Hyperbolic Chaos and Quasiperiodic Dynamics in Experimental Nonautonomous Systems of Coupled Oscillators

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Abstract— In the present paper in radiophysical experiments two different ways of destruction of invariant torus are shown. In the first case formation of chaotic attractor from four-frequency quasiperiodic torus involving torus doubling bifurcations is presented. In the second transformation of two-frequency torus into hyperbolic chaotic attractor was observed.

1. INTRODUCTION

The first view on the theory of turbulence has been presented by Landau and Hopf and based on the idea of sequential including of a growing number of noninteracting quasiperiodic motions [1, 2]. However, the results obtained in the theory of dynamical chaos and bifurcation have not proved this hypothesis. In the context of the onset of turbulence, possible occurrence of hyperbolic chaotic attractors and invariant torus was discussed four decades ago by Ruelle and Takens [3, 4], but they did not consider any concrete example of dynamical equations demonstrating the phenomenon. Shil'nikov and Turaev have advanced a scenario of formation of the Smale-Williams attractor in a kind of the so-called blue sky catastrophe [5]; an explicit example was suggested later [6]. Physically implementable systems with hyperbolic attractors have been discovered only recently in [7]. The scenarios of development of hyperbolic chaos is one of the interesting question. In [6] was shown the birth of hyperbolic chaos through bifurcation associated with catastrophe of blue sky. In [8] was described one more scenario for the onset or destruction of the Smale-Williams attractor is associated with a collision of two chaotic invariant sets, an attracting solenoid and a nonattracting one. In [9] two scenario of development of hyperbolic chaos after Feigenbaum cascade of period-doubling bifurcations and through destruction of invariant curve are demonstrated.

In the present paper using radiophysical experiment we made comparison analysis of formation chaotic dynamics in non-autonomous systems. We considered two different circuits, where were observed two different transformations of quasiperiodic dynamics. In Section 2 the first case, when we can see doubling of four-frequency torus and transition to chaos is presented. In Section 3 the transition from quasiperiodic dynamic to hyperbolic chaos is shown.

2. TRANSITION FROM QUASIPERIODICITY TO CHAOS IN COUPLED NON-AUTONOMOUS RLD-CONTOURS

Firstly, we would like to consider more simple system, which can demonstrate transition from high-dimensional torus to chaos. We consider system, consisting of two non-linear oscillating contours $R_5L_1D_1$ and $R_8L_2D_2$ (see Fig. 1). Non-linearity of oscillation circuits is due to nonlinearity of current-capacitance and current-voltage characteristics of diodes included in the circuit. Oscillating circuit are excited by the biharmonic signals due operational amplifiers (OA_1, OA_2). The first nonlinear oscillatory $R_5L_1D_1$-circuit is excited by the signal representing the sum of two harmonic components $A_1 \sin(\omega_1 t) + A_2 \sin(\omega_2 t)$, and the second $R_8L_2D_2$-circuit is excited by the signal $A_3 \sin(\omega_3 t) + A_4 \sin(\omega_4 t)$. In [10, 11] one can find some features of the quasiperiodically forced system. In our the experiment, all the frequencies of the harmonic components are in an irrational ratio (four-frequency excitation). In this case, the dynamics of the coupled system is invariant with respect to the phase difference or phase of harmonic components of the external force, and a four-dimensional torus is attractor. Research carried out for the dissipative coupling, the coupling element served as a variable resistor. The experiment was carried out the analysis of time realizations of voltage on the diode, the current in the diodes, projections of attractors on different phase plane, attractors in the multi-dimensional Poincaré section and Fourier spectrum analysis.

