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Complex Analytic Dynamics Phenomena in a System of Coupled Nonautonomous Oscillators with Alternative Excitation

O. B. Isaeva[a, *], S. P. Kuznetsov[a], and A. Osbaldestin[b]

[a] *Institute of Radio Engineering and Electronics (Saratov Branch), Russian Academy of Sciences, Saratov, 410019 Russia*

[b] *Department of Mathematics, University of Portsmouth, Portsmouth PO1 3HE, United Kingdom*

**e-mail: isaeva@rambler.ru*

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Abstract—A model system of two coupled nonautonomous oscillators is proposed, in which the phenomena of complex analytic dynamics (Mandelbrot and Julia sets, etc.) characteristic of complex logistic maps are realized. The idea underlying the model is based on the mechanism of alternative excitation transfer from one subsystem to another and on the method (well known in the theory of oscillations) of slowly varying complex amplitudes.

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The section of nonlinear dynamics engaged in investigation into the properties of iterative maps set by analytical functions of complex variables employs a profoundly developed mathematical apparatus [1]. A classical object for such investigations is offered by the quadratic complex map

$$z_{n+1} = c + z_n^2, \qquad (1)$$

which exhibits a trivial behavior for $c = 0$: the result of sequential iterations tends to infinity for $|z_0| > 1$ and remains within a finite region for $|z_0| < 1$, so that the unit circle $|z_0| = 1$ is a boundary between the two types of behavior. For other values of the complex parameter c, the boundary separating these types has, in the general case, a rather complicated structure and exhibits a fractal character, offering an example of so-called Julia sets (see Fig. 1).

On the other hand, once the initial condition $z_0 = 0$ is set and the behavior of z_n as a function of a complex parameter c is studied, the iterations also run to infinity for some values of c and remains within a finite region for the other. The set of points on the complex c plane, which correspond to the latter situation, is called the Mandelbrot set (Fig. 1a). The dynamics of complex variable z within a limited region may correspond to both periodic and chaotic regimes. The shape of the region of periodic dynamics on the complex plane (painted gray in Fig. 1a) resembles a cactus, representing a set of rounded subdomains occurring at the periphery of a large cardioid-shaped domain. Main lobes of the cactus correspond to the dynamics with periods 1, 2, 3, as indicated by the corresponding numbers in Fig. 1a. A fractal pattern (painted black in Fig. 1a) bounding the "Mandelbrot cactus" on the plane of parameter c corresponds to chaotic dynamics. Figures 1b–1f give examples of the Julie sets corresponding to the values of c indicated by points in Fig. 1a.

The principal question is whether the phenomena of complex mapping dynamics have physical realizations (see, e.g., [2]). One successful example of such a realization for the Mandelbrot set is offered by a system of specially chosen symmetrically coupled maps or nonlinear oscillators with an external periodic drive [3]. In this context, the Mandelbrot set was treated as a region of generalized partial synchronization [4] and some aspects of the realization of complex dynamics in autonomous flow systems were studied [5]. An analog electronic device modeling the dynamics of appropriately coupled quadratic maps was proposed [6], which allowed the Mandelbrot set to be observed in physical experiment for the first time.

This Letter describes an alternative approach, which makes possible the realization of dynamics characteristics of complex maps. A complex variable represents the amplitude of oscillations in a system of two coupled nonautonomous oscillators, which exhibit alternative excitation ant transmit it to each other it in a relay-race manner. Previously, an analogous idea was used for the physical realizations of some abstract models and phenomena with complex dynamics such as Bernoulli map, Smale–Williams attractor, Arnold's cat, and robust strange nonchaotic attractor [7–11].

Let us discuss the main principles of functioning of the proposed system, using the method of slow amplitudes that is well known in the theory of oscillations and waves. Consider an oscillatory process with frequency ω_0 and slowly varying complex amplitude $A(t)$: $x(t) = \mathrm{Re}[A(t)\exp(i\omega_0 t)]$. A transformation of this signal

