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Hamiltonization and Separation of Variables for a Chaplygin Ball on a Rotating Plane

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Abstract—We discuss a non-Hamiltonian vector field appearing in considering the partial motion of a Chaplygin ball rolling on a horizontal plane which rotates with constant angular velocity. In two partial cases this vector field is expressed via Hamiltonian vector fields using a nonalgebraic deformation of the canonical Poisson bivector on $e^*(3)$. For the symmetric ball we also calculate variables of separation, compatible Poisson brackets, the algebra of Haantjes operators and 2×2 Lax matrices.

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1. INTRODUCTION

The theory of integrable systems appeared as a family of mathematical methods which can be applied to find exact solutions of dynamical systems. The main motivation was to determine the scope of mathematical models of real physical processes. Explicit solutions allow us to test analytical and numerical schemes applied to a given mathematical model and to choose a reasonable approximation to solutions of the model.

In classical mechanics reparameterization of time is a standard tool for construction of explicit solutions of equations of motion according to Kepler [25], Jacobi, Maupertuis [28], Weierstrass [48], for discussions, see [27, 31, 32] and references therein. In nonholonomic mechanics Appel [1] and Chaplygin [11, 12] also used reparameterization of time for integrating certain nonholonomic systems with two degrees of freedom. In modern nonholomic mechanics we have many equations of motion taking a Hamiltonian form after suitable symmetry reduction and time reparameterization, see e. g. [2, 4, 5, 7, 8, 10, 15, 17–20, 24, 29] and references therein. Unfortunately, in most of these publications the authors discuss only the form of equations of motion instead of exact solutions of these equations.

The main aim of this note is to compare Hamiltonization and the modern method of separation of variables embedding elements of artificial intelligence such as machine learning and deep learning. Harnessing of modern computational abilities for studying integrable systems is naturally placed as a prominent avenue in contemporary classical and quantum mechanics. For instance, this allows us to automate validation of mathematical models of real physical process, see e. g. one of a collection of papers about machine learning in physics [13]. In classical mechanics computer modeling currently consists not only of approximate numerical calculations and visualization, but also of algorithmic reduction to quadratures [21, 22, 34, 47].

In this note we take a non-Hamiltonian vector field from the recent paper by Borisov, Mamaev and Bizyaev [6] and obtain Poisson bivectors, variables of separation and quadratures using only modern computer software. Our main motivation is to enlarge the known collection of

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deformations of canonical Poisson bivectors appearing both in Hamiltonian and non-Hamiltonian dynamics [3, 10, 21, 22], because sufficiently large data sets are an integral part of the field of machine learning.

1.1. Hamiltonization

In 1903 Chaplygin found quadratures in the mathematical model of an inhomogeneous balanced ball rolling without slipping on a horizontal plane [11]. These quadratures for the non-Hamiltonian model can be resolved after reparameterization of time similar to the well-known Weierstrass reparameterization of time in Hamiltonian mechanics [48]. Equations of motion in Hamiltonian form were not written until 2001 [9].

In 1911 Chaplygin introduced the reducing multiplier method and applied this method to integrate what would later become known as the Chaplygin sleigh [12]. He also remarked that his general procedure (using the reducing multiplier) for integrating certain nonholonomic systems with two degrees of freedom was “interesting from a theoretical standpoint as a direct extension of the Jacobi method to simple nonholonomic systems”.

The first part of Chaplygin’s theorem states that in the case of nonholonomic systems in two generalized coordinates $(q_1, q_2) \in Q$ possessing an invariant measure with density $N(q_1, q_2)$ equations of motion may be written in Hamiltonian form after the time reparameterization $\mathrm{d}\tau = N\mathrm{d}t$. The second part of this theorem says that if a nonholonomic system can be written in Hamiltonian form after time reparameterization $\mathrm{d}\tau = f(q_1, q_2)\mathrm{d}t$, then the original system has an invariant measure. Both functions N and f are known as the reducing multiplier, or simply the multiplier, see [19, 29] for historical remarks and discussions. The reduced phase space of Chaplygin’s system is isomorphic to the cotangent bundle T^*Q where reduced equations may be formulated as

$$\frac{\mathrm{d}z}{\mathrm{d}t} = N(q_1, q_2)\,\mathcal{P}_f\,\mathrm{d}H, \tag{1.1}$$

where $z = (q_1, q_2, p_1, p_2)$. Roughly speaking, Chaplygin considered conformally Hamiltonian vector fields Z associated with Turiel type deformations $\mathcal{P}_f$ of the canonical Poisson bivector on T^*Q [36, 45]. A discussion of symplectic and nonsymplectic diffeomorphisms associated with a conformally Hamiltonian vector field in Hamiltonian mechanics can be found in [27].

After introduction of Chaplygin’s theorem, subsequent research on the theorem resulted in, among other things, an extension to the quasi-coordinate context, a study of the geometry behind the theorem, discoveries of isomorphisms between nonholonomic systems, an example of a system in higher dimensions which was Hamiltonizable through a similar time reparameterization, determination of necessary conditions for Hamiltonization, the study of deformations of Poisson structures in nonholonomic systems, etc. More detailed discussions of various modern methods of Hamiltonization may be found in [2, 4, 5, 8, 10, 15, 17–20, 29] and references therein.

The main advantage of any type of Hamiltonization is that we identify the phase space with a Poisson or symplectic manifold. It allows us to study non-Hamiltonian systems using the standard machinery of Poisson or symplectic geometry.

The main disadvantage of any type of Hamiltonization is that Hamiltonization only works for a narrow class of nonholonomic systems; even if it works, the reduction to quadratures is not transparent. Another disadvantage is that we do not have an algorithmic procedure for constructing a cotangent bundle T^*Q with generalized coordinates or quasi-coordinates starting with original physical variables.

1.2. Separation of Variables

In [12] Chaplygin discussed a direct extension of the Jacobi method for simple nonholonomic systems. In fact, we do not need any extensions because the original geometric version of the Jacobi methods is independent of time and, therefore, it is directly applies both for Hamiltonian and non-Hamiltonian systems.

In 1837 Jacobi proved that m solutions $h_1 = H_1(x,y),\dots,h_m = H_m(x,y)$ of m separation relations

$$\Phi_j(x_j, y_j, h_1, \dots, h_m) = 0, \quad j = 1, \dots, m, \quad \det \left[\frac{\partial \Phi_j}{\partial h_k}\right] \neq 0, \tag{1.2}$$

are in involution with respect to the Poisson bracket

$$\{H_i, H_j\}_f = 0\,, \qquad i, j = 1, \dots, m,$$

defined by the Poisson bivector

$$\mathcal{P}_f = \sum_{j=1}^{m} f_j(x_j, y_j) \left(\frac{\partial}{\partial x_j} \wedge \frac{\partial}{\partial y_j}\right), \tag{1.3}$$

where $f_j(x_j, y_j)$ are arbitrary functions. Equations (1.2) define curves $X_1, \dots, X_m$ on a projective plane depending on m parameters $h_1, \dots, h_m$, so that the common level surface of functions $H_i(x, y) = h_i$ is a product of these plane curves

$$\mathcal{M}: \qquad X_1 \times X_2 \times \cdots X_m.$$

If $\mathcal{M}$ is a regular Lagrangian submanifold on phase space, we have a completely integrable system, but in the generic case $\mathcal{M}$ is a product of plane curves only.

