Complex dynamics of Chaplygin sleigh due to periodic switch of the nonholonomic constraint location

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The study of the complex dynamics of nonlinear systems, including dynamic chaos, is a fundamental interdisciplinary problem. A productive approach within this scientific direction consists in development of generalized models covering a particular range of phenomena and applicable for description of systems of different nature. As a sample, we refer to the standard Chirikov-Taylor 2D map relating to conservative Hamiltonian systems and a similar model called Zaslavsky map for systems with dissipation, characterized by contraction of the phase volume and occurrence of regular or chaotic attractors depending on the parameters [1].

In mechanics, in addition to systems described with the Hamiltonian formalism and systems with friction (dissipative), a special class of objects with nonholonomic constraints is also distinguished. It includes many tasks of practical importance, for example, analysis of mobile devices in robotics. Hierarchy of types of behavior of nonholonomic systems embraces a variety of situations from simple (integrable) to complex (nonintegrable) [2]. In a nonholonomic system conserving the mechanical energy and symmetric to time inversion, phase volume elements locally can undergo compression or expansion in some domains of the phase space, which makes it possible to observe behaviors similar to attractors in dissipative systems. Basing on a paradigmatic example of nonholonomic system, the Chaplygin sleigh [3], we propose setup that provides complex dynamics with conservation of mechanical energy and leads to a map that can pretend to be a nonholonomic analog of the Chirikov-Taylor and Zaslavsky mappings.

Consider motion of the Chaplygin sleigh with moment of inertia J and mass m on horizontal plane, assuming that the nonholonomic mechanical constraint is imposed turn by turn for time period T at points $A_{1,2,3}$ on the sleigh located at the vertices of an equilateral triangle at a distance r=1 from the center of mass C. At the moment of constraint switching and further, while it is on, the speed of the corresponding point is supposed to be oriented in fixed direction relative to the sleigh defined as a direction of initial local velocity of this point at the instant of the switch. One can imagine that the supports providing contact of the sled with the plane are fabricated as “knife-edges” installed on the hinges at $A_{1,2,3}$, each of which is fixed successively for the time period T. While no fixation, a “knife-edge” does not affect the motion of the sledge, but it always retains direction of the local velocity. With the fixation, rotation of the “knife-edge” stops and the nonholonomic constraint appears to be imposed at this point.

The description consists in consideration of three periodically repeating stages as the constraints are placed in turn at $A_{1,2,3}$. At all the stages, including the switching, the mechanical energy is conserved. At each stage, one can use equations of the Chaplygin sleigh in the form [3], for which the variables and parameters should be re-defined from stage to stage.

Taking into account isotropy in the plane and conservation of energy W, which can be regarded naturally as a parameter, the problem can be reduced to a two-dimensional map. As dynamical variables at a moment of switch we use a normalized angular velocity $w_n = \omega(nT)\sqrt{\mu / 2W}$ and an angle $\gamma_n = \arg[U_C(nT) + iV_C(nT)] - \alpha(nT)$ between direction of motion of center of mass and radius vector of the point where the constraint is currently supplied.