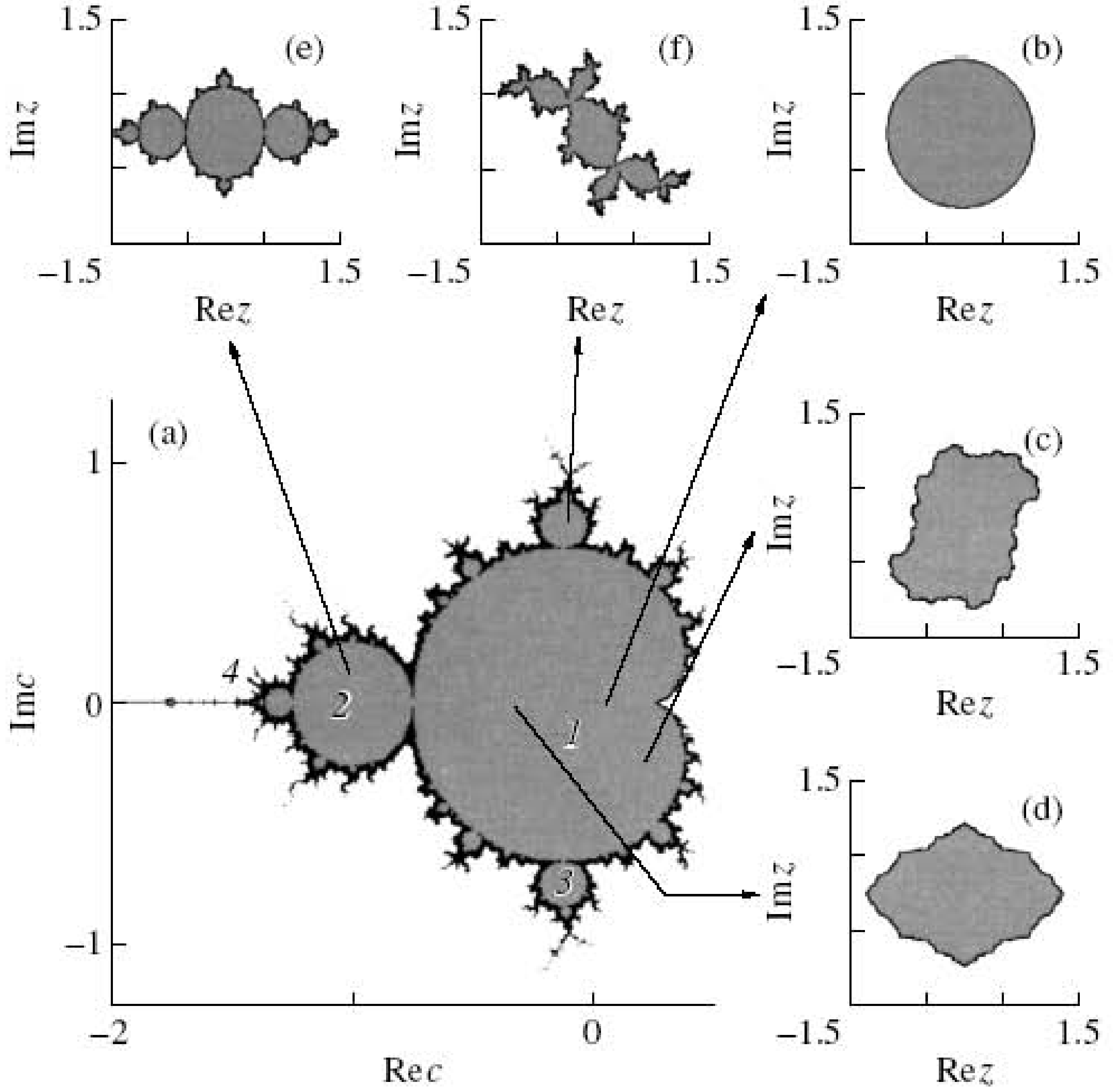

Fig. 1. Quadratic complex map (1): (a) the plane of complex parameter c showing Mandelbrot set (painted gray) corresponding to periodic dynamics (regions of periods 1, 2, and 3 indicated by the corresponding numbers), separated by a fractal pattern corresponding to chaotic dynamics in the phase space (painted black) from the region (painted white) in which the trajectories run to infinity; (b–f) the plane of complex variable z showing Julia sets for various values of parameter c indicated by points in (a): 0 (b); $0.2–0.3i$ (c); -0.4 (d); -0.8 (e); $-0.1–0.75i$ (f).

on an element with a quadratic nonlinearity yields $y(t) = x^2(t) = (1/2)|A(t)|^2 + (1/2)\mathrm{Re}[A(t)^2\exp(2i\omega_0 t)]$. As can be seen the double-frequency component has complex amplitude representing the square of the initial amplitude. Such a transformation, taking place during the evolution over a period of action in the proposed nonautonomous oscillatory system, constitutes a basis of the proposed approach to realization of the complex analytical dynamics.