How was mentioned above values of frequencies of external force we fixed irrational. Frequencies were chosen in the next way: ω_1 was close to linear resonance frequency of oscillatory circuit and equals 45 kHz, another frequencies were fixed in the next ratio:

$$\frac{\omega_2}{\omega_1} = \frac{\sqrt{5}-1}{2}\frac{7}{5}, \quad \frac{\omega_3}{\omega_1} = \frac{\sqrt[3]{13}-2}{2}, \quad \frac{\omega_4}{\omega_1} = \frac{\sqrt[3]{15}-2}{2}. \tag{1}$$

In the experiments amplitudes A_3, A_4 were fixed, all transformations of dynamic of the system was observed at varying of amplitudes A_1 and A_2. We put $A_3 = A_4 = 0.05V$. In this case in phase space we can observe four-frequency quasiperiodic oscillations. At varying amplitudes A_1 and A_2 we can see doubling of four-frequency torus and it destruction, with transition to chaos. For studying regimes of four-frequency torus we use the methodic of multi-fold Poincaré section [12]. In order to visualize invariant curve of four-frequency torus we have to realize triple Poincaré section. As the section variable we use second, third and forth frequencies ($\omega_2 t$, $\omega_3 t$, and $\omega_4 t$ correspondingly). Character of oscillations was analyzed by projection of phase portrait on plane ($A_1 \sin(\omega_1 t)$, i_D), where i_D is intensity of diode current and $A_1 \sin(\omega_1 t)$ represents phase of corresponding harmonic of external force. In Fig. 2 one can find examples of attractors in triple Poincaré section in projection on this plane. We use projection on this plane due to the fact that this plane shows the dependence of behavior of dynamical variable on the phase of external action. If we observe smooth invariant curve then we can talk about stability of dynamics. Appearance of breaks and blurring on phase portrait indicates local instability and in general the presence of phase-dependent dynamics.

A phase portrait in Fig. 2(a) corresponds to smooth doubled four-frequency torus. Phase por-

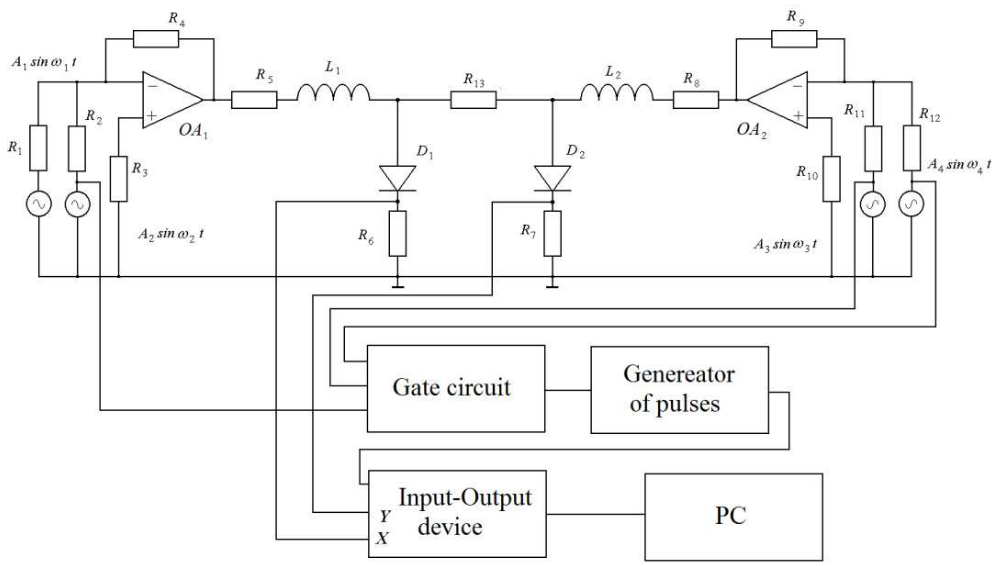

Figure 1. Scheme of electronic circuit of coupled biharmonicaly excited RLD-contours.

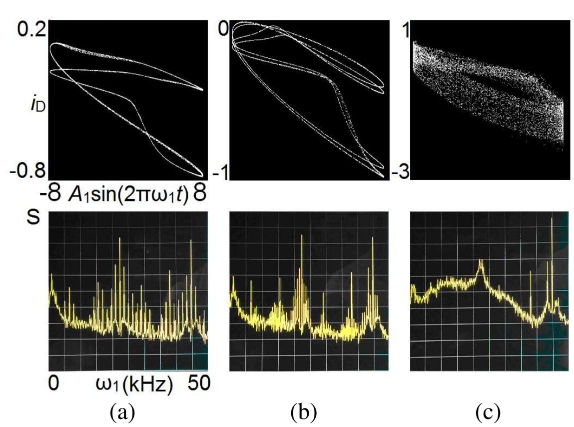

Figure 2. Illustration of transition from four-frequency quasiperiodicoscillations to chaos. Two-dimensional projections of phase portraits in triple stroboscopic section and the Fourier power spectra are shown.

