Smale-Williams attractor in a system of alternately oscillating coupled Froude pendulums

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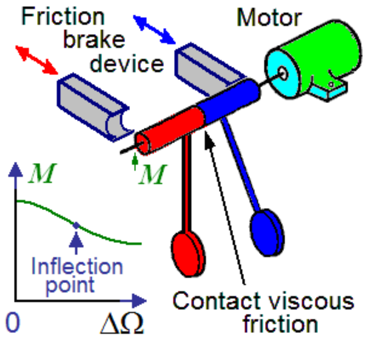

Fig. 1. A system of two Froude pendulums: the motion of one and the other pendulum is decelerated alternately by attaching a brake shoe providing suppression of the self-oscillations

Uniformly hyperbolic attractors are characterized by roughness, or structural stability, by virtue of which the generated chaos retains its features under small variations of the system parameters. Obviously, this property is desirable for any plausible application of chaos. From the point of view of clarity, among possible examples of hyperbolic chaos we should outline systems of mechanical nature as they are easily perceived and interpreted in a frame of our everyday experience [1]. Here we propose a mechanical system based on two Froude pendulums placed on a common shaft rotating at a constant angular velocity undergoing alternate brake by periodic application of frictional forces (Fig. 1). The pendulums are weakly connected with each other by viscous friction. Denoting the angular coordinate of the first and the second pendulum as x and y, we write down the equations

$$\begin{aligned}
&\ddot{x} = [a - d(t) - b\dot{x}^2]\dot{x} - \sin x + \mu + \varepsilon(\dot{y} - \dot{x}),\\
&\ddot{y} = [a - d(t + T/2) - b\dot{y}^2]\dot{y} - \sin y + \mu + \varepsilon(\dot{x} - \dot{y}),\\
&d(t) = \begin{cases} 0, \ t < T_0, \\ D, \ T_0 < t < \tfrac{1}{2}T, \quad d(t+T) = d(t). \\ 0, \ T/2 < t < \tfrac{1}{2}T. \end{cases}
\end{aligned} \qquad (1)$$

Parameters are assigned as follows:

$$\begin{aligned}
a = 0.36,\ b = 0.16,\ \mu = 0.087,\ \varepsilon = 0.0003,\\
D = 0.8,\ T = 250,\ T_0 = T/4
\end{aligned} \qquad (2)$$

To explain the functioning of the system, let's start with a situation when one pendulum exhibits self-oscillation, and the second is braked. The parameters are chosen in such way that at the end of the previous stage, the basic frequency of the developed self-oscillatory mode is half of frequency of small oscillations. Therefore, when the brake of the second pendulum is switched off, it will begin to swing in a resonant manner due to the action of the second harmonic from the first pendulum, and the phase of the oscillations that arise will correspond to the doubled phase of the first pendulum. As a result, when the second pendulum approaches the sustained self-oscillatory state, its phase appears to be doubled in comparison with the initial phase of the first pendulum. Further, the first pendulum undergoes braking, and at the end of this stage, its oscillations will be stimulated by the second harmonic from the second pendulum, and so on.

Fig. 2 shows a diagram for the phases determined at the end parts of successive stages of excitation of one of the pendulums, obtained in numerical calculations for a sufficiently large number of the modulation periods. As can be seen, the mapping for the phase in the topological sense is a quadruple expanding circle map. As volume contraction takes place along the remaining directions in the state space of the system, such transformation of the phases correspond to occurrence of the Smale-Williams solenoid as attractor of the Poincaré map.

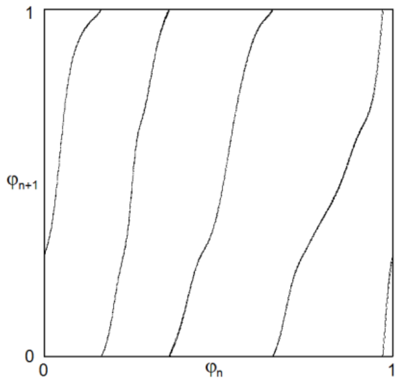

Fig. 2. A diagram illustrating transformation of the phases in successive stages of activity of one of the pendulums

According to calculations, the Lyapunov exponents for the attractor of the Poincaré map are $\Lambda_1 = 1.30$, $\Lambda_2 = -11.8$, $\Lambda_3 = -18.9$, $\Lambda_4 = -41.5$. The presence of a positive exponent indicates chaotic nature of the dynamics. Its value is close to a constant equal to ln 4=1.386, which agrees with the approximate description of the phase variable by the expanding circle map. An estimate of the Kaplan – Yorke dimension gives $D = 1 + \Lambda_1 / |\Lambda_2| \approx 1.11$.

The hyperbolicity of the chaotic attractor was tested with the help of criterion based on analysis of angles of intersection of stable and unstable invariant subspaces of small perturbation vectors with verification of the absence of tangencies between these subspaces.

Fig. 3 shows the Lyapunov exponents versus parameter *a*. As can be seen, in a neighborhood of the point corresponding to (2) there is a region where the positive Lyapunov exponent remains close to ln 4 and, as can be verified, correspondence in the topological sense with the Bernoulli map for the iteration diagrams for the phases takes place. Appearance of significant deviations of the Lyapunov exponent from ln 4, including drops to negative values ("windows of regularity") indicates violation of the hyperbolicity. In accordance with the expected structural stability, the same type of chaotic attractor should persist under variation of system parameters in some region, and this is indeed the case.

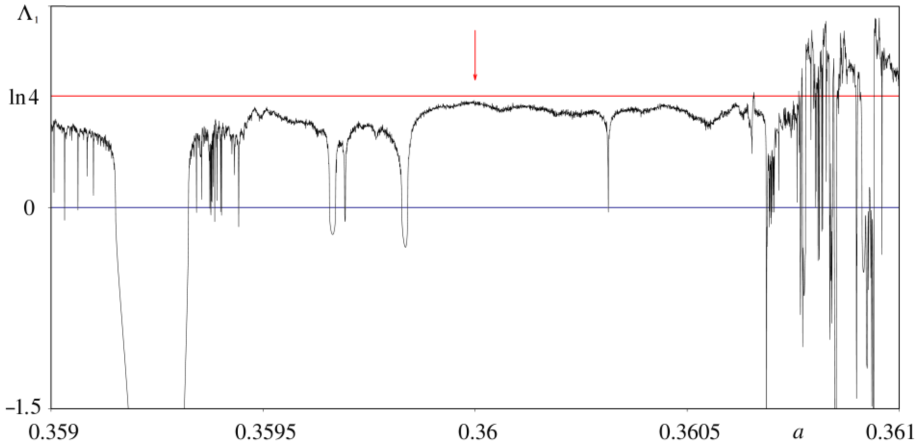

Fig. 3. Graph for the largest Lyapunov exponent of the Poincaré map versus the parameter *a*. An arrow indicates situation corresponding to the Smale-Williams attractor with parameters (2)

The work was supported by Russian Science Foundation, grant No. 15-12-20035.

References

[1] Kuznetsov S.P., Kruglov V.P. *On Some Simple Examples of Mechanical Systems with Hyperbolic Chaos*. Proc. Steklov Inst. of Mathematics, 2017, vol. 297, pp. 208–234.

[2] Kuznetsov S.P. *Dynamical chaos and uniformly hyperbolic attractors: from mathematics to physics*. Physics-Uspekhi, 2011, vol. 54, pp. 119-144.