Criticality of FQ type in the context of autonomous systems and systems with noise

A.P. Kuznetsov, S.P. Kuznetsov, I.R. Sataev, A.V. Savin.

The type of critical behavior at the onset of chaos combining two scenarios of transition to chaos is investigated for coupled period-doubling systems. It is demonstrated, that the codimension of this critical behavior in the dissipatively coupled autonomous auto-oscillatory systems is 3. Renormalization group analysis of this critical behavior in the noisy driven systems is presented. It is shown that to observe the scaling in noisy driven systems one should rescale the noise amplitude, the numerical value of the correspondent scaling constant is obtained.

1. Introduction

Since the works of Feigenbaum [1-2] the renormalization group (RG) analysis and the concepts of universality and scaling have become usual in nonlinear dynamics. Transition to chaos via universal Feigenbaum's period-doubling cascade has been observed in a great variety of nonlinear dissipative systems of different physical nature.

It is well known that the parameter space near the threshold of chaos (critical point) and attractor at this point demonstrate the property of self-similarity (or scaling). The constants of scaling are unique for each type of critical point. Nowadays a lot of different types of critical behavior is known (see [3-17]).

In this paper, we review a critical situation associated with period doublings, which require at least two-dimensional maps as models for representation of the dynamics. This situation may arise in the context of multiparameter analysis of transition to chaos in multidimensional systems. Earlier, this type of critical behavior was investigated for coupled invertible and non-invertible maps [13,15,27] and it was shown, that it may demonstrate codimension-2 properties, provided the dissipative coupling is chosen. In this paper we study FQ type of critical dynamics in the context of transition to chaos in autonomous dynamical systems.

The effect of external noise on the critical behavior is another problem for investigation. This problem has been studied for different types of critical behavior (see [18-26]). The general scenario in this case is that the external noise destructs the fine structure near the critical point so one should rescale the noise amplitude to observe the self-similar structure, and correspondent scaling factor is unique for each type of critical behavior and can be obtained by renormalization group analysis. It means that experimentally we can observe only a finite number of rescalings, it depends both on noise amplitude and scaling factor: the larger they are the fewer rescalings can be observed.

In present paper we provide a RG analysis for the noise in the case of FQ type of critical behavior [13-15]) which typically arise in the asymmetrically coupled systems with period-doublings. In section 2 there are some brief information on this type of critical behavior. In section 3 the influence of external noise is investigated, and in section 4 the critical behavior of the FQ-type in coupled ordinary differential equation systems is studied.

2. The critical behavior of the FQ-type: the model system.

Let's consider the systems of two coupled nonidentical logistic maps:

$$x_{n+1}=1-\lambda x_n^2-Cy_n^2,$$
$$y_{n+1}=1-Ay_n^2-Bx_n^2. \qquad (1)$$

In the fig.1 we can see the structure of it's parameter plane. The period-doubling bifurcation curve meets the Neimark-Sacker bifurcation curve at the point which may be called *the period-doubling terminal* (PDT) *point*. It was shown ([13]) that these points accumulate at some critical point called FQ-point (Feigenbaum+Quasiperiodicity) because both Feigenbaum transition to chaos and the breakup of quasiperiodical regimes may be observed in any small vicinity of this point. The renormalization group analysis of this critical behavior was made by Kuznetsov *et al* [13-15] and scaling factors for the parameter and phase space were obtained. The correspondent RG equation has three relevant eigenvalues (δ_1=6.32631925…, δ_2=3.44470967…, δ_3=–1.900071670…), therefore, the full codimension of this critical behavior is three (it means that in general situation we should adjust three parameters to obtain this behavior). But in the system of coupled maps with dissipative coupling (1) this critical behavior realizes as the phenomenon with the codimension two due to some internal symmetry (see [13-15] for detailes). At the fig. 2 the self-similar structures of the vicinity of the FQ-point in the parameter plane and of the critical attractor are demonstrated.

3. The critical behavior of the FQ-type in the noisy driven system.

Now let's consider the noisy driven system:

$$x_{n+1}=1-\lambda x_n^2-Cy_n^2+\varepsilon\xi_n^{(1)},$$
$$y_{n+1}=1-Ay_n^2-Bx_n^2+\varepsilon\xi_n^{(2)}. \qquad (2)$$

Here $\overline{\xi}_n=(\xi_n^{(1)},\xi_n^{(2)})$ is a random vector, whose components are random values distributed uniformly on the segment [–0.5;0.5] in our numerical simulation.