Realizations of the common level set $\mathcal{M}$ as a product of curves is independent of parameterization of trajectories living on $\mathcal{M}$, i. e., independent of time and of the form of equations of motion. It is a pure geometric fact. We can find this realization without reduction of equations of motion to Hamiltonian form.

Compatible Poisson bivectors $\mathcal{P}_f$ and $\mathcal{P}_g$ associated with two sets of functions $f_1, \dots, f_m$ and $g_1, \dots, g_m$ are related to each other

$$\mathcal{P}_g = N\mathcal{P}_f$$

by the formal recursion operator

$$N = \mathcal{P}_g \mathcal{P}_f^{-1} = \sum_{j=1}^{m} g_j f_j^{-1} L_j, \tag{1.4}$$

where

$$L_j = \frac{\partial}{\partial x_j} \otimes dx_j + \frac{\partial}{\partial y_j} \otimes dy_j$$

form the so-called algebra of Haantjes operators with vanishing Nijenhuis torsion.

Functions $H_1, \dots, H_m$ and compatible Poisson bivectors $\mathcal{P}_f$ and $\mathcal{P}_g$ satisfy the equation

$$\mathcal{P}_g \mathbf{dH} = \mathcal{P}_f \mathbf{F}_{fg} \mathbf{dH}, \qquad \mathbf{dH} = (dH_1, dH_2, \dots, dH_m), \tag{1.5}$$

where $\mathbf{F}_{fg}$ is the so-called control matrix. Eigenvalues of the control matrix are functions on variables of separation, i. e.,

$$\lambda_j = \lambda_j(x_i, y_i).$$

If one of the compatible Poisson bivectors P_f or P_g is nondegenerate, we can calculate variables of separation using the recursion operator N. If both bivectors are degenerate, we can calculate variables of separation using the control matrix $\mathbf{F}$.

Now we are ready to discuss the application of the geometric Jacobi method to integration of the equations of motion

$$\dot{z}_i = Z_i(z_1, \dots, z_n), \qquad i = 1, \dots, n, \tag{1.6}$$

when the number of equations is not very large. First integrals of the vector field Z could be obtained by the brute force method, i. e., by solving the equation

$$\dot{H}(z) = 0, \tag{1.7}$$

using some ansatz for $H(z)$. If we find some first integrals $H_1,\ldots,H_m$, we can try to solve the algebraic equations

$$\sum \mathcal{P}_{f\,ij}\frac{\partial H_k}{\partial z_i}\frac{\partial H_\ell}{\partial z_j}=0,\qquad k,\ell=1,\ldots,m \tag{1.8}$$

with respect to the Poisson bivector $\mathcal{P}_f$. Because a priori these equations have infinitely many solutions of the form (1.3), we have to restrict the space of solutions, i. e., use a suitable ansatz in order to get a partial solution. Instead of (1.8) we can solve the equation

$$Z=\mathrm{f}_1(z)\mathcal{P}_f\,\mathrm{d}H_1+\ldots+\mathrm{f}_m(z)\mathcal{P}_f\,\mathrm{d}H_m. \tag{1.9}$$

It is easy to see that conformally Hamiltonian vector fields (1.1) belong to a very restricted subspace of solutions for this equation.

If we suppose that equations of motion are reducible to quadratures (completely or partially), then there is also another decomposition

$$Z=\mathrm{g}_1(z)\mathcal{P}_g\,\mathrm{d}H_1+\ldots+\mathrm{g}_m(z)\mathcal{P}_g\,\mathrm{d}H_m, \tag{1.10}$$

where $\mathcal{P}_g$ is a Poisson bivector compatible with $\mathcal{P}_f$. A pair of compatible Poisson bivectors determines variables of separation, which allows us to reduce the equations of motion to quadratures. In the generic case it could be a complete or partial separation of variables.

Thus, if the equations of motion (1.6) can be reduced to quadratures in the framework of the Jacobi method, we have an algorithm of reduction:

- solve Eq. (1.7) with respect to functionally independent first integrals $H_1,\ldots,H_m$;
- solve Eqs. (1.9), (1.10) with respect to compatible Poisson bivectors $\mathcal{P}_f$ and $\mathcal{P}_g$;
- find eigenvalues of the corresponding matrix F_{fg};
- calculate quadratures associated with variables of separation.

Some results of application of this algorithm in Hamiltonian and non-Hamiltonian mechanics are given in [21–23, 33–35, 47].

The main advantage of the Jacobi method is that, using variables of separation, we obtain not only quadratures, but also families of compatible Poisson brackets, recursion operators, algebras of Haantjes operators, master symmetries, Lax matrices, new integrable systems and exact discretization of the original equations of motion [42–44].

The main disadvantage of the Jacobi method is that we can solve Eqs. (1.7) and (1.9), (1.10) only using an ansatz for the solution. We hope that the selection of a suitable ansatz can be automated using elements of artificial intelligence.

1.3. Equations of Motion for Chaplygin Ball on a Rotating Plane

Let us consider the following equations of motion:

$$\dot{\boldsymbol\gamma}=\boldsymbol\gamma\times\boldsymbol\omega,\qquad \dot{\mathbf{K}}=\Omega\boldsymbol\gamma\times\mathbf{K}+(\mathbf{K}-d\Omega\boldsymbol\gamma)\times\boldsymbol\omega. \tag{1.11}$$

Here the vectors $\boldsymbol\gamma=(\gamma_1,\gamma_2,\gamma_3)$ and $\mathbf{K}=(K_1,K_2,K_3)$ are variables on the phase space, $x\times y$ is a vector product in $\mathbb{R}^3$, d and Ω are some parameters, $\mathbf{A}=\mathrm{diag}(a_1,a_2,a_3)$ is a diagonal matrix, and

$$\boldsymbol\omega=\mathbf{A}\mathbf{K}+\frac{(\boldsymbol\gamma,\mathbf{A}\mathbf{K})}{d^{-1}-(\boldsymbol\gamma,\mathbf{A}\boldsymbol\gamma)}\mathbf{A}\boldsymbol\gamma.$$

These equations (1.11) describe a partial case of motion of an inhomogeneous balanced ball rolling without slipping on a horizontal plane rotating with constant angular velocity Ω, see Eqs. (22) in [6]. We use the same notation as in [6], where the reader can find a complete description of variables $z=(\gamma_1,\gamma_2,\gamma_3,K_1,K_2,K_3)$, definitions of parameters and a list of the necessary references.

According to [6], the equations of motion (1.11) possess two geometric integrals of motion

$$C_1 = \boldsymbol{\gamma}^2 = 1, \qquad C_2 = (\mathbf{K}, \boldsymbol{\gamma}), \tag{1.12}$$

an integral of motion similar to the Jacobi integral

$$H = \frac{1}{2}(\mathbf{K}, \mathbf{A}\mathbf{K}) - \frac{\mathbf{K}^2}{2d} + \frac{d}{2(1-d(\boldsymbol{\gamma}, \mathbf{A}\boldsymbol{\gamma}))}(\mathbf{A}\mathbf{K}, \boldsymbol{\gamma})^2, \tag{1.13}$$

and an invariant measure

$$\mu = \rho\,\mathrm{d}\mathbf{K}\mathrm{d}\boldsymbol{\gamma}, \quad \text{where} \quad \rho = \big(1 - d(\boldsymbol{\gamma}, \mathbf{A}\boldsymbol{\gamma})\big)^{-\frac{1}{2}}. \tag{1.14}$$

The Jacobi integral has been introduced in [15] for the n-dimensional Chaplygin ball, and in general for a system with affine non-holonomic constraints in [16].