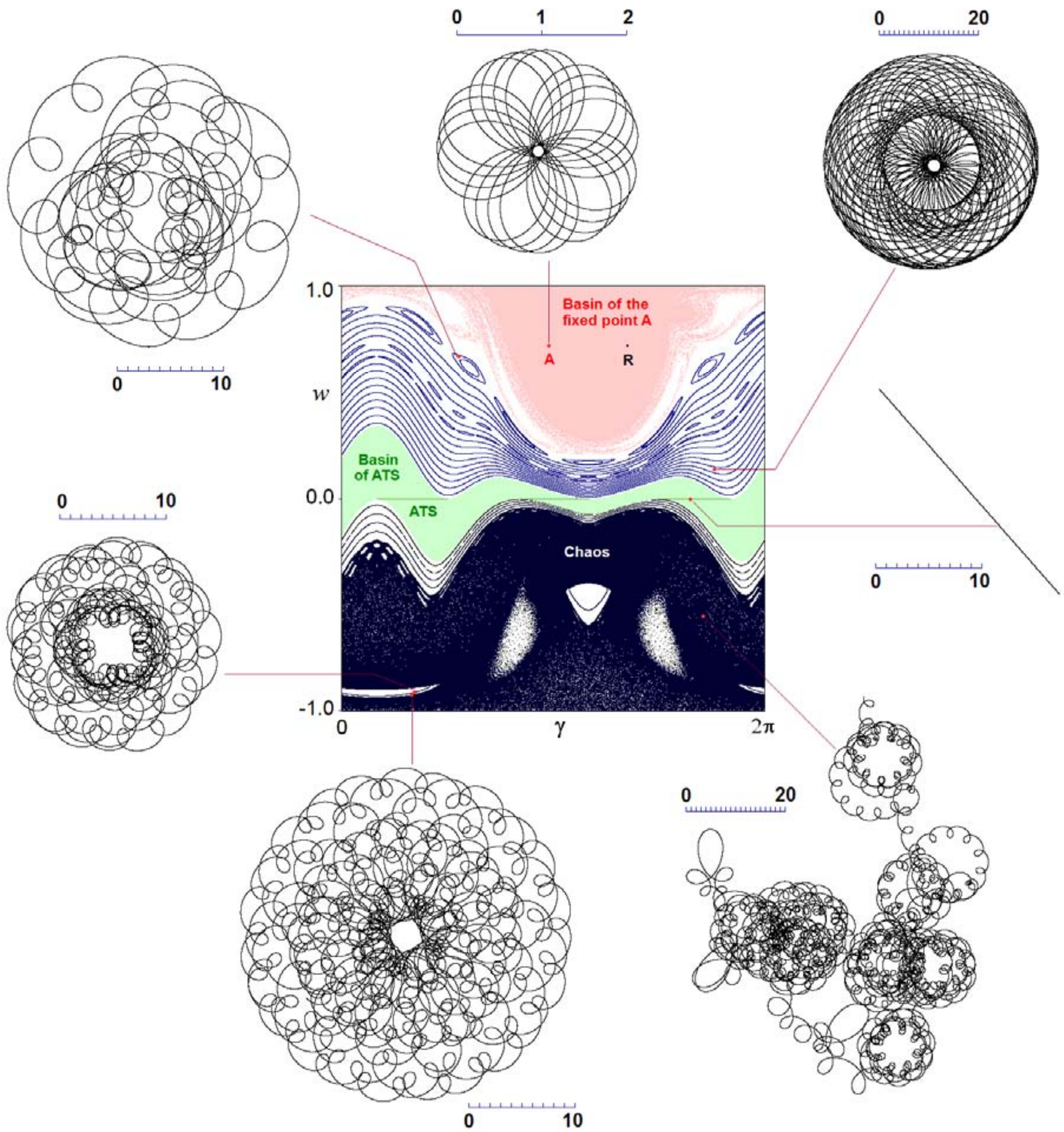

Fig. 1. At the center is the phase portrait of the mapping (1) on the plane (γ, ω) at T=5, μ=3, 2W=1.5129. Blue lines represent quasiperiodic motions corresponding to quasi- conservative dynamics; the same curves appear when solving the problem in reverse time. A vast area of dark blue in the lower part is "chaotic sea". A light green area is a basin of attraction of trivial motions on the axis w=0. The pink region is the attraction basin of the fixed point A. The fixed point R is repeller. On the periphery, the trajectories of the spatial displacement of the center of mass of the sleigh corresponding to the representative points of the phase plane are shown.

Using results of analytical solution of the problem, a map describing the state transformation over a period T can be represented as

$$\gamma_{n+1} = \arg\left[U\left(1 - G^2 - \frac{2iaG}{\sqrt{\mu + a^2}}\right)\right] - \frac{2\pi}{3}, \quad w_{n+1} = \frac{2G}{\left(1+G^2\right)\sqrt{1+\mu^{-1}a^2}}, \qquad (1)$$

where

$$U=\sqrt{1-w_n^2}\,\exp(i\gamma_n)+\frac{iw_n}{\sqrt{\mu}},\quad \mu=\frac{J}{mr^2},\quad a=|U|^{-1}\,\mathrm{Re}\,U,\quad G=\frac{w_n\sqrt{\mu+a^2}\,\exp\left(-\dfrac{aT\sqrt{2W}}{\mu+a^2}\right)}{\mu^{1/2}+\mu^{1/2}\,|U|-w_n\,|U|^{-1}\,\mathrm{Im}\,U}$$

Figure 1 shows a diagram in the plane (γ, ω) obtained from iterations of the map with different initial conditions. As for the Chirikov-Taylor map, one can see invariant curves either going from the left to the right edge, or appearing as closed formations, and a "chaotic sea" located at the bottom part of the picture.

A fundamental attribute is presence of objects intrinsic rather to dissipative dynamics. In particular, there is a fixed point *A*, which is an attractor. Also, there is a repeller, the point R, to which the map iterations in reverse time approach. Additionally, there is a trivial attracting set ATS on the axis w=0. It corresponds to motions of the sleigh without rotation, with a constant speed of the translational motion, when switching the “knife-edges” occurs without changing their orientation in the laboratory frame. This is a set of fixed points of the two-dimensional map defined for the total switching period 3T. On periphery, the diagram shows trajectories of the center of mass in the laboratory reference frame corresponding to representative types of motion. Note a chaotic, similar to a diffusive migration of the sleigh on trajectory containing numerous curls in the right bottom picture, which corresponds to the "chaotic sea".

Summing up, we considered a nontraditional Chaplygin sleigh modification of with periodically switched place of the nonholonomic constrains, and derived 2D map, which appears as an analog of the Chirikov-Taylor map for the nonholonomic system. Depending on parameters and initial conditions, regular and chaotic motions take place as demonstrated computationally.

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[2] Borisov A.V., Mamaev I.S., Bizyaev I.A. // Rus. J. Nonl. Dyn., 2013, vol. 9, no. 2, pp. 141-202.

[3] Borisov A.V., Mamaev I.S. // J. Appl. Math. & Mech., 2009, vol. 73, no 2, pp. 156-161.