# Hyperchaotic pseudohyperbolic attractors and quasiattractors in minimal ensembles of oscillators


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***Abstract* — We present a study of two minimal ensembles of oscillators of principally different types. The first ensemble describes the dynamics of two coupled classical Lorenz system. We demonstrate that it can generate robust chaotic dynamics associated with pseodohyperbolic attractors by Turaev-Shilnikov. The second ensemble describes dynamics of two coupled generators of quasiperiodic oscillations. It exhibits non-robust chaotic dynamics associated with quasiattractors of Afraimovich-Shilnikov type. Hyperchaotic dynamics are shown for both types of ensembles. Characteristic structures of parameter planes are demonstrated.**



***Keywords— minimal ensembles of oscillators, hyperchaos, pseuduhyperbolic attractors, quasiattractors.***


## I. INTRODUCTION

The dynamics of ensembles of interacting oscillators is very rich and diverse. Interaction in ensembles initiates various phenomena such as: synchronization [1], clustering [2], chimeras [3], etc. To observe the phenomena of chimeras and clustering, the minimal ensemble should consist of three oscillators so that they can form groups. An increase in the size of the ensemble can lead to more complex dynamics and the formation of new phenomena. Another way to complicate dynamics is complication of the dynamics of the base element. In this case, the variety of dynamics is usually limited by the features of the base model. However, under certain conditions, even in minimal ensembles consisting of two elements, new types of regimes may arise. To implement chaotic behavior, the minimum dimension of the phase space of a dynamical system must be equal to three. In this case, the system is characterized by three Lyapunov exponents: one positive, one zero and one negative. In the interaction of two such oscillators, from the point of view of the phase space dimension, new possibilities arise, for example, with weak coupling, one can observe hyperchaos, a chaotic regime in which two or more Lyapunov exponents in the spectrum are positive. Chaos may occur with additional zero exponents of Lyapunov and others.

In accordance to *PQ hypothesis* (pseudohyperbolic or quasiattractor) [4] chaotic attractors can be divided into two groups: *pseudohyperbolic attractors* and *quasiattractors*. The term "pseudohyperbolic" was introduced in [5, 6] for robust attractors which chaotic dynamics persist under small perturbations. In fact, pseudohyperbolicity is a weak version of hyperbolicity for which expansion along each direction in an expanding subspace is replaced by volume-expanding condition. In contrast to hyperbolic attractors pseudohyperbolic ones are not structurally stable. Nevertheless, each orbit of such attractor has positive Lyapunov exponent and this property remains for all close systems. The term *quasiattractor* was introduced by Afraimovich and Shilnikov in [7] for non-robust chaotic attractors. Unlike pseudohyperbolic attractors, quasiattractors either contain stable periodic orbit (with narrow, invisible in numerics, absorbing domains) or such orbits appear under arbitrarily small perturbations (parameter changing). For systems with quasiattractors, one can never be sure that increasing the accuracy or the computation time would not make the maximal Lyapunov exponent vanish. In parameter planes for this systems regions with chaotic dynamics alternate with stability windows with stable periodic orbit.

The very important property of systems with pseudohyperbolic attractors is that the minimal ensemble of such systems interacting via weak coupling also demonstrate pseudohyperbolic attractors [6]. Moreover, since both systems demonstrate chaotic dynamics and the coupling is weak the resulting attractor should be robustly hyperchaotic. In the first part of this paper, we demonstrate this effect on the example of two coupled classical Lorenz systems. It is known that in the parameter space of this system there exist a quite large region corresponding to pseudohyperbolic Lorenz attractor [8-9]. Choosing parameters from this region, using diagrams of Lyapunov exponents we show that for weak couplings this ensemble indeed demonstrate robust hyperchaotic dynamics i.e. its strange attractor is a good candidate to be pseudohyperbolic and hyperchaotic. To the best of our knowledge, there are no examples of systems demonstrating pseudohyperbolic (but not structurally stable) hyperchaotic attractors. Moreover, we demonstrate for the minimal ensemble of Lorenz systems another interesting phenomenon. We show, that for some detuning parameter the ensemble can demonstrate hyperchaotc attractor with three positive Lyapunov exponents. It means that the dynamics in this system can become more complicate only due to coupling.