Consider a system of two coupled nonautonomous oscillators described by the following set of equations:

$$\ddot{x} + \omega_0^2 x + F\left(\gamma + \sin\frac{2\pi}{T}t\right)\dot{x} = \varepsilon y\sin\omega_0 t + \lambda\sin(\omega_0 t + \varphi),$$
$$\ddot{y} + (2\omega_0)^2 y + F\left(\gamma - \sin\frac{2\pi}{T}t\right)\dot{y} = \varepsilon x^2, \qquad (2)$$

where x and y denote the generalized coordinate in the first and second subsystem, respectively, and F, γ, λ, and ε are the parameters. The natural frequency of the first oscillator is ω, and that of the second oscillator is twice as large. The coefficients at the first derivative, which control the dissipation in both subsystems, slowly vary with the time in counterphase with the period T. Let this period be equal to an integer of the periods of intrinsic oscillations: $T = 2\pi n/\omega_0$ ($N = 1, 2, 3, \ldots$). The parameter γ is a positive quantity smaller than unity (e.g., $\gamma = 0.5$). In this case, the dissipation (remaining positive on averaging over the period) periodically becomes negative for one or another oscillator. In this stage, the oscillator is active (oscillations increase), while the rest of the period it is dissipative (oscillations decay).

Let us assume that, at the onset of the active stage in the second oscillator, the first oscillator has complex amplitude A and the corresponding variable is $x(t) \sim \mathrm{Re}[A(t)\exp(i\omega_0 t)]$. This signal acts on the second oscillator via a nonlinear quadratic element, in which a "seed" for the arising fluctuations is the second-harmonic component $\mathrm{Re}[A(t)^2\exp(2i\omega_0 t)]$. For this reason, the complex amplitude of oscillations for the second oscillator in the active stage is proportional to A^2. In the stage of reverse action (see the right-hand part of the first equation in (2), the mixing with a reference signal leads to the appearance of component with a difference frequency ω_0 and an amplitude proportional to A^2. This component, together with the additional oscillatory term $\lambda\sin(\omega_0 t + \varphi)$ (determined by the parameters of