Lets introduce the renormalization group analysis for the noisy driven system. We'll re-write the random vector as $\overline{\xi}_n=(\xi_n\varphi(x_n,y_n),\xi_n\psi(x_n,y_n))$ where $\varphi(x,y)$ and $\psi(x,y)$ are differentiable functions and ξ_n is a random value with Gaussian distribution[1]. Thus, the system at the FQ critical point will be

$$x_{n+1}=g(x_n,y_n)+\varepsilon\xi_n\varphi(x_n,y_n),$$
$$y_{n+1}=f(x_n,y_n)+\varepsilon\xi_n\psi(x_n,y_n), \qquad (3)$$

where $g(x,y)$ and $f(x,y)$ are the fixed point of the RG map for the FQ critical behavior. In fact it is 2D map driven by two random signals $\Phi_n(x,y)=\xi_n\varphi(x,y)$ and $\Psi_n(x,y)=\xi_n\psi(x,y)$ with second moments

$$P(x,y)=\left\langle\Phi_n^2(x,y)\right\rangle=\varphi^2(x,y),$$
$$Q(x,y)=\left\langle\Phi_n(x,y)\Psi_n(x,y)\right\rangle=\varphi(x,y)\psi(x,y),$$
$$S(x,y)=\left\langle\Psi_n^2(x,y)\right\rangle=\psi^2(x,y).$$

If the noise amplitude ε is small the twice iterated map can be written as follows

$$x_{n+2}=g(g(x_n,y_n),f(x_n,y_n))+\varepsilon[\xi_n\varphi(x_n,y_n)g'_x(g(x_n,y_n),f(x_n,y_n))+$$
$$\xi_n\psi(x_n,y_n)_n g'_y(g(x_n,y_n),f(x_n,y_n))+\xi_{n+1}\varphi(g(x_n,y_n),f(x_n,y_n))]$$
$$y_{n+2}=f(g(x_n,y_n),f(x_n,y_n))+\varepsilon[\xi_n\varphi(x_n,y_n)f'_x(g(x_n,y_n),f(x_n,y_n))+$$
$$\xi_n\psi(x_n,y_n)f'_y(g(x_n,y_n),f(x_n,y_n))+\xi_{n+1}\psi(g(x_n,y_n),f(x_n,y_n))]. \qquad (4)$$

(Here g'_x denotes $\frac{\partial g}{\partial x}$).

After standard renormalization procedure we'll get the RG map for the second moments of random functions:

[1] The empirical results show that the only requirement to the distribution of the random value is its finiteness. The shape of distribution doesn't effect the results.

$$P_{n+1}(x,y)=\alpha^2\left\{\left(g'_x(g(\frac{x}{\alpha},\frac{y}{\beta}),f(\frac{x}{\alpha},\frac{y}{\beta}))\right)^2 P_n(\frac{x}{\alpha},\frac{y}{\beta})+\right.$$
$$+2g'_x(g(\frac{x}{\alpha},\frac{y}{\beta}),f(\frac{x}{\alpha},\frac{y}{\beta}))g'_y(g(\frac{x}{\alpha},\frac{y}{\beta}),f(\frac{x}{\alpha},\frac{y}{\beta}))Q_n(\frac{x}{\alpha},\frac{y}{\beta})+$$
$$\left.+\left(g'_y(g(\frac{x}{\alpha},\frac{y}{\beta}),f(\frac{x}{\alpha},\frac{y}{\beta}))\right)^2 S_n(\frac{x}{\alpha},\frac{y}{\beta})+P_n(g(\frac{x}{\alpha},\frac{y}{\beta}),f(\frac{x}{\alpha},\frac{y}{\beta}))\right\}$$
$$Q_{n+1}(x,y)=\alpha\beta\left\{g'_x(g(\frac{x}{\alpha},\frac{y}{\beta}),f(\frac{x}{\alpha},\frac{y}{\beta}))f'_x(g(\frac{x}{\alpha},\frac{y}{\beta}),f(\frac{x}{\alpha},\frac{y}{\beta}))P_n(\frac{x}{\alpha},\frac{y}{\beta})\right.$$
$$+\left[g'_x(g(\frac{x}{\alpha},\frac{y}{\beta}),f(\frac{x}{\alpha},\frac{y}{\beta}))f'_y(g(\frac{x}{\alpha},\frac{y}{\beta}),f(\frac{x}{\alpha},\frac{y}{\beta}))+\right.$$
$$\left.+g'_y(g(\frac{x}{\alpha},\frac{y}{\beta}),f(\frac{x}{\alpha},\frac{y}{\beta}))f'_x(g(\frac{x}{\alpha},\frac{y}{\beta}),f(\frac{x}{\alpha},\frac{y}{\beta}))\right]Q_n(\frac{x}{\alpha},\frac{y}{\beta}) \quad (5)$$
$$\left.+g'_y(g(\frac{x}{\alpha},\frac{y}{\beta}),f(\frac{x}{\alpha},\frac{y}{\beta}))f'_y(g(\frac{x}{\alpha},\frac{y}{\beta}),f(\frac{x}{\alpha},\frac{y}{\beta}))S_n(\frac{x}{\alpha},\frac{y}{\beta})+Q_n(g(\frac{x}{\alpha},\frac{y}{\beta}),f(\frac{x}{\alpha},\frac{y}{\beta}))\right\}$$
$$S_{n+1}(x,y)=\beta^2\left\{\left(f'_x(g(\frac{x}{\alpha},\frac{y}{\beta}),f(\frac{x}{\alpha},\frac{y}{\beta}))\right)^2 P_n(\frac{x}{\alpha},\frac{y}{\beta})+\right.$$
$$+2f'_x(g(\frac{x}{\alpha},\frac{y}{\beta}),f(\frac{x}{\alpha},\frac{y}{\beta}))f'_y(g(\frac{x}{\alpha},\frac{y}{\beta}),f(\frac{x}{\alpha},\frac{y}{\beta}))Q_n(\frac{x}{\alpha},\frac{y}{\beta})+$$
$$\left.+\left(f'_y(g(\frac{x}{\alpha},\frac{y}{\beta}),f(\frac{x}{\alpha},\frac{y}{\beta}))\right)^2 S_n(\frac{x}{\alpha},\frac{y}{\beta})+S_n(g(\frac{x}{\alpha},\frac{y}{\beta}),f(\frac{x}{\alpha},\frac{y}{\beta}))\right\},$$