In Section 2 we present a Poisson bivector $\mathcal{P}$ which allows us to rewrite the vector field Z (1.11) in conformally Hamiltonian form

$$Z = \mathrm{f}(z)\,\mathcal{P}\mathrm{d}H(z)$$

at $C_2 = 0$. This bivector $\mathcal{P}$ is a linear deformation of the standard Lie–Poisson bivector on algebra $e^*(3)$ involving nonalgebraic functions, i. e., the so-called Turiel type deformation [45].

In Section 3 we discuss a counterpart of the heavy symmetric top at $a_1 = a_2$ in (1.11). In this case we have one more first integral [6]:

$$H_2 = \rho K_3 - \frac{da_1\rho\,(\mathbf{K},\boldsymbol{\gamma})\gamma_3}{da_1 - 1} + \frac{da_1 - 1}{a_1\sqrt{d(a_1 - a_3)}}\,\Omega \ln\left(\sqrt{d(a_1-a_3)}\,\gamma_3 + \rho^{-1}\right).$$

Using this nonalgebraic first integral, we can decompose the vector field (1.11) into Hamiltonian vector fields

$$Z = \mathrm{f}_1(z)\,\mathcal{P}'\mathrm{d}H(z) + \mathrm{f}_2(z)\,\mathcal{P}'\mathrm{d}H_2(z)$$

and find second Poisson bivectors $\mathcal{P}''$ compatible with $\mathcal{P}'$ so that

$$Z = \mathrm{g}_1(z)\,\mathcal{P}''\mathrm{d}H(z) + \mathrm{g}_2(z)\,\mathcal{P}''\mathrm{d}H_2(z).$$

This allows us to calculate variables of separation for the equations of motion (1.11) on a computer.

The same variables of separation may be obtained more easily using a counterpart of the Lagrange calculations for the symmetric heavy top. In the framework of the Jacobi method these variables of separation determine compatible Poisson brackets, recursion operators, the algebra of Haantjes operators and 2×2 Lax matrices for the vector field (1.11).

2. CONFORMALLY HAMILTONIAN VECTOR FIELD AT $(\boldsymbol{\gamma}, \mathbf{K}) = 0$

Let us substitute the vector field Z (1.11), its geometric integrals $C_{1,2}$ (1.12) and the Jacobi integral H (1.13) into the following system of algebraic equations:

$$Z = \mathrm{f}(z)\mathcal{P}\mathrm{d}H \qquad \text{and} \qquad \mathcal{P}\mathrm{d}C_{1,2} = 0. \tag{2.1}$$

The desired Poisson bivector $\mathcal{P}$ must also satisfy the Jacobi identity, i. e., the system of differential equations

$$[\![\mathcal{P}, \mathcal{P}]\!] = 0 \tag{2.2}$$

encoded in a shortened form using the Schouten bracket

$$[\![A, B]\!]_{ijk} = -\sum_{m=1}^{6}\left(B_{mk}\frac{\partial A_{ij}}{\partial z_m} + A_{mk}\frac{\partial B_{ij}}{\partial z_m} + \mathrm{cycle}(i,j,k)\right).$$

In our case $z = (\boldsymbol{\gamma}, \mathbf{K})$.

Substituting the linear ansatz for entries of the Poisson bivector

$$\mathcal{P}_{ij}=\sum_{m=1}^{3} u_{ij}^m(\boldsymbol{\gamma})\,K_m+v_{ij}(\boldsymbol{\gamma})$$

and the function $\mathrm{f}(z)=\mathrm{f}(\boldsymbol{\gamma})$ into (2.1), one gets an inconsistent system of algebraic equations, which has a solution only at $C_2=(\boldsymbol{\gamma},\mathbf{K})=0$. This solution of algebraic equations depends on an arbitrary function $\mathrm{f}(\boldsymbol{\gamma})$ on variables γ_1,γ_2 and γ_3.

Substituting this partial solution into the Jacobi identity and solving the resulting differential equations, we obtain the desired Poisson bivector. It took us only a few seconds to solve both algebraic and differential equations (2.1)–(2.2) using one of the modern computer algebra systems.

Proposition 1. *At $\boldsymbol{\gamma}^2=1$ and $(\boldsymbol{\gamma},\mathbf{K})=0$ the vector field Z (1.11) is a conformally Hamiltonian vector field*

$$Z=\mathrm{f}(z)\mathcal{P}\mathrm{d}H$$

with a conformal factor depending only on variables γ_1,γ_2 and γ_3

$$\mathrm{f}(z)=2\rho\,\delta,$$

which is a product of functions ρ from (1.14) and

$$\delta=(1-da_1)a_2a_3\gamma_1^2+(1-da_2)a_1a_3\gamma_2^2+(1-da_3)a_1a_2\gamma_3^2.$$

The proof consists of straightforward verification of the algebraic and differential equations (2.1)–(2.2) by using an explicit form of the Poisson bivector $\mathcal{P}$.

In variables $z=(\boldsymbol{\gamma},\mathbf{K})$ the bivector $\mathcal{P}$ is equal to

$$\mathcal{P}=\frac{d}{2\delta\mu}\begin{pmatrix}0 & \boldsymbol{\Upsilon}\\ -\boldsymbol{\Upsilon}^\top & \Omega\,\boldsymbol{\Gamma}\end{pmatrix}+\frac{d\mu}{2\delta}\begin{pmatrix}0 & 0\\ 0 & \boldsymbol{\Pi}\end{pmatrix}, \tag{2.3}$$

where $\boldsymbol{\Upsilon}$ is the 3×3 matrix

$$\boldsymbol{\Upsilon}=\boldsymbol{\Gamma}_3-\boldsymbol{\Gamma}\mathbf{A}(d\mathbf{A}-\mathbf{I}d)^{-1},$$

the matrix $\boldsymbol{\Gamma}$ is equal to

$$\boldsymbol{\Gamma}=\begin{pmatrix}0 & \gamma_3 & -\gamma_2\\ -\gamma_3 & 0 & \gamma_1\\ \gamma_2 & -\gamma_1 & 0\end{pmatrix}$$

and

$$\boldsymbol{\Gamma}_3=\begin{pmatrix}\dfrac{\gamma_1\gamma_2\gamma_3(a_3-a_2)}{(da_3-1)(da_2-1)} & \dfrac{\gamma_2^2\gamma_3(a_3-a_2)}{(da_3-1)(da_2-1)} & \dfrac{\gamma_2\gamma_3^2(a_3-a_2)}{(da_3-1)(da_2-1)}\\[2ex] \dfrac{\gamma_1^2\gamma_3(a_1-a_3)}{(da_3-1)(da_1-1)} & \dfrac{\gamma_1\gamma_2\gamma_3(a_1-a_3)}{(da_3-1)(da_1-1)} & \dfrac{\gamma_1\gamma_3^2(a_1-a_3)}{(da_3-1)(da_1-1)}\\[2ex] \dfrac{\gamma_1^2\gamma_2(a_2-a_1)}{(da_2-1)(da_1-1)} & \dfrac{\gamma_1\gamma_2^2(a_2-a_1)}{(da_2-1)(da_1-1)} & \dfrac{\gamma_1\gamma_2\gamma_3(a_2-a_1)}{(da_2-1)(da_1-1)}\end{pmatrix},$$