trait in Fig. 2(b) corresponds to quadruple four-frequency torus. In Fig. 2(c) chaotic oscillations occurring via torus break down are illustrated. In Figs. 2(a)–(c), in bottom, Fourier spectra are shown for four-frequency doubled and quadruple torus and for chaotic oscillations. How one can see in Figs. 2(a) and (b) there are discrete power spectrum. When invariant curve is destroying we can see continuous spectrum. Low-frequency part of power spectrum goes upper. But we can clear distinguish main components of high-frequency part of spectrum.

3. TRANSITION FROM QUASIPERIODICITY TO HYPERBOLIC CHAOS

Now let us consider the system with hyperbolic chaos. The first real model with hyperbolic chaos was suggested in [7] and can be described by the next mathematical equations:

$$\begin{aligned} \ddot{x} - \left(h_1 + A_1 \cos\left(\frac{2\pi t}{T}\right) - x^2\right)\dot{x} + \omega_0^2 x = \varepsilon_1 y \cos\omega t, \\ \ddot{y} - \left(h_2 + A_2 \cos\left(\frac{2\pi t}{T}\right) - y^2\right)\dot{y} + (2\omega_0)^2 x = \varepsilon_2 x^2. \end{aligned} \tag{2}$$

The model was composed of two two subsystems, the van der Pol oscillators with characteristic frequencies ω_0 and $2\omega_0$. The parameter responsible for the limit cycle birth is forced to vary slowly in opposite phases in one and the other subsystem with period T and amplitudes A_1 and A_2 around the mean values h_1 and h_2. Due to the modulation, the subsystems become active or suppressed turn by turn. The coupling of subsystems is characterized by coefficients ε_1 and ε_2.

Schematic diagram of the circuit is shown in Fig. 3. It is composed of two partial self-oscillatory subsystems containing the inductor-capacitor circuits L_1C_1 and L_2C_2, supplemented by negative resistance elements arranged with the operational amplifiers DA_6 and DA_{10}. As dynamical variables in the experiment we use voltages U_1 and U_2 across the capacitors C_1 and C_2. Quadratic coupling responsible for the transfer of excitation from the first oscillator to the second is provided by a multiplier DA_8 and an operational amplifier DA_9. The circuit containing the multiplier DA_4 and the operational amplifier DA_5 provides the effect of the second oscillator onto the first one via the product of the signal with the reference sinusoidal signal from generator of frequency close to the natural frequency of the first oscillator. Modulation of the activity in the subsystems in counterphase is provided by variation of resistances of the field-effect transistors T_1 and T_2 controlled by signal from the frequency divider producing a sinusoidal voltage of period 8 times larger than the period of the reference signal. The amplitudes of parameter modulation in the subsystems can be regulated by the potentiometers R_{18} and R_{19}. The mean activity parameters of the subsystems are controlled by the potentiometers R_{15} and R_{25} determining a level of the resistive energy loss in the oscillatory circuits L_1C_1 and L_2C_2.

In the experiment we observed projection of phase portrait on the plane "voltage-current" in stroboscopic section corresponding to processing of a sequence of states followed by the modulation period and power spectrum, iteration diagram for the phases on successive modulation periods, and power Fourier spectrums. The phases were computed from the recorded data defined by the relation $\varphi = \arg[c(U_1/a + \dot{U}_1/b) + id(U_1/a + \dot{U}_1/b)$ with empirically chosen constants $a = 2.2$, $b = 0.7$, $c = 0.7$, $d = 1.8$. The iteration diagram of Smale-Williams solenoid for phases of oscillations on successive stages of activity should correspond to the topology of the Bernoulli map. For hyperbolic chaos Fourier power spectrum is continuous and is characterized by roughly constant spectral density in some frequency range.