In the second part of this paper we study a system of two dissipatively coupled generators of quasiperiodic oscillations. The base model is also three-dimensional [10]. In the autonomous case, the system demonstrates two-frequency quasiperiodic regimes which are characterized by two zero Lyapunov exponents. We demonstrate, that the interaction of two such systems may cause to both chaotic and hyperchaotic oscillations with two positive Lyapunov exponents [11]. Moreover, we demonstrate that the observed chaotic regimes in this model cannot be robust. These regimes are associated with quasiattractors. For this model a scenario of the formation of hyperchaotic attractors is briefly described.

## II. Hyperchaotic pseudohyperbolic attractors: Coupled Lorenz systems

### A. Mathematicla model of coupled Lorenz systems

Classical Lorenz system

$$\dot{x} = \sigma(y - x),$$
$$\dot{y} = x(r - z) - y,$$
$$\dot{z} = xy - bz,$$

where $x$, $y$, and $z$ are dynamical variables, σ, $b$, and $r$ are parameters is one of the basic dissipative systems demonstrating chaotic dynamics. Strange attractors in this system were discovered yet in 1963 by E. Lorenz [12]. However, the first theories of Lorenz attractors were proposed much later. Among them the theory developed by Afraimovich-Bykov-Shiknikov [13, 14] is the most completed and applied. It provides a set of effectively verifiable conditions. If an attractor satisfies these conditions (this fact can be checked numerically) one can conclude that this attractor is indeed singular-hyperbolic and, thus, pseudohyperbolic.

Afraimovich-Bykov-Shiknikov conditions for classical Lorenz attractor, when

$$\sigma=10,\ b=8/3,\ r=28, \tag{1}$$

were successfully verified by Tucker in [8] where it was proved that this attractor is indeed singular-hyperbolic, i.e. it is robustly chaotic. Pseudohyperbolicity of Lorenz attractors implies a very important conclusion [6]. The ensemble of two identical weak coupled Lorenz systems with a pseudohyperbolic attractor also demonstrate pseudohyperbolic attractors. Since pseudohyperbolicity is the robust property, we can expect that two coupled Lorenz systems with not identical but close parameters should demonstrate pseudohyperbolic hyperchaotic attractors. In this section we try to demonstrate this property using charts of Lyapunov exponents.

Two coupled Lorenz systems can be written as the next system of differential equations:

$$\begin{aligned}
\dot{x}_1 &= \sigma(y_1 - x_1) + M_C(x_2 - x_1),\\
\dot{y}_1 &= rx_1 - x_1 z_1 - y_1,\\
\dot{z}_1 &= x_1 y_1 - bz_1,\\
\dot{x}_2 &= \sigma(y_2 - x_2) + M_C(x_1 - x_2),\\
\dot{y}_2 &= (r+\Delta)x_2 - x_2 z_2 - y_2,\\
\dot{z}_2 &= x_2 y_2 - bz_2.
\end{aligned} \tag{2}$$

Here $M_C$ is a coupling strength, and Δ is detuning in the parameter $r$ between oscillators. We fix parameters σ, $b$, $r$ in accordance with (1), i.e. both uncoupled subsystems demonstrate pseudohyperbolic Lorenz attractor. Increasing of detuning Δ, lead to changing of parameter $r$ in the second oscillator. But it will stay in the area of pseudohyperbolic chaos.

### B. Numerical Simulations

Let us analyse dynamics of model (2) with the spectrum of Lyapunov exponents, which was calculated by algorithm presented in [15]. In the parameter plane (Δ, $M_C$) two type of hyperchaos were detected: with two (blue color) and three (red color) positive Lyapunov exponents.