so that

$$\boldsymbol{\Gamma}_3\boldsymbol{\Gamma}=0.$$

The skew-symmetric 3×3 matrix $\mathbf{\Pi}$ has a more cumbersome form

$$\mathbf{\Pi} = \frac{1}{\det(d\mathbf{A} - \mathbf{Id})} \begin{pmatrix} 0 & \dfrac{\alpha_1 d\gamma_1\gamma_2\gamma_3(\boldsymbol{\gamma} \times \mathbf{K})_3 + \beta_3\mathbf{K}_3}{\gamma_1^2 + \gamma_2^2} & \dfrac{\alpha_2 d\gamma_1\gamma_2\gamma_3(\boldsymbol{\gamma} \times \mathbf{K})_2 + \beta_2\mathbf{K}_2}{\gamma_1^2 + \gamma_3^2} \\ * & 0 & \dfrac{\alpha_1 d\gamma_1\gamma_2\gamma_3(\boldsymbol{\gamma} \times \mathbf{K})_1 + \beta_1\mathbf{K}_1}{\gamma_2^2 + \gamma_3^2} \\ * & * & 0 \end{pmatrix},$$

where

$$\begin{aligned}
\alpha_3 &= (a_2 - a_1)\big(d(a_2 - a_3)(a_3 - a_1)\gamma_3^2 + a_3(1 - da_2)(1 - da_1)\big),\\
\alpha_2 &= (a_1 - a_3)\big(d(a_2 - a_3)(a_1 - a_2)\gamma_2^2 + a_2(1 - da_3)(1 - da_1)\big),\\
\alpha_1 &= (a_2 - a_3)\big(d(a_1 - a_3)(a_2 - a_1)\gamma_1^2 + a_1(1 - da_3)(1 - da_2)\big)
\end{aligned}$$

and

$$\begin{aligned}
\beta_1 = {}& d(a_1 - a_2)^2(1 - da_3)\gamma_2^6 + d(a_1 - a_3)^2(1 - da_2)\gamma_3^6\\
&- d(a_1 - a_2)(da_1a_2 + 2da_1a_3 - 2da_2a_3 - da_3^2 - 3a_1 + a_2 + 2a_3)\gamma_2^4\gamma_3^2\\
&- d(a_1 - a_3)(2da_1a_2 + da_1a_3 - da_2^2 - 2da_2a_3 - 3a_1 + 2a_2 + a_3)\gamma_2^2\gamma_3^4\\
&+ (a_1 - a_2)(2d^3a_1a_2a_3 - 3d^2a_2a_3 - 2da_1 + da_2 + da_3 + 1)\gamma_2^4\\
&+ (a_1 - a_3)(2d^3a_1a_2a_3 - 3d^2a_2a_3 - 2da_1 + da_2 + da_3 + 1)\gamma_3^4\\
&+ (2a_1 - a_2 - a_3)(2d^3a_1a_2a_3 - 3d^2a_2a_3 - 2da_1 + da_2 + da_3 + 1)\gamma_2^2\gamma_3^2\\
&+ (1 - da_1)(2d^2a_1a_2a_3 - d^2a_2^2a_3 - da_1a_3 - da_2a_3 - a_1 + a_2 + a_3)\gamma_2^2\\
&+ (1 - da_1)(2d^2a_1a_2a_3 - d^2a_2a_3^2 - da_1a_2 - da_2a_3 - a_1 + a_2 + a_3)\gamma_3^2,
\end{aligned}$$

$$\begin{aligned}
\beta_2 = {}& d(a_1 - a_2)^2(da_3 - 1)\gamma_1^6 + d(a_2 - a_3)^2(da_1 - 1)\gamma_3^6\\
&- d(a_1 - a_2)(da_1a_2 - 2da_1a_3 + 2da_2a_3 - da_3^2 + a_1 - 3a_2 + 2a_3)\gamma_1^4\gamma_3^2\\
&- d(a_2 - a_3)(da_1^2 - 2da_1a_2 + 2da_1a_3 - da_2a_3 - 2a_1 + 3a_2 - a_3)\gamma_1^2\gamma_3^4\\
&+ (a_1 - a_2)(2d^3a_1a_2a_3 - 3d^2a_1a_3 + da_1 - 2da_2 + da_3 + 1)\gamma_1^4\\
&+ (a_3 - a_2)(2d^3a_1a_2a_3 - 3d^2a_1a_3 + da_1 - 2da_2 + da_3 + 1)\gamma_3^4\\
&+ (a_1 - 2a_2 + a_3)(2d^3a_1a_2a_3 - 3d^2a_1a_3 + da_1 - 2da_2 + da_3 + 1)\gamma_1^2\gamma_3^2\\
&+ (da_2 - 1)(2d^2a_1a_2a_3 - d^2a_1^2a_3 - da_1a_3 - da_2a_3 + -a_1 - a_2 + a_3)\gamma_1^2\\
&+ (da_2 - 1)(2d^2a_1a_2a_3 - d^2a_1a_3^2 - da_1a_2 - da_1a_3 + a_1 - a_2 + a_3)\gamma_3^2,
\end{aligned}$$

$$\begin{aligned}
\beta_3 = {}& d(a_1 - a_3)^2(da_2 - 1)\gamma_1^6 + d(a_2 - a_3)^2(da_1 - 1)\gamma_2^6\\
&- d(a_3 - a_1)(2da_1a_2 - da_1a_3 + da_2^2 - 2da_2a_3 - a_1 - 2a_2 + 3a_3)\gamma_1^4\gamma_2^2\\
&- d(a_3 - a_2)(da_1^2 + 2da_1a_2 - 2da_1a_3 - da_2a_3 - 2a_1 - a_2 + 3a_3)\gamma_1^2\gamma_2^4\\
&+ (a_1 - a_3)(2d^3a_1a_2a_3 - 3d^2a_1a_2 + da_1 + da_2 - 2da_3 + 1)\gamma_1^4\\
&+ (a_2 - a_3)(2d^3a_1a_2a_3 - 3d^2a_1a_2 + da_1 + da_2 - 2da_3 + 1)\gamma_2^4\\
&+ (a_1 + a_2 - 2a_3)(2d^3a_1a_2a_3 - 3d^2a_1a_2 + da_1 + da_2 - 2da_3 + 1)\gamma_1^2\gamma_2^2\\
&+ (1 - da_3)(d^2a_1^2a_2 - 2d^2a_1a_2a_3 + da_1a_2 + da_2a_3 - a_1 - a_2 + a_3)\gamma_1^2\\
&+ (1 - da_3)(d^2a_1a_2^2 - 2d^2a_1a_2a_3 + da_1a_2 + da_1a_3 - a_1 - a_2 + a_3)\gamma_2^2.
\end{aligned}$$