In Fig. 4 stroboscopic phase portraits, iteration phases diagrams, and Fourier power spectra illustrating transition to hyperbolic attractor from quasi-periodic dynamics are shown. In diagram 4(a) one can see an invariant curve representing the quasi-periodic attractor. The iteration map has in this case one monotonous branch. Panel 4(b) shows formation of a singularity that is irremovable by variable changes: the curve corresponding to the phase map tends to undergo a break there and to transform to Bernoulli-type map with two branches. Diagram 4(c) shows situation where hyperbolic chaos is realized. The phase portrait for this case should be Smale-Williams solinoid, but solenoid due to a strong compression in the transverse direction is difficult to distinguish from the invariant curve. Phase map looks very close to Bernoulli map. The Fourier power spectrum in the course of these transformations tends to smooth out.

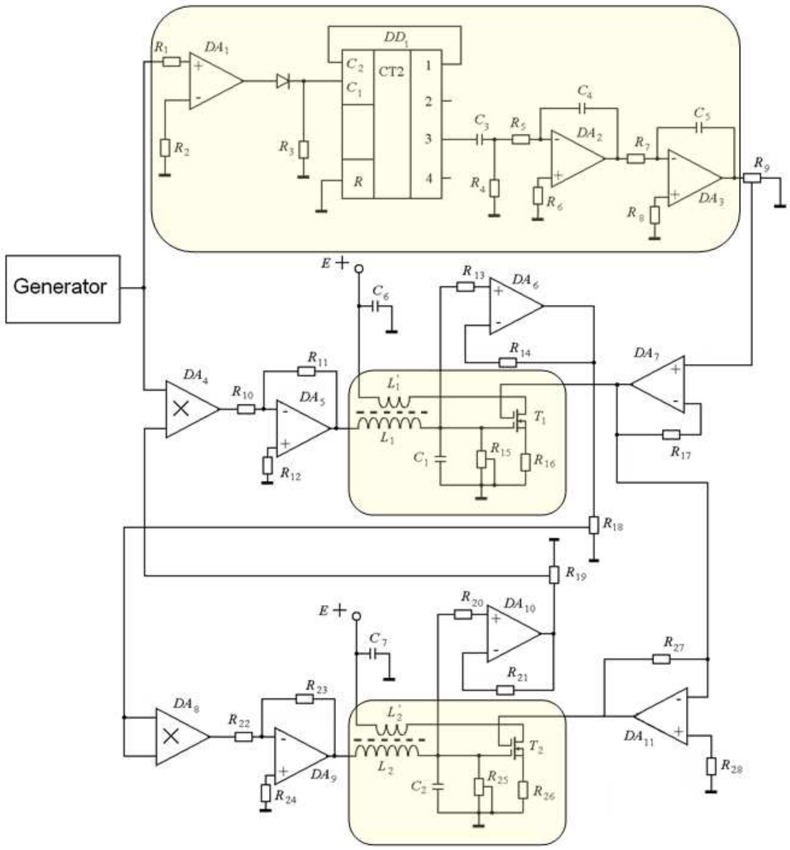

Figure 3. Scheme of electronic circuit of generator of hyperbolic chaos.

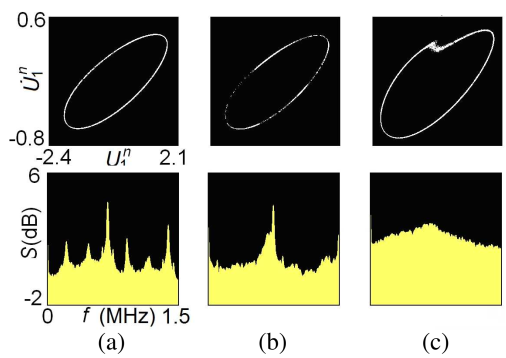

Figure 4. Illustration of transition from quasiperiodicity to hyperbolic chaos. Stroboscopic portraits of attractors, plots of iteration maps for phases at successive stages of activity and the Fourier power spectra are shown.

4. CONCLUSION

Thus, in the present paper we consider two ways of destruction torus and formation of complex chaotic dynamics. In the first case we observe transition from four-frequency quasiperiodic dynamics to chaos. And we can see that part of Fourier spectrum became continuous, but on the other part we can distinguish main frequency components. In the second case we observe transition from quasiperiodic oscillations to hyperbolic chaos, where all power spectrum is continuous.

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