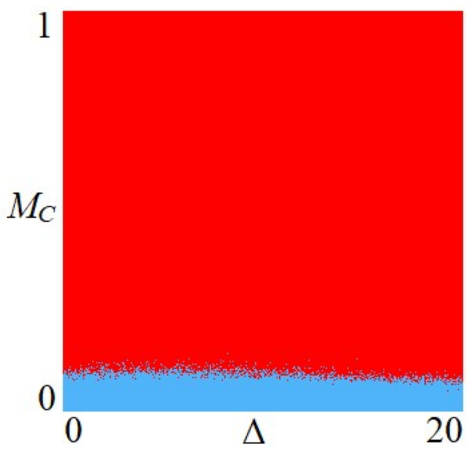


Fig. 1. Chart of Lyqapunov exponents for coupled Lorenz systems (1). σ=10, $r$=28, $b$=8/3.

The theory of Turaev and Shilnikov [6] works at small coupling strength $M_C$ and detuning parameter Δ. For such parameter we observe hyperchaotic attractors with two positive Lyapunov exponents. The absence of stability windows inside the corresponding regions confirm hypothesis that these attractors can be pseudohyperbolic.

Increasing strength of coupling $M_C$, one can observe another type of hyperchaotic attractors with three positive Lyapunov exponents. As in the previous case, there are no stability widows in the corresponding region of Lyapunov diagram. Since Turaev, Shilnikov theory works only for small couplings we can only assume that this hyperchaos can be also pseudohyperbolic. The graph of four maximal exponents presented in Fig. 2 shows that these exponents smoothly depend on parameter $M_C$ which confirms our assumption.

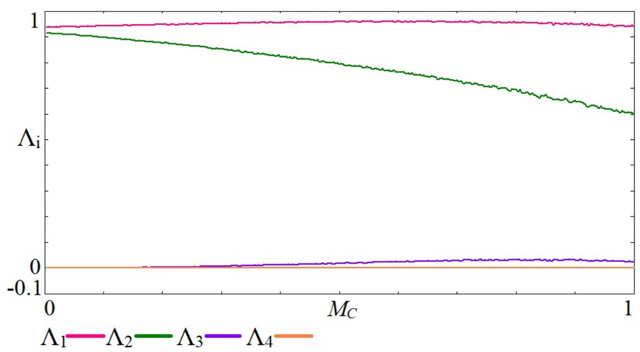


Fig. 2. Plots of the four largest Lyapunov exponents for Lorenz systems (1). σ=10, $r$=28, $b$=8/3, Δ=1.

In future papers, we plan to verify pseudohyperbolicity of the observed hyperchaotic attractors employing the method of verification of continuity for directions-contracting and volume-expanding subspaces proposed in [9] and method based on the computation of angles between these subspaces proposed in [16, 17].

## III. Hyperchaotic quasiattractors: Coupled quasiperiodic generators

### A. Mathematical model

As an example of model with quasiattractor we will consider generator of quasiperiodic oscillations [10]. This generator can demonstrate autonomous quasiperiodic oscillation. Two coupled generators will be written in the next form:

$$\begin{aligned}
&\dot{x}_1 = y_1,\\
&\dot{y}_1 = (\lambda + z_1 + x_1{}^2 - \beta x_1{}^4)y_1 - \omega_0^2 x_1 + M_C(y_2 - y_1),\\
&\dot{z}_1 = b(\varepsilon - z_1) - k y_1^2,\\
&\dot{x}_2 = y_2,\\
&\dot{y}_2 = (\lambda + z_2 + x_2{}^2 - \beta x_2{}^4)y_2 - (\omega_0 + \Delta)^2 x_2 + M_C(y_1 - y_2),\\
&\dot{z}_2 = b(\varepsilon - z_2) - k y_2^2,
\end{aligned} \tag{3}$$

where $x_1$, $y_1$, $z_1$, $x_2$, $y_2$, $z_2$ are dynamical variables, λ is parameter of excitation, $\omega_0$, μ are parameters, responded for base incommeasure frequencies of each oscillator, Δ is frequency detuning between oscillators, $M_C$ is coupling strength. In [11] various types of dynamical behavior were demonstrated, including chaos, chaos with additional zero Lyapunov exponent in the spectrum and hyperchaos. Let us consider in detail features of hyperchaos.

