot (c).

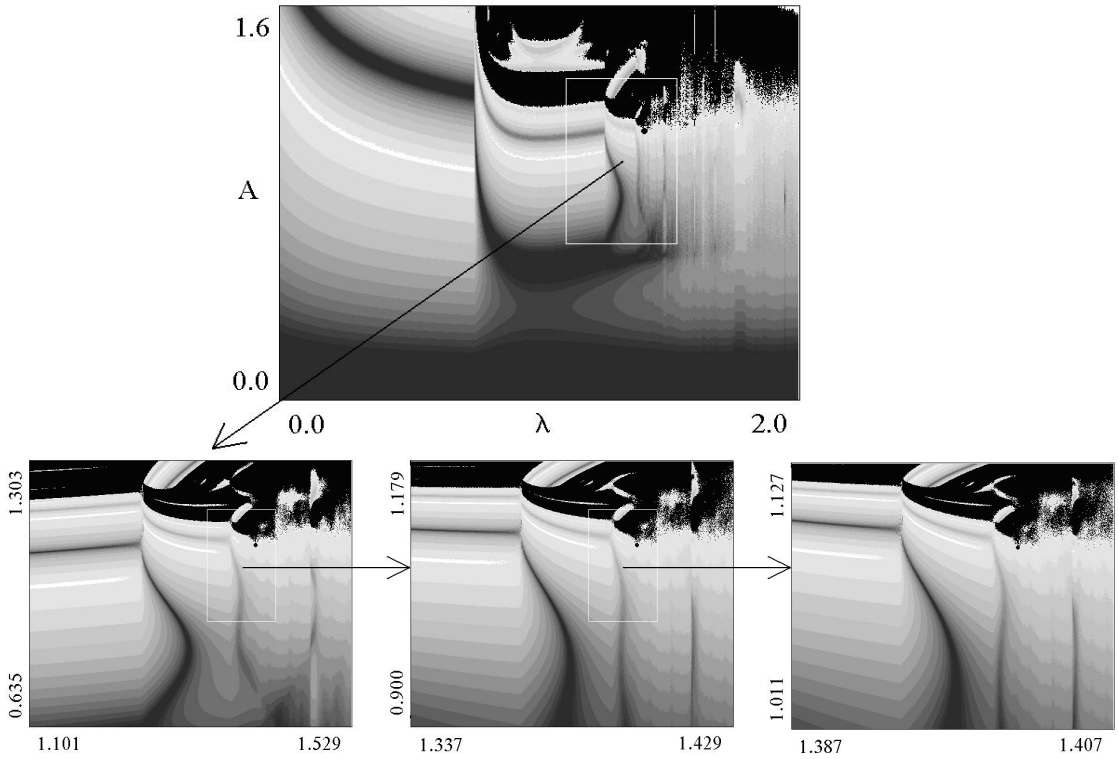

FIG. 7. Scaling for the Lyapunov chart in the case of bicritical dynamics in presence of the external noise at $\gamma = 0$ and $\varepsilon = 0.005$. $B = 0.45$.

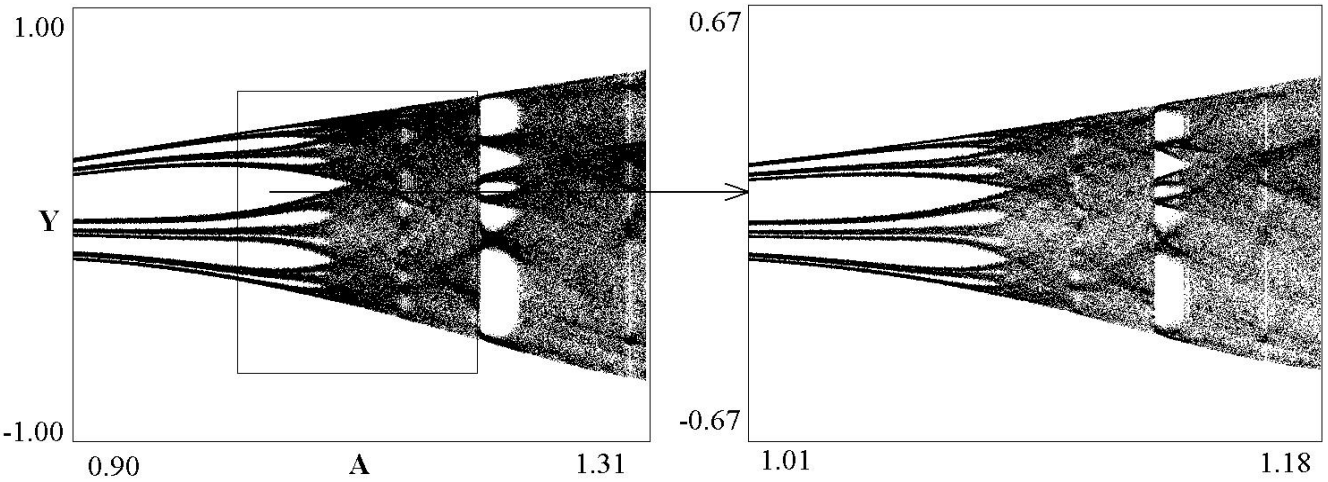

FIG. 8. Scaling for the bifurcation tree of the map (18) at $B = 0.45$, $\lambda = 1.4011552$. Noise amplitude is $\gamma = 0$ and $\varepsilon = 0.0015$.

illustrations of scaling properties are given for each case along the respective lines. The rescaling rule accounts the noise constants μ_T, μ_S or μ_E, respectively (for example, in fig.6 the scaling is illustrated for the line s_2).