Here constants α and β are the FQ scaling factors in the phase space (α=–1.900071670…, β=–4.00815849…). We have found the fixed point of this map numerically and it's largest eigenvalue γ=8.206143… is the scaling factor for the noise amplitude.

Some computer illustrations of scaling in the noisy driven system are shown at fig.3 (in the parameter plane near the FQ point) and fig.4 (at the critical attractor). We can see that the original and rescaled fragments are practically identical.

4. The FQ critical behavior in the coupled Chua's circuits.

The original system (1) is non-invertible map and special investigation of this critical behavior in the invertible systems should be provided. Recently it was shown that such behavior may occur in a systems of two dissipatively coupled invertible Henon maps and its codimension is two for that system [27]. To reveal the properties of this critical behavior in a more generic situation we consider next a system of ordinary differential equations. We've chosen the well-known Chua's circuit [28-29] for the partial system.

Lets consider two Chus's circuit with dissipative coupling:

$$\begin{aligned}
\dot{x}_1 &= \alpha_1(y_1-h(x_1))+c_1(x_2-x_1),\\
\dot{y}_1 &= x_1-y_1+z_1+c_1(y_2-y_1),\\
\dot{z}_1 &= -\beta_1 y_1+c_1(z_2-z_1),\\
\dot{x}_2 &= \alpha_2(y_2-h(x_2))+c_2(x_1-x_2),\\
\dot{y}_2 &= x_2-y_2+z_2+c_1(y_1-y_2),\\
\dot{z}_2 &= -\beta_2 y_2+c_2(z_1-z_2),
\end{aligned} \quad (6)$$

Here $c_{1,2}$ are coupling parameter which will be fixed during our investigations (c_1=0,1; c_2=–0,05) and $h(x)$ is a piece-wise linear function which we assume to be

$$h(x)=\begin{cases}\frac{2x}{7}-\frac{3}{7}, & x\ge 1,\\ -\frac{x}{7}, & -1<x<1,\\ \frac{2x}{7}+\frac{3}{7}, & x\le -1.\end{cases} \tag{7}$$

according to the tradition [29].

First lets try to obtain the FQ point by adjusting two parameters α_1 and α_2, other parameters are fixed ($\beta_{1,2}$=10). The structure of the parameter plane is shown in the fig. 5. The general structure seems to be very similar to that of the system (1) but the fine structure near the possible critical point (fig. 5b) is quite different and unusual. The period-doubling lines become non-smooth there and it is impossible to obtain the PDT-point for the period-16 cycle. It means that there is no FQ point in this plane.

We can suppose next reasons for such phenomenon. As it was mentioned earlier the FQ critical behavior can realize as the phenomenon of codimension two only due to the symmetry of equations arising from the special type of coupling between the systems of identical structure. But for the system under consideration this symmetry is broken. It can be explained both formally and physically.

The formal explanation is based on that fact that the Poincare map for the system (6) is 5D map so it in no way can be considered as two coupled systems of identical structure. This destructs the symmetry needed to observe the FQ critical behavior as the phenomenon with codimension two.

Physically the independent phase dynamics in subsystems is the reason for such destruction. We can introduce the phase for each subsystem and they will evolve independently so the points of the Poincare map corresponds to different phases, unless we are in a phase-locked region. Nevertheless, we must take into account phase dynamics and this is the reason for the breakup of symmetry.

Therefore, in the case of autonomous ODE systems dissipative coupling does not lead to reduction of codimension and the FQ critical behavior must be regarded as the phenomenon of codimension three. We should adjust three parameters now to come at the critical point (β_2=10).

It is known that in the critical point unstable cycles of all periods exist and their multipliers converge rapidly to some fixed values (called universal multipliers) unique for each type of critical behavior while the cycle period grow. Hence, the sequence of points with multipliers equal to universal should rapidly converge to the critical point (if that exists). There are two universal multipliers (μ_1=–1.579739…, μ_2=–1.057149…) [13] for the FQ type and we should adjust three parameters. This problem can be solved as follows.