Because $\boldsymbol{\gamma}^2 = 1$ and $(\boldsymbol{\gamma}, \mathbf{K}) = 0$ we can simplify these expressions by using algebraic transformation variables $K_i \to L_i$ defined by equations of the form

$$
\begin{aligned}
L_1 &= \frac{2\rho(1-da_1)\Big(\big(a_2a_3(\gamma_2^2+\gamma_3^2)d - a_3\gamma_2^2 - a_2\gamma_3^2\big)K_1 - (a_3-a_2)\gamma_1\gamma_2\gamma_3\big(\boldsymbol{\gamma}\times\mathbf{K}\big)_1\Big)}{d(\gamma_2^2+\gamma_3^2)},\\
L_2 &= \frac{2\rho(1-da_2)\Big(\big(a_1a_3(\gamma_1^2+\gamma_3^2)d - a_3\gamma_1^2 - a_1\gamma_3^2\big)K_2 - (-a_3+a_1)\gamma_1\gamma_2\gamma_3\big(\boldsymbol{\gamma}\times\mathbf{K}\big)_2\Big)}{d(\gamma_1^2+\gamma_3^2)},\\
L_3 &= \frac{2\rho(1-da_3)\Big(\big(da_1a_2(\gamma_1^2+\gamma_2^2) - \gamma_1^2a_2 - a_1\gamma_2^2\big)K_3 - (a_2-a_1)\gamma_1\gamma_2\gamma_3\big(\boldsymbol{\gamma}\times\mathbf{K}\big)_3\Big)}{d(\gamma_1^2+\gamma_2^2)}.
\end{aligned}
\tag{2.4}
$$

Here ρ (1.14) is an algebraic function of variables γ_1, γ_2 and γ_3.

In variables $z = (\boldsymbol{\gamma}, \mathbf{L})$ the Poisson bivector $\mathcal{P}$ (2.3) becomes a quite visible object

$$
\mathcal{P} = \begin{pmatrix}
0 & 0 & 0 & 0 & \gamma_3 & -\gamma_2\\
0 & 0 & 0 & -\gamma_3 & 0 & \gamma_1\\
0 & 0 & 0 & \gamma_2 & -\gamma_1 & 0\\
0 & \gamma_3 & -\gamma_2 & 0 & L_3 & -L_2\\
-\gamma_3 & 0 & \gamma_1 & -L_3 & 0 & L_1\\
\gamma_2 & -\gamma_1 & 0 & L_2 & -L_1 & 0
\end{pmatrix}
+ \frac{2\rho(da_1-1)(da_2-1)(da_3-1)\Omega}{d}
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & \boldsymbol{\gamma}_3 & -\boldsymbol{\gamma}_2\\
0 & 0 & \gamma_1 & -\boldsymbol{\gamma}_3 & 0 & \boldsymbol{\gamma}_1\\
0 & 0 & 0 & \boldsymbol{\gamma}_2 & -\boldsymbol{\gamma}_1 & 0
\end{pmatrix}.
\tag{2.5}
$$

In the next subsection we prove that this bivector is a nonalgebraic deformation of the Lie–Poisson bivector on $e^*(3)$

$$
\mathcal{P}_0 = \begin{pmatrix} 0 & \mathbf{\Gamma} \\ -\mathbf{\Gamma}^\top & \mathbf{M} \end{pmatrix},
\tag{2.6}
$$

where

$$
\mathbf{\Gamma} = \begin{pmatrix} 0 & \gamma_3 & -\gamma_2 \\ -\gamma_3 & 0 & \gamma_1 \\ \gamma_2 & -\gamma_1 & 0 \end{pmatrix}, \qquad
\mathbf{M} = \begin{pmatrix} 0 & M_3 & -M_2 \\ -M_3 & 0 & M_1 \\ M_2 & -M_1 & 0 \end{pmatrix}.
$$

This means that we cannot reduce the bivector $\mathcal{P}$ (2.5) to the Lie–Poisson bivector $\mathcal{P}_0$ (2.6) using only algebraic transformations of variables.

2.1. Deformation of the Canonical Poisson Bivector Linear in Momenta

Let Q be an n-dimensional smooth manifold endowed with a (1,1) tensor field $\Lambda(q)$ with vanishing Nijenhuis torsion and with a vector field $\nu(q)$. The canonical Poisson bracket on its cotangent bundle T^*Q

$$\{q_i, q_j\} = 0, \qquad \{q_i, p_j\} = \delta_{ij}\,, \qquad \{p_i, p_j\} = 0,$$

where q_i, p_i are fibered coordinates, after the linear transformation of momenta

$$p_j \to \sum_{i=1}^{n} \Lambda_j^i(q) p_i + \nu_j(q)$$

looks like

$$\{q_i, q_j\}' = 0\,, \qquad \{q_i, p_j\}' = \Lambda_j^i\,, \qquad \{p_i, p_j\}' = \sum_{k=1}^{n} \left(\frac{\partial \Lambda_i^k}{\partial q_j} - \frac{\partial \Lambda_j^k}{\partial q_i} \right) p_k + \frac{\partial \nu_j}{\partial q_i} - \frac{\partial \nu_i}{\partial q_j}.$$

It is the so-called Turiel type deformation of the canonical Poisson bracket on T^*Q, see [36, 45] and references therein.

The bivector $\mathcal{P}$ (2.5) determines the standard Poisson brackets $\{\gamma_i, \gamma_j\} = 0$ and $\{\gamma_i, L_j\} = \varepsilon_{ijk}\gamma_k$ and nonstandard brackets $\{L_i, L_j\}$ between momenta. Let us consider a change of variables

$$\mathbf{L} = (L_1, L_2, L_3) \to \mathbf{M} = (M_1, M_2, M_3) \qquad L_i \to M_i = L_i + \eta_i(\boldsymbol{\gamma}), \tag{2.7}$$

which reduces these nonstandard Poisson brackets to canonical form

$$\{M_1, M_2\} = M_3, \qquad \{M_2, M_3\} = M_1, \qquad \{M_3, M_1\} = M_2 \tag{2.8}$$

at

$$(\boldsymbol{\gamma}, \mathbf{M}) = \eta_1\gamma_1 + \eta_2\gamma_2 + \eta_3\gamma_3 = 0 \qquad \text{and} \qquad \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1.$$

These algebraic equations hold if we use the following ansats:

$$\eta_1 = \gamma_2\eta_4 - \frac{\gamma_1}{\gamma_1^2 + \gamma_2^2}\,\gamma_3\eta_3, \qquad \eta_2 = -\gamma_1\eta_4 - \frac{\gamma_2}{\gamma_1^2 + \gamma_2^2}\,\gamma_3\eta_3$$

for functions $\eta_{1,2}$ depending on unknown functions η_3 and η_4. We substitute this ansats into (2.8) and solve the resulting differential equations.

The first partial solution of Eqs. (2.8) reads as $\eta_3 = 0$ and

$$\eta_4 = \frac{2(1 - da_1)(1 - da_2)(1 - da_3)\Omega}{d^{3/2}\sqrt{a_1 - a_2}\sqrt{\gamma_3^2 - 1}}\, F\left(\sqrt{\frac{d(a_2 - a_1)\gamma_2^2}{d(a_1 - a_2)\gamma_3^2 - da_1 + 1}}, \sqrt{\frac{d(a_1 - a_2)\gamma_3^2 - da_1 + 1}{d(a_2 - a_2)(\gamma_1^2 + \gamma_2^2)}} \right),$$

where F is an incomplete elliptic integral of the first kind.