5 Bicritical dynamics. 2D method for determination of noise constants

As mentioned, the suggested method for determination of the noise rescaling constants admits an expansion. In particular, we present its generation for some class of 2D maps. This is a set of two unidirectionally coupled 1D maps in presence of noise, added into the second equation:

$$\begin{aligned} X_{n+1} &= f(X_n, Y_n), \\ Y_{n+1} &= g(X_n, Y_n) + \varepsilon\xi_n. \end{aligned} \tag{13}$$

After taking the derivatives with respect to ε we have

$$\begin{aligned} (X_{n+1})'_\varepsilon &= f'_X(X_n, Y_n)(X_n)'_\varepsilon \\ &+ f'_Y(X_n, Y_n)(Y_n)'_\varepsilon, \\ (Y_{n+1})'_\varepsilon &= g'_X(X_n, Y_n)(X_n)'_\varepsilon \\ &+ g'_Y(X_n, Y_n)(Y_n)'_\varepsilon + \xi_n \end{aligned} \tag{14}$$

and then the averaging yields

$$\begin{aligned} \langle(X_{n+1})'_\varepsilon\rangle &= f'_X(X_n, Y_n)\langle(X_n)'_\varepsilon\rangle \\ &+ f'_Y(X_n, Y_n)\langle(Y_n)'_\varepsilon\rangle, \\ \langle(Y_{n+1})'_\varepsilon\rangle &= g'_X(X_n, Y_n)\langle(X_n)'_\varepsilon\rangle \\ &+ g'_Y(X_n, Y_n)\langle(Y_n)'_\varepsilon\rangle + \langle\xi_n\rangle. \end{aligned} \tag{15}$$

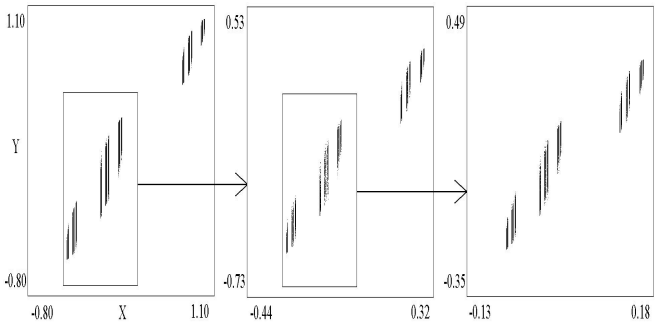

FIG. 9. Scaling for the bicritical attractor (18). $B = 0.45$, $\lambda = 1.4011552$, $A = 1.0900943$. Noise amplitude is $\gamma = 0$ and $\varepsilon = 0.01$.

Next, we take the square of the both parts of the

equation (14) and average the result

$$\begin{gathered}\langle(X_{n+1})'^2_\varepsilon\rangle = f'^2_X(X_n,Y_n)\langle(X_n)'^2_\varepsilon\rangle \\ +2f'_X(X_n,Y_n)f'_Y(X_n,Y_n)\langle(X_n)'_\varepsilon\rangle\langle(Y_n)'_\varepsilon\rangle \\ +f'^2_Y(X_n,Y_n)\langle(Y_n)'^2_\varepsilon\rangle, \\ \langle(Y_{n+1})'^2_\varepsilon\rangle = g'^2_X(X_n,Y_n)\langle(X_n)'^2_\varepsilon\rangle \\ +2g'_X(X_n,Y_n)g'_Y(X_n,Y_n)\langle(X_n)'_\varepsilon\rangle\langle(Y_n)'_\varepsilon\rangle \\ +g'^2_Y(X_n,Y_n)\langle(Y_n)'^2_\varepsilon\rangle + \langle\xi_n^2\rangle.\end{gathered} \tag{16}$$

Now we can undertake iterations the set of equations (13), (15) and (16) at critical point, and the noise constant μ may be found as the ratio

$$\frac{\langle(Y_{2N})^2_\varepsilon\rangle}{\langle(Y_N)^2_\varepsilon\rangle} \to \frac{\mu^2}{\alpha^2}, \tag{17}$$

where α is the scaling factor for variable Y and $N \to \infty$.

A particular example of the map (13) is the system of two unidirectionally coupled logistic maps in the presence of the external noise:

$$\begin{gathered}X_{n+1} = 1 - \lambda X_n^2 + \gamma\eta_n, \\ Y_{n+1} = 1 - AY_n^2 - BX_n^2 + \varepsilon\xi_n.\end{gathered} \tag{18}$$

Here γ and ε designate the intensities of noise, η_n and ξ_n are random sequences. Without noise the system demonstrates so-called bicritical behaviour [5],[6] which corresponds to the threshold to chaos in both subsystems simultaneously. Above mentioned extended method allows computing the noise rescaling constants for this 2D system: $\mu_F = 6.61903$ for the intensity γ and $\mu_B = 2.7136$ for the intensity ε. With the use of these constants we obtained the computer illustrations of scaling for the chart of Lyapunov exponent (fiq.7), for the bifurcation tree (fig.8), and for the phase portrait (fig.9).

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