The multipliers are the roots of the characteristic equation (8):

$$\mu^5 - Sp\mu^4 + I_2\mu^3 - I_3\mu^2 + I_4\mu - Det = 0. \tag{8}$$

where coefficients are the invariants of the Jacobian of the Poincare map. Near the critical point dynamics is effectively two-dimensional and three multipliers of the five are practically equal to zero, so the two non-zero multipliers obey the next equations:

$$Sp=\mu_1+\mu_2,\ I_2=\mu_1\mu_2 \tag{8}$$

We trace the sequence of point where multipliers of n-cycle obey (8) and multipliers of $2n$-cycle obey the first equation in (8), that gives us three conditions to adjust three parameters. The results are shown in Table 1. The PDT points sequence converges to the point α_1=6,6330061623, α_2=6,5859306384, β_1=10,1980230965. We calculate the cycles existing at this point (there first elements are presented in Table 2) and their multipliers (Table 3). The multipliers converges to the universal values, and this is an evidence for it's really the FQ point. Also we estimated the scaling factor for the attractor which rescales the distance between the elements of n and $2n$ cycles (Table 4). It converge to the value α=–1.900… which is universal for the FQ-type.

5. Conclusion

An attempt of complete “physical” investigation of the FQ-type critical behavior was undertaken in our work. We have studied the influence of noise on this type of critical behavior and have shown that its amplitude should be rescaled to observe the scaling near the critical point. Numerical value of corresponding scaling factor for noise amplitude was obtained in the course of RG analyses for the noisy system.. Also we have studied the FQ critical behavior in dissipatively coupled autonomous differential equation system and have shown that its codimension is equal to full codimension three in such system unlike the model system where it is equal to two. Thus it was shown once more that one should be extremely careful while expanding the results obtained for the maps onto generic case system.

Acknowledgments

Authors thank Prof. A. Pikovsky (University of Potsdam) for useful and fruitful collaborations and instant discussions. Authors also thank the RFBR (grants 06-02-16773 and 04-02-04011) for financial support.

The bibliography

[1] Feigenbaum M. J. Quantitative universality for a class of nonlinear transformations. //J. of Stat. Phys., 1978, v.19, №1, pp.25-52.

[2] Feigenbaum M. J. The universal metric properties of nonlinear transformations. //J. of Stat. Phys.,1979, v.26, №6, pp.669-706.

[3] Chang S.J., Wortis M., Wright J.A. Iterative properties of a one-dimensional quartic map. Critical lines and tricritical behavior. //Phys. Rev., 1981, v.A24, pp.2669-2684.

[4] MacKey R.S, Tresser C. Some flesh on skeleton: The bifurcation structure of bimodal maps. //Physica D, 1987, v. 27, №3, pp.412-422.

[5] MacKey R.S., Tresser C. Boundary of topological chaos for bimodal maps of the interval. //J. London Math. Soc., 1988, v.37, №1, pp.164-181.

[6] Schell M., Fraser S., Kapral R. Subharmonic bifurcations in the sine map: an infinite of bifurcations. //Phys. Rev., 1983, v.A28, №1, pp.373-378.

[7] Mackey R.S, van Zeijts J.B.J. Period doubling for bimodal maps: a horseshoe for a renormalization operator. //Nonlinearity, 1988, v.1, pp.253-277.

[8] Feigenbaum M.J., Kadanoff L.P., Shenker S.J. Quasiperiodicity in dissipative systems: A renormalization group analysis. // Physica, 1982, v.D5, p.370-386.

[9] Rand D., Ostlund S., Sethna J., Siggia E.D. Universal transition from quasiperiodicity to chaos in dissipative systems. //Phys. Rev. Lett., 1982, v.49, №2, pp.132-.135

[10] Ostlund S., Rand D., Sethna J., Siggia E.D. Universal properties of the transition from quasiperiodicity to chaos in dissipative systems. //Physica, 1983, v.D8, №3, pp.303-342.

[11] Rand D. Existence, non-existence and universal breakdown of dissipative goldeninvariant tori: I. Golden critical circle maps. //Nonlinearity, 1992, v.5, pp.639-662.

[12] Kuznetsov S.P. A variety of critical phenomena associated with the golden mean quasiperiodicity. //Izvestiya VUZ: Applied Nonlinear Dynamics, 2002, v.10, №3, cc.22-39.

[13] Kuznetsov S.P., Sataev I.R. Period-doubling for two-dimensional non-invertible maps: renormalization group analysis and quantitative universality. //Physica D, 1997, v.101, p.249-269.

[14] Kuznetsov A.P., Kuznetsov S.P., Sataev I.R. A variety of period-doubling universality classes in multiparameter analysis of transition to chaos. //Physica D, 1997, v.109, p.91-112.

[15] Kuznetsov S.P., Sataev I.R. Universality and scaling in non-invertible two-dimensional maps. //Physica Scripta, 1996, v.T67, p.184-187.