### B. *Numerical simulations*

As a main tool for analysis we will use spectrum of Lyapunov exponents. In Fig.3 a fragment of chart of dynamical regimes based on the spectrum Lyapunov exponents demonstrating chaotic behavior is shown. Red and yellow colors correspond to periodic and quasiperiodic dynamics. Domains of chaos and hyperchaos are marked with white and grey colors, accordingly. Quasi-periodic oscillation are base for development of chaos, and also for periodic oscillations occurring as a result of synchronization. Structure of the chart is complex in domain of quasiperiodicity there are different tongues of periodicity. Intersection of tongues and different bifurcations inside them lead to chaos formation. Such complex structure of parameter plane is character for quasiattractors, which are sensitive for changing parameters.

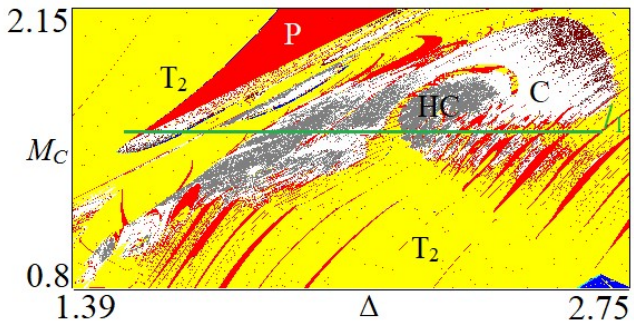


Fig. 3. Chart of dynamical regimes for coupled quasiperiodic generators (3) in the domain of chaos. λ=-1.0, β=1/18, $b$=1, ε=4, $k$=0.02.

For further analysis we fixed coupling strength $M_C$=1.57, in corresponding with line $l_1$ in Fig.3, and varied frequency detuning. In Fig.4a plots of the fours largest Lyapunov exponents on the frequency detuning is presented. Intervals of hyperchaos were denoted with grey color. Figure 4b represents zoomed fragment of plot, where weak hyperchaos is shown.

In Fig.4a one can distinguish two big intervals of hyperchaos. Hyperchaotic attractors for different domains can have various structure. Let us consider different chaotic attractors with phase portraits in Poincarè section, which are presented in Fig.5. As Poincarè cross-section we use hyperplane $y_1$=0.

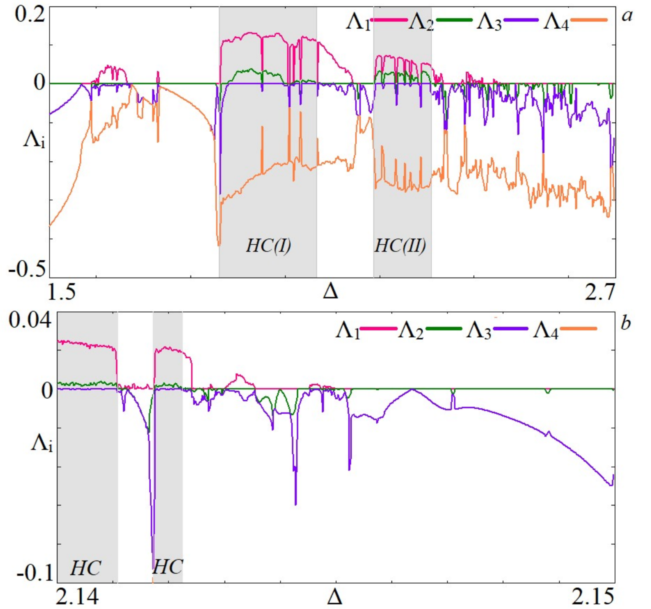


Fig. 4. Plots of the four largest Lyapunov exponents for coupled quasiperiodic generators (3). λ=-1.0, β=1/18, $b$=1, ε=4, $k$=0.02, $M_C$=1.57.