The second partial solution of Eqs. (2.8) looks like $\eta_4 = 0$ and

$$\eta_3 = \frac{2(1 - da_1)(1 - da_2)(1 - da_3)\Omega}{d^{3/2}\kappa}\, \ln\left(\sqrt{d}\,\kappa\boldsymbol{\gamma}_3 + \frac{1}{\rho} \right),$$

where the function ρ defines the invariant measure (1.14) and

$$\kappa = \sqrt{\frac{(a_1 - a_3)\boldsymbol{\gamma}_1^2 + (a_2 - a_3)\boldsymbol{\gamma}_2^2}{\boldsymbol{\gamma}_1^2 + \boldsymbol{\gamma}_2^2}}.$$

Thus, in variables $z = (\boldsymbol{\gamma}, \mathbf{M})$ the vector field Z (1.11) becomes a conformally Hamiltonian vector field

$$Z = \mathrm{f}(\boldsymbol{\gamma})\mathcal{P}\mathrm{d}H$$

defined by Hamiltonian H (1.13), which is a nonalgebraic function in variables $\mathbf{M}$ (2.7) on the cotangent bundle of the two-dimensional sphere $T^*\mathbb{S}^2$, which is symplectomorphic to symplectic leaf of $e^*(3)$ defined by $(\boldsymbol{\gamma}, \boldsymbol{\gamma}) = 1$ and $(\boldsymbol{\gamma}, \mathbf{M}) = 0$. Here $\mathcal{P}_0$ is given by (2.6).

This result was obtained by directly solving algebraic and differential equations on a computer. Now we can apply this result in order to prove that bivector $\mathcal{P}$ is the Turiel type deformation of canonical Poisson brackets. As we identify phase space with $T^*\mathbb{S}^2$, we can introduce spherical coordinates and non-standard momenta

$$\phi=\arctan\gamma_1/\gamma_2,\quad p'_\phi=-K_3,\quad \theta=\arccos\gamma_3,\quad p'_\theta=-\frac{\gamma_2K_1-\gamma_1K_2}{\sqrt{\gamma_1^2+\gamma_2^2}}.$$

Proposition 2. *The transformation of momenta*

$$p_\phi=\Lambda_1^1(\phi,\theta)p'_\phi+\Lambda_1^2(\phi,\theta)p'_\theta+\nu_1(\phi,\theta),$$
$$p_\theta=\Lambda_2^1(\phi,\theta)p'_\phi+\Lambda_2^2(\phi,\theta)p'_\theta+\nu_2(\phi,\theta)$$

reduces the Poisson bracket associated with bivector $\mathcal{P}$ (2.3) to a canonical Poisson bracket on the cotangent bundle of the two-dimensional sphere

$$\{\phi,\theta\}=0,\qquad \{\phi,p_\phi\}=\{\theta,p_\theta\}=1,\qquad \{p_\phi,p_\theta\}=0,$$

if

$$\Lambda=\frac{2(da_3-1)\rho}{d}\begin{pmatrix}(a_1-a_2)\cos^2\phi-da_1a_2+a_2 & \dfrac{(a_1-a_2)\sin2\phi\sin2\theta}{4}\\[2ex] \dfrac{(a_1-a_2)\sin2\phi\cos\theta}{2\sin\theta} & \dfrac{\lambda}{da_3-1}\end{pmatrix}$$

where

$$\lambda=-(da_3-1)(a_1-a_2)\cos^2\theta\cos^2\phi+(da_2-1)(a_1-a_3)\cos^2\theta-(da_1-1)(da_2-1)a_3$$

and functions $\nu_{1,2}(\phi,\theta)$ satisfy the differential equation

$$\frac{\partial\nu_1}{\partial\theta}-\frac{\partial\nu_2}{\partial\phi}=2\Omega(1-da_1)(1-da_2)(1-da_3)\rho\sin\theta.$$

Partial solutions of this equation are an incomplete elliptic integral of the first kind and a logarithmic function, which have been obtained by the brute force method before.

The proof consists of straightforward verification.

It is easy to prove that the (1,1) tensor field $\Lambda(q)$ has zero Nijenhuis torsion and, therefore, the bivector $\mathcal{P}$ (2.3) is a Turiel type deformation of the canonical Poisson bivector.

In nonholonomic mechanics functions $L^i_j(q)$ and $\nu_j(q)$ are usually algebraic functions on configuration space Q. For instance, see linear deformations appearing for:

- reduced motion of the Chaplygin ball on the plane [35];
- reduced motion of the Chaplygin ball on the sphere [37];
- nonholonomic Veselova system and its generalizations [38];
- reduced motion of the Routh ball on the plane [3];
- nonholonomic motion of a body of revolution on the plane [41];
- nonholonomic motion of a homogeneous ball on the surface of revolution [41];
- nonholonomic Heisenberg type systems [23];
- other non-Hamiltonian systems associated with Turiel type deformations [36].

In contrast to these systems with two degrees of freedom, one gets nonalgebraic deformations involving elliptic integrals or logarithms for reduced motion of the Chaplygin ball on the rotating plane (1.11).

In fact, the apparition of nonalgebraic deformations is the main result because it allows us to essentially enlarge the list of possible Turiel type deformations appearing in the mathematical description of real physical systems. We suppose that similar nonalgebraic deformations of the canonical Poisson brackets appear also in other models of rigid body motion on rotating surfaces [14, 26, 30, 46, 49].

Nonalgebraic deformations for non-Hamiltonian systems with three degrees of freedom are also discussed in [8, 40].

3. THE SUM OF HAMILTONIAN VECTOR FIELDS

At the symmetric case $a_1 = a_2$ and the vector field Z (1.11) has a formal integral of motion

$$H_2 = \rho K_3 - -\frac{da_1\rho\,(\boldsymbol{\gamma},\mathbf{K})\gamma_3}{da_1-1} + \frac{da_1-1}{a_1\sqrt{d(a_1-a_3)}}\,\Omega\ln\left(\sqrt{d(a_1-a_3)}\,\gamma_3+\rho^{-1}\right), \tag{3.1}$$

with a logarithmic term, see [6] for a discussion.

Proposition 3. *At $a_1 = a_2$ the vector field Z (1.11) is a sum of two Hamiltonian vector fields*

$$Z = \mathrm{f}_1(z)\mathcal{P}'\,\mathrm{d}H + \mathrm{f}_2(z)\mathcal{P}'\,\mathrm{d}H_2 \tag{3.2}$$

with coefficients

$$\mathrm{f}_1 = 2\rho\,\delta, \qquad \textit{and} \qquad \mathrm{f}_2 = \frac{2a_1(a_1-a_3)\gamma_3\cdot(\boldsymbol{\gamma},\mathbf{K})}{d}.$$

Here H is a Jacobi integral and the Poisson bivector P' is equal to

$$\mathcal{P}' = \mathcal{P} + \frac{d^2\rho\cdot(\boldsymbol{\gamma},\mathbf{K})}{2\Big((a_3-a_1)\gamma_3^2+(da_1-1)a_3\Big)(da_1-1)}\begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&\sigma_3&-\sigma_2\\ 0&0&0&-\sigma_3&0&\sigma_1\\ 0&0&0&\sigma_2&-\sigma_1&0 \end{pmatrix}, \tag{3.3}$$

where

$$\sigma_1 = \left(a_1 - \frac{(a_1-a_3)\gamma_3^2}{1-\gamma_1^2}\right)\gamma_1, \qquad \sigma_2 = \left(a_1 - \frac{(a_1-a_3)\gamma_3^2}{1-\gamma_2^2}\right)\gamma_2,$$

and

$$\sigma_3 = \left(a_1a_3(da_1-1) - (a_1-a_3)^2\gamma_3^2 + \frac{a_3-a_1}{d}\right)\frac{\gamma_3}{da_3-1}.$$

The proof is a straightforward calculation.