[16] Kuznetsov S.P. Tricriticality in two-dimensional maps. //Phys. Lett., 1992, v.A169, p.438-444.

[17] Kuznetsov A.P., Kuznetsov S.P., Mosekilde E., Turukina L.V. Two-parameter analysis of the scaling behavior at the onset of chaos: tricritical and pseudo-tricritical points. //Physica A, 2001, v.300, p.367-385.
[18] B. Shraiman, C.E. Wayne, P.C. Martin. Scaling theory for noisy period-doubling transition to chaos. //Phys. Rev. Lett., 1981, v.46, №14, pp.935-939.
[19] J.P. Crutchfield, M. Nauenberg, J.Rudnik. Scaling for external noise at the onset of chaos. //Phys. Rev. Lett., 1981, v.46, №14, p.933-935.
[20] A. Hamm, R. Graham. Scaling for small random perturbations of golden critical circle maps. //Phys. Rev. A, 1992, v.46, №10, pp.6323-6333.
[21] G. Gyorgyi, N. Tishby. Scaling in a stochastic Hamiltonian systems: a renormalization approach. //Phys. Rev. Lett., 1987, v.58, №6, pp.527-530.
[22] G. Gyorgyi, N. Tishby. Path integrals in Hamiltonian systems: breakup of the last KAM tori due to random forces. //Phys. Rev. Lett., 1989, v. 62, №4, pp.353-356.
[23] J.V. Kapustina, A.P. Kuznetsov, S.P. Kuznetsov, E. Mosekilde. Scaling properties of bicritical dynamics in unidirectionally coupled period-doubling systems in the presence of noise. //Phys. Rev. E, 2001, v.64, №6, 006207 (12 pages).
[24] O.B. Isaeva, S.P. Kuznetsov, A.H. Osbaldestin. Effect of noise on the dynamics of a complex map at the period-tripling accumulation point. //Phys. Rev. E, 2004, v.69, 036216.
[25] A.P. Kuznetsov, S.P. Kuznetsov, J.V. Sedova. Effect of noise on the critical golden-mean quasiperiodic dynamics in the circle map. //Physica A, 2006, v.359, pp.48-64.
[26] S.P. Kuznetsov. Effect of noise on the dynamics at the torus-doubling terminal point in a quadratic map under quasiperiodic driving. //Phys. Rev. E, 2005, v.72, 026205.
[27] Kuznetsov A.P., Savin A.V., Kim S.-Y. On the Criticality of the FQ-Type in the System of Coupled Maps with Period-Doubling. // Nonlinear Phenomena in Complex Systems, 2004, v.7, №1, pp.69-77.
[28] Matsumoto T, Chua L.O., Komuro M. The double scroll. // IEEE Transactions on Circuits and Systems, 1985, v. CAS-32, №8, pp. 797-818.
[29] Chua L.O., Komuro M., Matsumoto T. The double scroll family. //IEEE Transactions on Circuits and Systems, 1986, v. CAS-33, №11, pp. 1073-1118.

Table 1

n	α_1	α_2	β_1
16	6.6322759493	6.5855783800	10.1959699945
32	6.6346848740	6.5867114343	10.2028396484
64	6.6338779890	6.5863313934	10.2005199736
128	6.6331492820	6.5859952453	10.1984333086
256	6.6330061623	6.5859306384	10.1980230965

Table 2

n	y_1	z_1	x_2	y_2	z_2
1	0.24659319	-0.35980782	0.76903361	0.20376135	-0.13991670
2	0.25243738	-0.53159549	0.75594684	0.22675477	-0.29124826
4	0.08677779	-0.24240962	0.88795905	0.18936405	-0.21109082
8	0.20582640	-0.25576499	0.78319741	0.15150264	-0.07601758
16	0.15922796	-0.22339862	0.84635857	0.09630167	-0.11829855
32	0.17372657	-0.22851192	0.82723572	0.11353928	-0.09961583
64	0.17189928	-0.22765608	0.83040306	0.11309123	-0.10231508
128	0.17861078	-0.23111979	0.81826776	0.11950256	-0.09245976
256	0.17515818	-0.22922687	0.82475859	0.11528035	-0.09756806
512	0.17616889	-0.22975642	0.82286977	0.11651880	-0.09604593