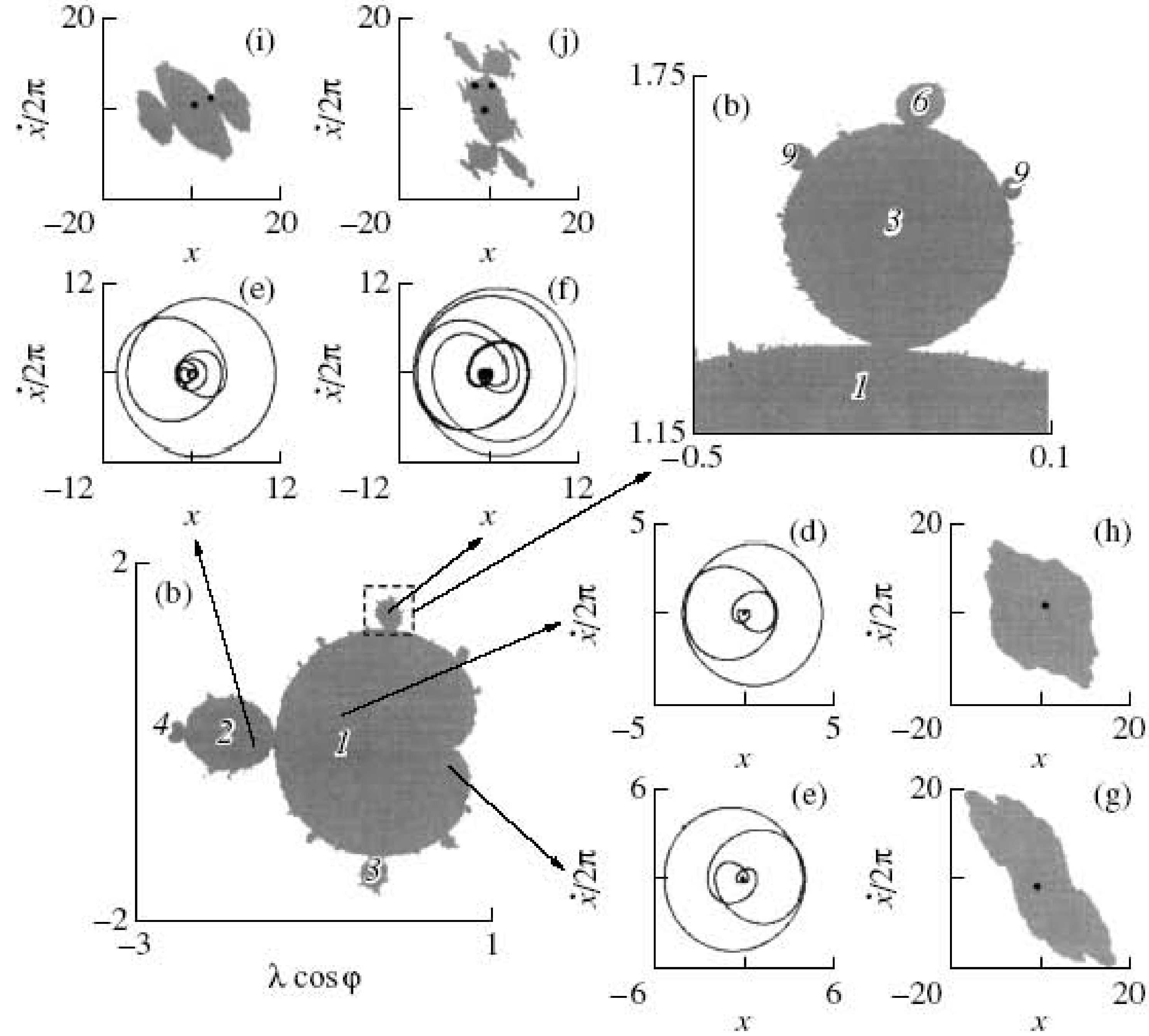

Fig. 2. A system of two coupled nonautonomous oscillators (2) with $\omega_0 = 2\pi$, $T = 10$, $F = 7$, $\gamma = 0.5$, and $\varepsilon = 1$: (a, b) Mandelbrot set (painted gray) and its magnified fragment on the plane of parameters $\lambda\cos\varphi$, $\lambda\sin\varphi$ corresponding to the real and imaginary parts of the parameter c of complex map (1), where numbers *1*, *2*, *3*, etc. indicate the regions of periodic dynamics with periods T, $2T$, $3T$, etc., and regions painted white correspond to infinite growth; (c–j) the plane of variables $(x, \dot{x}/2\pi)$ showing the projections of limit cycles and the corresponding sections of the basins of attraction for various values of parameters indicated by points in (a): $\lambda\cos\varphi = 0.5$, $\lambda\sin\varphi = -0.2$ (c, g); $\lambda\cos\varphi = -0.7$, $\lambda\sin\varphi = 0.4$ (d, h); $\lambda\cos\varphi = -1.7$, $\lambda\sin\varphi = 0.1$ (e, i); $\lambda\cos\varphi = -0.2$, $\lambda\sin\varphi = 1.5$ (f, h); points in the basins of attraction indicate the projections of the Poincaré map.

amplitude λ and phase φ), provides a seed for the complex amplitude of the first oscillator. Therefore, the mapping of the complex amplitude of the first oscillator corresponds, in a certain approximation, to a complex quadratic map. Indeed, the real and imaginary parts of z correspond to the generalized coordinate x and velocity $\dot{x}/\omega_0$, while the complex parameter c is represented by a complex quantity with the modulus λ and the argument φ.

Depending on the choice of this parameter, $c = \lambda e^{i\varphi}$, a solution to the equations describing the coupled nonautonomous oscillators either remains finite (for the initial conditions set I a certain region of the phase space) or runs to infinity. Figures 2a and 2b show the diagrams obtained by numerical calculations, where regions painted gray correspond to the observation of dynamics within a finite domain and white regions correspond to infinite growth. The calculations were performed for $\omega_0 = 2\pi$, $T = 10$, $F = 7$, $\gamma = 0.5$, and $\varepsilon = 1$. The patterns in these diagrams are apparently similar to the Mandelbrot set for a complex quadratic map. The regions with numbers *1*, *2*, *3*, etc. correspond to the lobes featuring the dynamics with periods T, $2T$, $3T$, etc., respectively.