In this case we cannot use any type of Hamiltonization for reduction of equations of motion to Hamiltonian form. Nevertheless, we can calculate variables of separation for equations of motion (1.11) directly solving Eq. (1.10) in the framework of the Jacobi method.

Because we know the solution (3.2) of Eq. (1.9), we have to solve (1.10) together with the equation

$$[\![\mathcal{P}',\mathcal{P}'']\!] = 0,$$

which guarantees compatibility of bivectors $\mathcal{P}'$ and $\mathcal{P}''$. Solving these equations on a computer, we use the following ansats:

- entries of $\mathcal{P}''$ are linear functions in momenta;
- the first coefficient g_1 is a function on γ_3 similar to (3.2);
- the second coefficient g_2 is a linear function in momenta similar to (3.2).

After a few seconds one gets the following answer:

$$\mathcal{P}'' = \begin{pmatrix} 0 & \mathbf{\Upsilon}'' \\ -\mathbf{\Upsilon}''^{\top} & \mathbf{\Pi}'' \end{pmatrix},$$

where

$$\mathbf{\Upsilon}'' = \begin{pmatrix} -\dfrac{\gamma_1\gamma_2\gamma_3}{\gamma_1^2+\gamma_2^2}\left(\dfrac{1}{\rho} - \dfrac{1}{a_1(\gamma_1^2+\gamma_2^2)}\right) & -\dfrac{\gamma_3}{\gamma_1^2+\gamma_2^2}\left(\dfrac{\gamma_2^2}{\rho} + \dfrac{\gamma_1^2}{a_1(\gamma_1^2+\gamma_2^2)}\right) & \dfrac{\gamma_2}{\rho} \\ \dfrac{\gamma_3}{\gamma_1^2+\gamma_2^2}\left(\dfrac{\gamma_1^2}{\rho} + \dfrac{\gamma_2^2}{a_1(\gamma_1^2+\gamma_2^2)}\right) & \dfrac{\gamma_1\gamma_2\gamma_3}{\gamma_1^2+\gamma_2^2}\left(\dfrac{1}{\rho} - \dfrac{1}{a_1(\gamma_1^2+\gamma_2^2)}\right) & -\dfrac{\gamma_1}{\rho} \\ -\dfrac{\gamma_2}{a_1(\gamma_1^2+\gamma_2^2)} & \dfrac{\gamma_1}{a_1(\gamma_1^2+\gamma_2^2)} & 0 \end{pmatrix}$$

and the entries of the skew-symmetric matrix $\mathbf{\Pi}''$ are equal to

$$\mathbf{\Pi}''_{12} = \frac{\gamma_3}{\gamma_1^2+\gamma_2^2}\left(-\frac{(\gamma_1K_1+\gamma_2K_2)}{\rho} + \frac{(da_1-1)\Omega - a_1K_3}{a_1^2} + \frac{\rho^2\left(da_3\gamma_3K_3 - \frac{(x_1K_1+x_2K_2)(da_3\gamma_3^2-1)}{\gamma_1^2+\gamma_2^2}\right)}{a_1}\right),$$

$$\mathbf{\Pi}''_{13} = \frac{K_2}{\rho} - \left(\frac{(da_1-1)\Omega}{a_1} - d\Big(a_1(\gamma_1K_1+\gamma_2K_2) + a_3\gamma_3K_3\Big)\rho^2\right)\frac{\gamma_2}{a_1(\gamma_1^2+\gamma_2^2)},$$

$$\mathbf{\Pi}''_{23} = -\frac{K_1}{\rho} + \left(\frac{(da_1-1)\Omega}{a_1} + d\Big(a_1(\gamma_1K_1+x_2K_2) + a_3\gamma_3K_3\Big)\rho^2\right)\frac{\gamma_1}{a_1(\gamma_1^2+\gamma_2^2)}.$$

In this case the coefficients in decomposition (1.10) are equal to

$$\mathrm{g}_1(z) = \frac{a_1d(1-\gamma_3^2)}{2(1-da_1)}$$

and

$$\mathrm{g}_2(z) = \left(\frac{a_1^2(\gamma_1^2+\gamma_2^2)}{(da_1-1)\rho} + da_1^2a_3(\gamma_1^2+\gamma_2^2)\rho - a_3(da_1-1)\rho^2\right)K_3$$
$$- a_1\gamma_3\left(\frac{a_1}{(da_1-1)\rho} + \frac{da_1^2(da_3-1)\rho}{(da_1-1)} - \frac{(da_3-1)\rho^2}{\gamma_1^2+\gamma_2^2}\right)(\gamma_1K_1+\gamma_2K_2).$$

Both bivectors $\mathcal{P}'$ and $\mathcal{P}''$ are degenerate and, therefore, we cannot determine the recursion operator (1.4), but we can easily calculate the control matrix $\mathbf{F}$ (1.5) and its eigenvalues, which are desired variables of separation. In our case we have to solve the algebraic equations

$$\mathcal{P}'\mathrm{d}H = \mathbf{F}_{11}\mathcal{P}''\mathrm{d}H + \mathbf{F}_{12}\mathcal{P}''\mathrm{d}H_2,$$
$$\mathcal{P}'\mathrm{d}H_2 = \mathbf{F}_{21}\mathcal{P}''\mathrm{d}H + \mathbf{F}_{22}\mathcal{P}''\mathrm{d}H_2$$

with respect to four entries of matrix $\mathbf{F}$. In our case $\mathbf{F}_{21} = 0$ and we can easily calculate the eigenvalues of $\mathbf{F}$

$$\lambda_1 = \mathbf{F}_{21} == \frac{d}{2a_1(da_3-1)(da_1-1)} \qquad \text{and} \qquad \lambda_2 = \frac{d(1-\gamma_3^2)a_1\sqrt{d(a_1-a_3)\gamma_3^2 - a_1d + 1}}{2(da_1-1)(da_1a_3 - (a_1-a_3)\gamma_3^2 - a_3)}.$$

There is only one nontrivial eigenvalue λ_2 which is the desired variable of separation. Now we have to validate that our algorithm yields the quadrature

$$\frac{\mathrm{d}\lambda_2}{\mathrm{d}t} = F(\lambda_2),$$

i. e., an equation which includes only the variable λ_2 and its differential. This quadrature will be discussed in the following subsection.