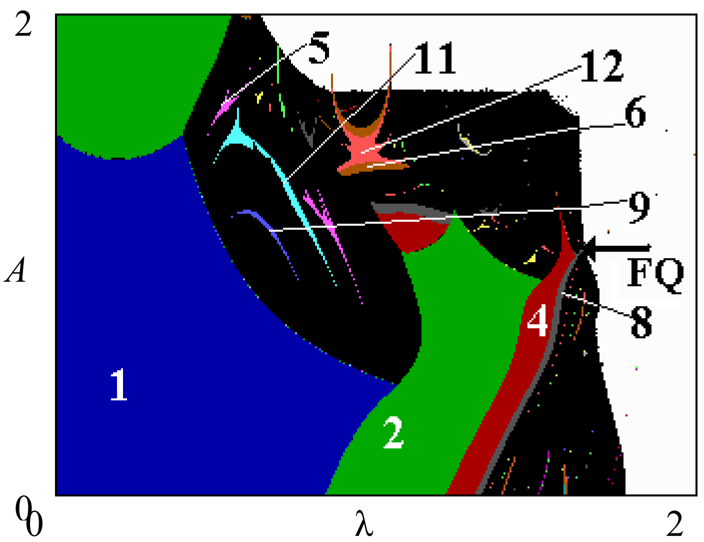
Table 3

n	μ_1	μ_2	μ_3	μ_4	μ_5
1	-1.61578	-1.27869	0.501664	$-4.961\cdot10^{-3}$	$-3.432\cdot10^{-3}$
2	-1.60888929	-1.06054871	0.36300185	$-2.557\cdot10^{-5}$	$-9.031\cdot10^{-6}$
4	-1.61128177	-1.02294567	0.07014112	$-1.126\cdot10^{-9}$	$-1.811\cdot10^{-10}$
8	-1.59305331	-1.03700271	0.01367606	$-1.788\cdot10^{-14}$	$5.713\cdot10^{-15}$
16	-1.55741487	-1.08679156	$-5.327\cdot10^{-6}$	$-6.750\cdot10^{-12}$	$1.270\cdot10^{-13}$
32	-1.58643546	-1.06785783	$1.088\cdot10^{-8}$	$3.107\cdot10^{-12}$	$1.083\cdot10^{-13}$
64	-1.59408160	-1.02517953	$-6.950\cdot10^{-11}$	$1.067\cdot10^{-11}$	$-6.000\cdot10^{-14}$
128	-1.56291148	-1.07897649	$1.686\cdot10^{-10}$	$3.379\cdot10^{-11}$	$1.563\cdot10^{-13}$
256	-1.57981941	-1.05708041	$-9.226\cdot10^{-10}$	$-1.669\cdot10^{-10}$	$-7.195\cdot10^{-12}$
512	-1.55838483	-1.07176663	$-1.428\cdot10^{-8}$	$1.984\cdot10^{-9}$	$-2.947\cdot10^{-12}$

Table 4

N	d	α
2	0.411097569236	
4	0.224659995593	1.829865473605
8	0.092598547703	2.426171912691
16	0.056513128423	1.638531617112
32	0.030592580029	1.847282196187
64	0.016537895766	1.849847191073
128	0.008637914483	1.914570443890
256	0.004639464262	1.861834469499
512	0.002442409060	1.899544322031

a)



b)

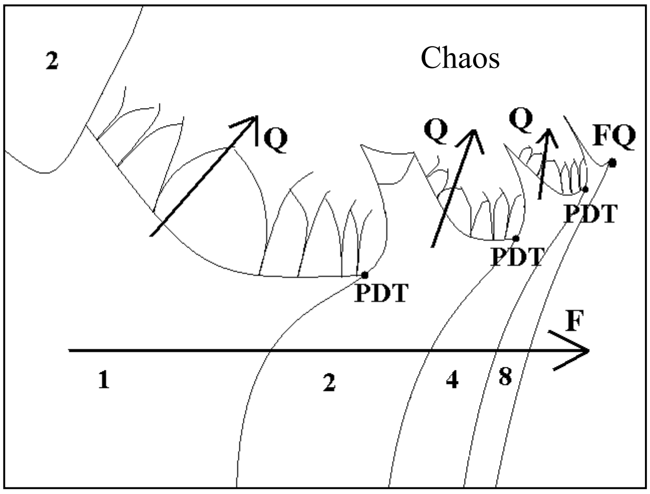


Fig. 1. The structure of the parameter plane of system (1) (a) and its schematic representation (b).
a) Coupling constants are fixed *B*=0.375, *C*=-0.25. The stability regions of different cycles are marked by different gray shades, the periods of cycles subscribed.
b) PDT means period-doubling point, Q means the destruction of quasiperiodicity, F mean the Feigenbaum cascade.