Figure 3 illustrates the character of dynamics for system (2) in a regime realized in the lobe of period 3. This pattern represents a time series of variables x and e in the coupled nonautonomous oscillators.

Apparently, the type of a dynamical regime the system under consideration is determined by the form of a "seeding" signal in the region of minimum amplitude of oscillations of the first oscillator. This signal represents a superposition of the external signal and a signal from the second oscillator transformed in the nonlinear element. Here, details of the relationship between phases of the two signals are important and determine the fine structure of the cactus lobes on the plane of parameters.

Figure 2 also shows diagrams on the plane of variables of the first oscillator $(x, \dot{x}/\omega_0)$ for the points indicated on the cactus in Fig. 2a, which are analogous to the portraits of Julia sets for the quadratic map in Fig. 1.

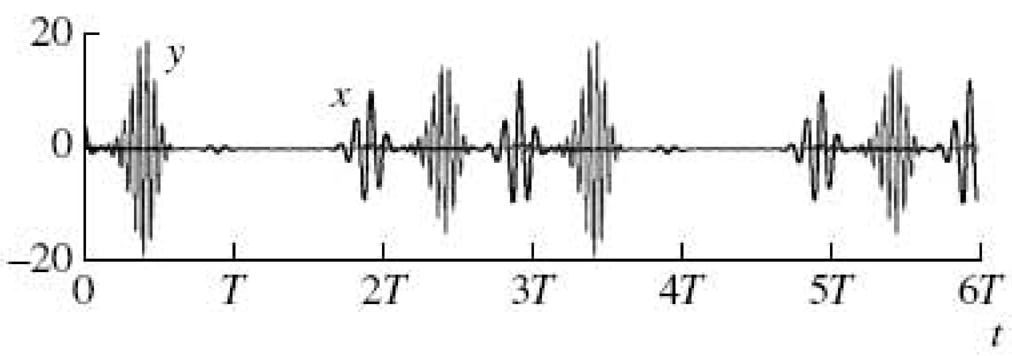

Fig. 3. Time series of the dynamics of coupled nonautonomous oscillators (2) with the parameters $\omega = 2\pi$, $T = 10$, $F = 7$, $\gamma = 0.5$, and $\varepsilon = 1$, $\lambda\cos\varphi = -0.2$, and $\lambda\sin\varphi = 1.5$, which correspond to a cycle of period 3 in the Poincaré section.

In the four-dimensional phase space $(x, \dot{x}/\omega_0, y, \dot{y}/2\omega_0)$, the attractor in the Poincaré section is situated rather close to the $(x, \dot{x}/\omega_0)$ plane, but not exactly in this plane. The basin of attraction represents a four-dimensional fractal object. Figures 2g–2j show sections of the basins of attraction by the plane $y = 0$, $\dot{y} = 0$ for the attractors situated in finite regions of the phase space and indicate the points corresponding to the attractor in a stroboscopic cross section, while Figs. 2c–2f show the phase portraits of these attractors projected onto the $(x, \dot{x}/\omega_0)$ plane.

As can be seen from the data presented above, the proposed model system demonstrates phenomena characteristic of the complex analytic dynamics, reproducing (at least, in the general form) objects such as the Mandelbrot and Julia sets. A more thorough analysis, which will be reported in subsequent publications, shows that this analogy can be broken on deeper levels of resolution of the fine-scale fractal structure. This is related to the fact that, in the description of dynamics within the framework of the method of slowly varying amplitudes, all terms in the equations are analytical (in the sense of the theory of functions of complex variables) only to the first approximation. Subsequent approximations involve nonanalytic additives, which lead to the breakage of small-scale details in the mapped objects (analogs of the Mandelbrot and Julia sets). Thus, the correspondence with the complex analytic dynamics is essentially asymptotic with respect to the parameter T, that is, it improves with increasing ratio of the time scales of the period of slow modulation of the dissipation parameter and the period of intrinsic oscillations.

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