Of course, this variable of separation may be obtained by directly using the symmetries of the ball similar to the Lagrange approach to a symmetric heavy top. More complicated, but algorithmic computer calculations of variables of separation were given in order to underline the importance of studying Eqs. (1.9)–(1.10) in nonholonomic mechanics.

3.1. Separation of Variables

It is easy to prove that the equation of motion

$$\dot\gamma_3^2 = a_1^2\,(\gamma_2 M_1 - \gamma_1 M_2)^2 = F(\gamma_3) \tag{3.4}$$

on the common level surface of the first integrals

$$C_1 = 1, \qquad C_2 = \ell, \qquad H_2 = k, \qquad 2H = h \tag{3.5}$$

includes only one variable with its own differential. Here

$$\begin{aligned} F(x) = {}& \frac{a_1^2 d(x^2-1)}{da_1 - 1}\,h + \frac{a_1^2(x^2-1)\big(dx^2(a_1-a_3)+2d^2a_1a_3-2da_1-da_3+1\big)}{(da_3-1)(da_1-1)^2}\,\ell^2 \\ & - \left(\frac{a_1 x\sqrt{dx^2(a_1-a_3)-da_1+1}}{(da_1-1)\sqrt{da_3-1}}\,\ell - a_1\sqrt{da_3-1}\,k \right. \\ & \left. + \frac{(da_1-1)\sqrt{da_3-1}\,\ln\Big(\sqrt{d(a_1-a_3)}x+\sqrt{dx^2(a_1-a_3)-da_1+1}\Big)\Omega}{\sqrt{d(a_1-a_3)}}\right)^2. \end{aligned} \tag{3.6}$$

Thus, at $a_1 = a_2$ the solution of the six equations of motion (1.11) is reduced to one nontrivial quadrature

$$\int^{\gamma_3} \frac{\mathrm{d}x}{\sqrt{F(x)}} = t$$

involving the nonalgebraic function $F(x)$ (3.6). We hope that the study of such non-Abelian quadratures can be carried out by means other than numerical simulations.

Following Lagrange and Jacobi, we can introduce two pairs of independent variables of separation

$$x_1 = \gamma_3,, \quad y_1 = a_1(\gamma_1 K_2 - \gamma_2 K_1), \qquad \text{and} \qquad x_2 = \phi(\gamma_1/\gamma_2), \quad y_2 = H_2, \tag{3.7}$$

so that

$$\{x_1, x_2\}' = \{y_1, y_2\}' = \{y_1, x_2\}' = \{x_1, y_2\}' = 0$$

and

$$\begin{aligned} \{x_1, y_1\}' &= f_1(x_1) \equiv \frac{a_1\sqrt{d(a_1-a_3)x_1^2 - a_1 d + 1}(x_1^2-1)}{d(a_1-a_3)\Big(a_3(da_1-1)-(a_1-a_3)x_1^2\Big)}, \\ \{x_2, y_2\}' &= f_2(x_2) \equiv \frac{\psi(x_2)}{da_1(a_1-a_3)(da_3-1)}, \qquad \psi = \frac{\mathrm{d}\,\phi(\gamma_1/\gamma_2)}{\mathrm{d}(\gamma_1/\gamma_2)}. \end{aligned} \tag{3.8}$$

Here $\phi(\gamma_1/\gamma_2)$ is an arbitrary function and $\{.,.\}'$ is the Poisson bracket associated with the Poisson bivector $\mathcal{P}'$ (3.3).

In the framework of the Jacobi method we identify these variables of separation with affine coordinates of divisors. The first divisor $P_1 = (x_1, y_2)$ belongs to the plane curve X_1' defined by a nonalgebraic equation of the form

$$X_1': \qquad y^2 - F(x) = 0. \tag{3.9}$$

The second divisor $P_2 = (x_2, y_2)$ lies on the horizontal line X_2' defined by the equation

$$X_2': \qquad y - k = 0. \tag{3.10}$$

Thus, we can formulate the following proposition.

Proposition 4. *At $a_1 = a_2$ the common level surface of the first integrals* (3.5) *of the non-Hamiltonian vector field Z* (1.11) *is a product of two plane curves $X_1' \times X_2'$* (3.9), (3.10).

The proof is a straightforward calculation.

Divisors P_1 and P_2 determine compatible Poisson brackets, recursion operators, the algebra of Haantjes operators with vanishing Nijenhuis torsion and 2×2 Lax matrices. For instance, we can easily recover the Poisson bivector $\mathcal{P}'$ (3.3) obtained by the brute force method in the previous section

$$\mathcal{P}' = f_1(x_1)\left(\frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y_1}\right) + f_2(x_2)\left(\frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial y_2}\right),$$

where $f_{1,2}$ are given by (3.8).

Constructions of the Poisson brackets (1.3), recursion operators (1.4) and the algebra of Haantjes operators

$$L_0 = Id, \qquad L_i = \frac{\partial}{\partial x_i} \otimes dx_i + \frac{\partial}{\partial y_i} \otimes dy_i, \quad i = 1, 2,$$

are independent of the form of the plane curves X_1' and X_2' and the time variable, in contrast to the construction of the 2×2 Lax matrices

$$\mathcal{L} = \begin{pmatrix} V & U \\ W & -V \end{pmatrix} \quad \text{and} \quad \mathcal{A} = \begin{pmatrix} 0 & 1 \\ S & 0 \end{pmatrix}, \quad \text{so that} \quad \dot{\mathcal{L}} = [\mathcal{L}, \mathcal{A}],$$

here x is a spectral parameter, and the polynomials in x

$$U(x) = x_2(x - x_1)\,, \qquad V(x) = \frac{1}{2}\,\dot{U}(x)$$

are the standard Jacobi polynomials on a product of the plane curves X_1' and X_2' and

$$W(x) = \frac{F(x) - V^2(x)}{U(x)}, \qquad S(x) = -\frac{\dot{W}(x)}{2V(x)}$$

are functions on x.

Unfortunately, the notion of compatible Poisson brackets, recursion operators, the algebra of Haantjes operators with vanishing Nijenhuis torsion and Lax matrices cannot help us in the search for real trajectories of motion.

4. CONCLUSION

In this note we discuss a non-Hamiltonian vector field appearing in considering the motion of a Chaplygin ball rolling on a horizontal plane which rotates with constant angular velocity [6]. In two partial cases we present the division of this vector field by Hamiltonian vector fields using brute force computer calculations. In the first case one reduces equations of motion to Hamiltonian form which can also be done in the framework of Hamiltonization theory. In the second case the vector field is a sum of two Hamiltonian vector fields which cannot be obtained by using any type of Hamiltonization. In both cases we obtain Turiel type deformations of canonical Poisson brackets on the cotangent bundle to the sphere.

Calculation of at least two different representations of a given vector field via Hamiltonian vector fields is a crucial part of finding variables of separation in the Jacobi method. In any existing theory of Hamiltonization the main aim is to obtain one very special representation, which may not exist for the given mathematical model of a physical process, and, therefore, these theories are not applicable to the algorithmic search for variables of separation. We prefer to develop a computer version of the Jacobi method which could be applicable both to Hamiltonian and to non-Hamiltonian vector fields with relatively low numbers of equations of motion. Of course, these computer algorithms are not applicable to abstract nonholonimic systems with nonfixed arbitrary numbers of degrees of freedom.

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CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest.

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