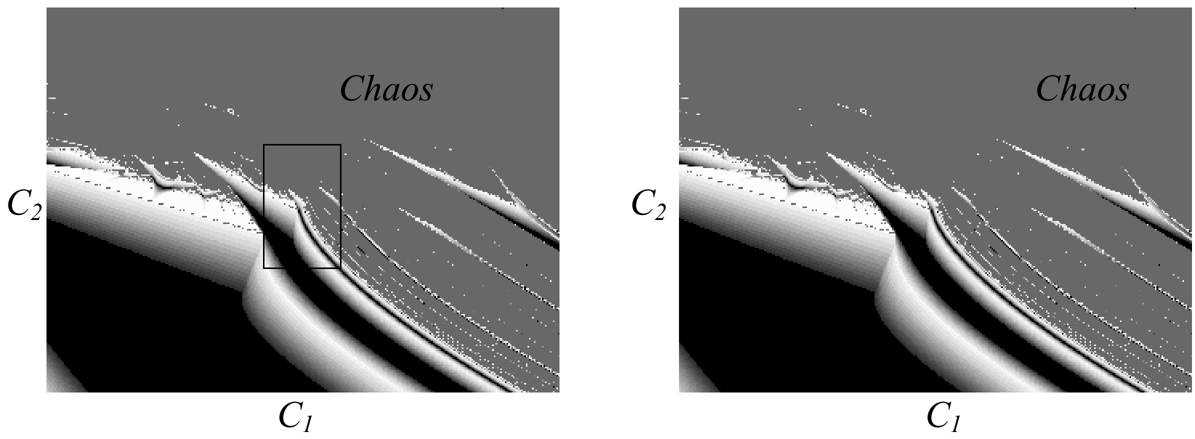


Fig. 2. The self-similar structure of the parameter plane near the FQ point in the system (1). The gray color codes the value of the largest Lyapunov exponent: white for zero, black for infinitely large negative, the darker is the color the larger is the negative value of exponent. Positive values are marked with middle grey. Right figure is the rescaled rectangle from the left figure. C_1 and C_2 are special scaling coordinates (see [27] for details), the scaling factors are δ_1 by C_1 and δ_2 by C_2.

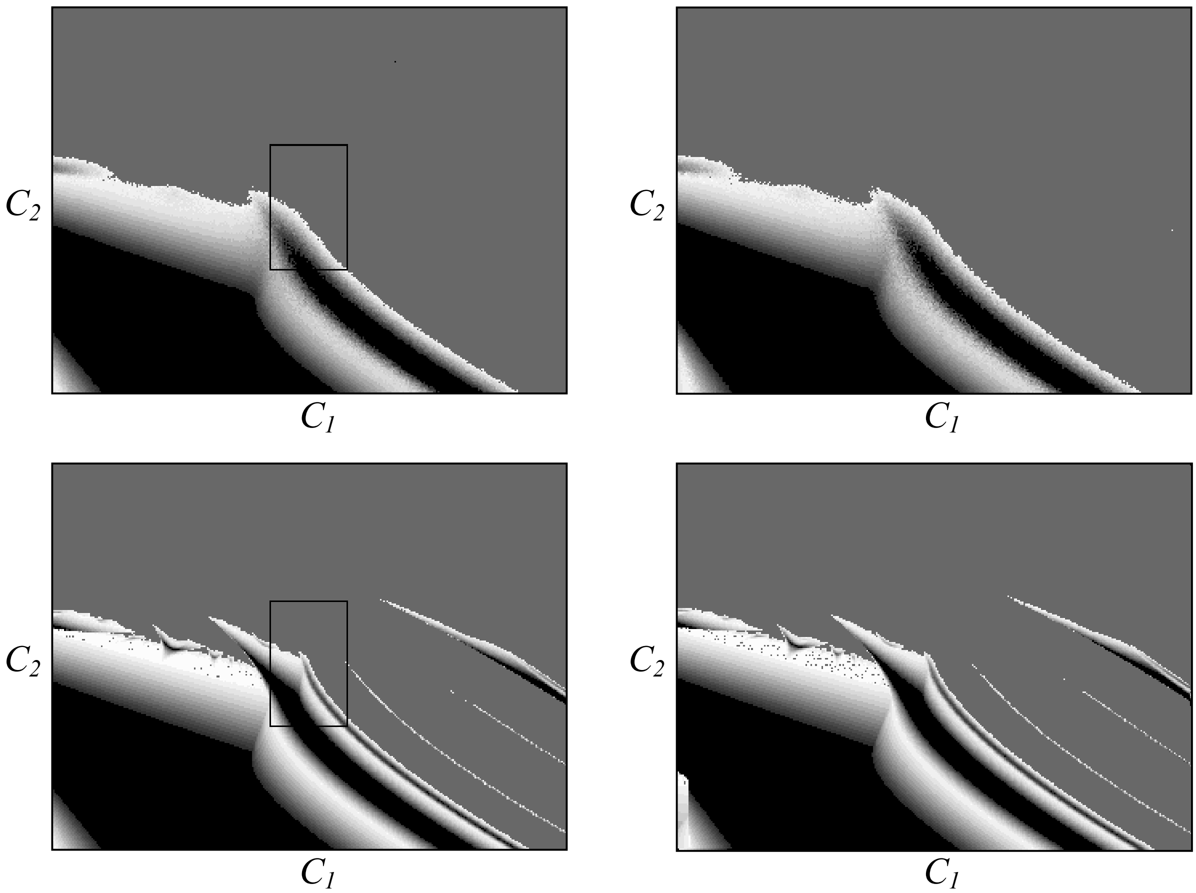


Fig. 3. Scaling near the FQ critical point in the parameter plane of noisy driven system (2). The initial noise amplitude is 0.01 for upper figures and 0.001 for lower figures.