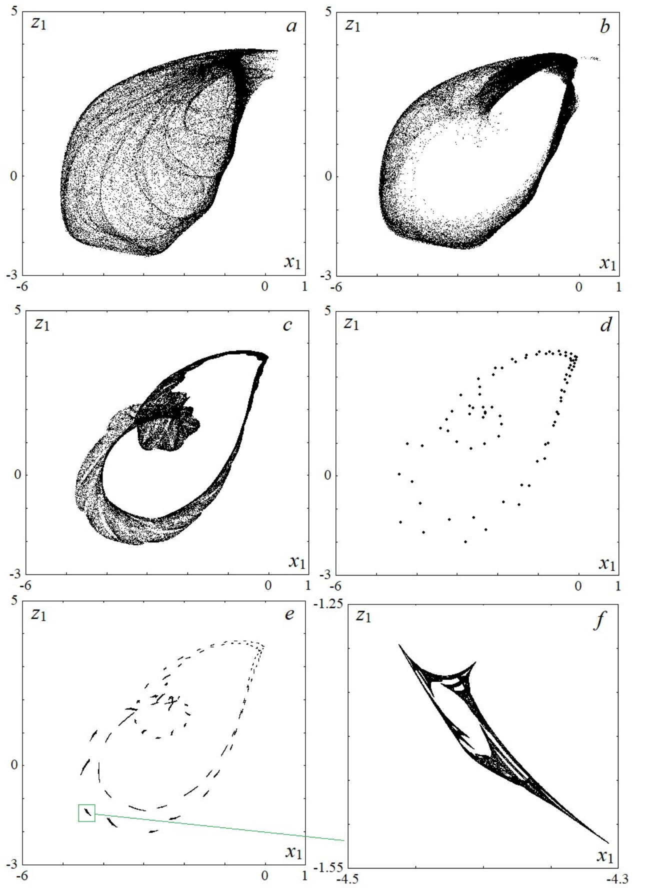


Fig. 5. Phase portraits in Poincarè section by hypersurface $y_1$=0 for coupled quasiperiodic generators (3). λ=-1.0, β=1/18, $b$=1, ε=4, $k$=0.02, $M_C$=1.57; a) Δ=1.8; b) Δ=2.2; c) Δ=2.141; d) Δ=2.1414; e) Δ=2.1411.

Chaotic attractors in Fig.5a-5c correspond to hyperchaos. For Δ=1.88 (domain I) the two largest Lyapunov exponents equal $\Lambda_1$=0.114, $\Lambda_2$=0.0044 (Fig.5a). For another domain (II) of hyperchaos, Δ=2.2 values of Lyapunov exponent are smaller and equal $\Lambda_1$= 0.07, $\Lambda_2$= 0.032 (Fig. 5b). These attractors are correspond to destruction of two-frequency torus. But inside area of hyperchaos there is weak hyperchaos, which characterized by smaller values of positive Lyapunov exponents. In Fig.5c example of weak hyperchaotic attractor is presented. Two largest Lyapunov exponents of this attractor equal $\Lambda_1$= 0.02, $\Lambda_2$= 0.0026. In the area of weak hyperchaos it is possible to observe chaos, occuried via secondary Neimark-Sacker bifurcation. In Fig. 6d example of limit cycle in Poincarè section is shown. In Fig. 5e chaos occurring on the base of this cycle is presented. In zoomed fragment of attractor (Fig.5f) the structure of the cycle persists.

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