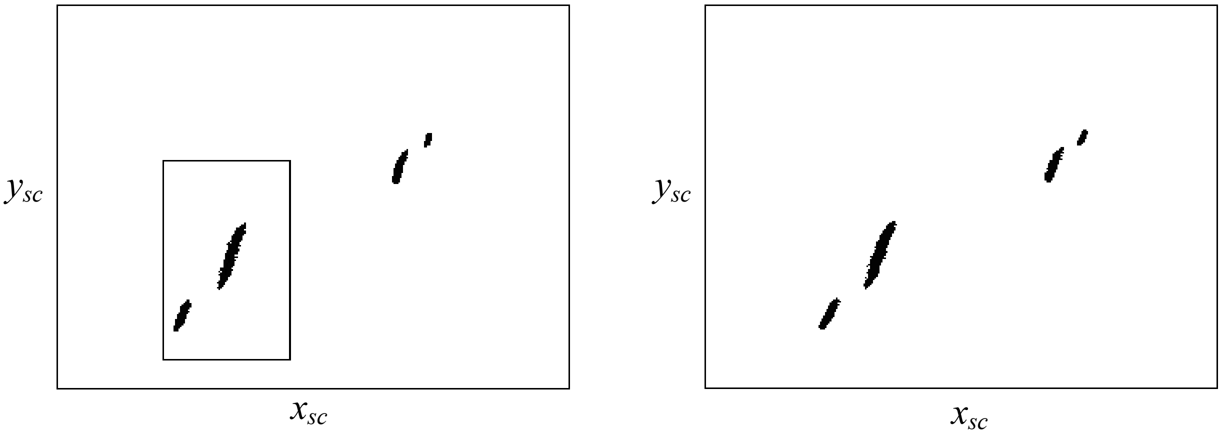


Fig. 4. Scaling on the FQ critical attractor in noisy driven system (2). The initial noise amplitude is 0.001, the scaling factors are =–1.900071670…for the x_{sc} and β=–4.00815849… for the y_{sc}.

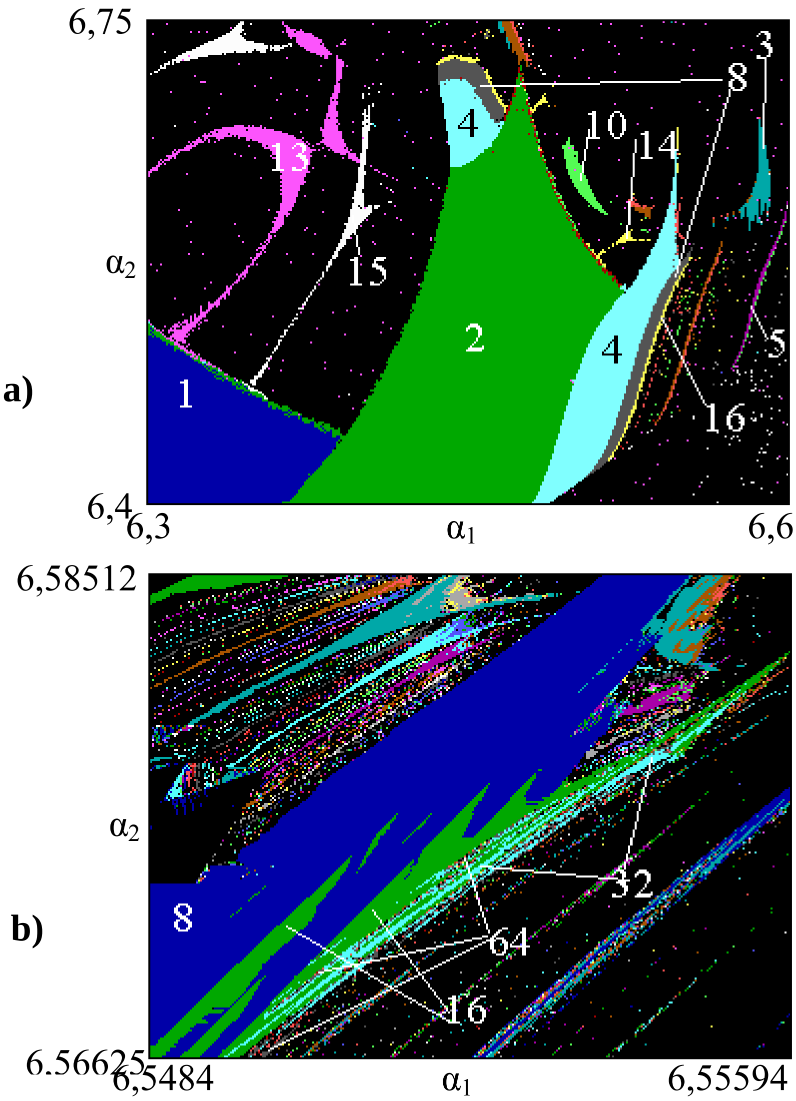


Fig. 5. The structure of the parameter plane of system (6).